Number 334

Manfred Droste

Structure of partially ordered sets with transitive automorphism groups

Memoirs
of the American Mathematical Society

Providence · Rhode Island · USA

September 1985 · Volume 57 · Number 334 (end of volume) · ISSN 0065-9266

Memoirs of the American Mathematical Society
Number 334

Manfred Droste

Structure of partially ordered sets with transitive automorphism groups

Published by the
AMERICAN MATHEMATICAL SOCIETY
Providence, Rhode Island, USA

September 1985 · Volume 57 · Number 334 (end of volume)

MEMOIRS of the American Mathematical Society

SUBMISSION. This journal is designed particularly for long research papers (and groups of cognate papers) in pure and applied mathematics. The papers, in general, are longer than those in the TRANSACTIONS of the American Mathematical Society, with which it shares an editorial committee. Mathematical papers intended for publication in the Memoirs should be addressed to one of the editors:

PREPARATION OF COPY. Memoirs are printed by photo-offset from camera-ready copy prepared by the authors. Prospective authors are encouraged to request a booklet giving detailed instructions regarding reproduction copy. Write to Editorial Office, American Mathematical Society, Box 6248, Providence, RI 02940. For general instructions, see last page of Memoir.

SUBSCRIPTION INFORMATION. The 1985 subscription begins with Number 314 and consists of six mailings, each containing one or more numbers. Subscription prices for 1985 are $188 list, $150 institutional member. A late charge of 10% of the subscription price will be imposed on orders received from nonmembers after January 1 of the subscription year. Subscribers outside the United States and India must pay a postage surcharge of $10; subscribers in India must pay a postage surcharge of $15. Each number may be ordered separately; *please specify number* when ordering an individual number. For prices and titles of recently released numbers, see the New Publications sections of the NOTICES of the American Mathematical Society.

BACK NUMBER INFORMATION. For back issues see the AMS Catalogue of Publications.

Subscriptions and orders for publications of the American Mathematical Society should be addressed to American Mathematical Society, Box 1571, Annex Station, Providence, RI 02901-1571. *All orders must be accompanied by payment.* Other correspondence should be addressed to Box 6248, Providence, RI 02940.

MEMOIRS of the American Mathematical Society (ISSN 0065-9266) is published bimonthly (each volume consisting usually of more than one number) by the American Mathematical Society at 201 Charles Street, Providence, Rhode Island 02904. Second Class postage paid at Providence, Rhode Island 02940. Postmaster: Send address changes to Memoirs of the American Mathematical Society, American Mathematical Society, Box 6248, Providence, RI 02940.

The paper used in this journal is acid-free and falls within the guidelines established to ensure permanence and durability.

Library of Congress Cataloging-in-Publication Data

Droste, Manfred, 1956-
Structure of partially ordered sets with transitive automorphism groups.

(Memoirs of the American Mathematical Society, ISSN 0065-9266; no. 334 (Sept. 1985))
"Volume 57, number 334 (end of volume)."
Bibliography: p.
1. Partially ordered sets. 2. Groups, Multiply transitive. 3. Automorphisms. I. Title. II. Series: Memoirs of the American Mathematical Society; no. 334.
QA3.A57 no. 334 [QA171.485] 510s [511.3'2] 85-15625
ISBN 0-8218-2335-3

ABSTRACT

In this paper, we study the structure of infinite partially ordered sets $(\Omega,\leq)$ under suitable transitivity assumptions on their group $A(\Omega) = \mathrm{Aut}(\Omega,\leq)$ of all order-automorphisms of $(\Omega,\leq)$.

Let $k \in \mathbb{N}$. We call $A(\Omega)$ k-transitive (k-homogeneous) if whenever $A,B \subseteq \Omega$ are two subsets of Ω each with k elements and $\varphi: (A,\leq) \to (B,\leq)$ is an isomorphism, then there exists an automorphism $\alpha \in A(\Omega)$ which maps A onto B (which extends φ), respectively. $A(\Omega)$ is ω-transitive (ω-homogeneous), if $A(\Omega)$ is k-transitive (k-homogeneous) for each $k \in \mathbb{N}$.

We show that under the assumption that $A(\Omega)$ is k-transitive or k-homogeneous for some $2 \leq k \in \mathbb{N}$ various sufficiently complicated structures $(\Omega,\leq)$ exist, and we give a classification and characterization of these structures. As one of many consequences we obtain that for each $k \geq 2$, k-transitivity of $A(\Omega)$ is indeed weaker than k-homogeneity, but, surprisingly, for any partially ordered set $(\Omega,\leq)$, $A(\Omega)$ is ω-transitive iff $A(\Omega)$ is ω-homogeneous.

1980 Mathematics Subject Classification:

Primary: 06A10, 20B22

Secondary: 20B27, 06A12

Key words and phrases: Partially ordered set, order-automorphism, k-transitive automorphism group, k-homogeneous automorphism group, order-preserving permutation, linearly ordered set, chains, antichains.

ACKNOWLEDGEMENTS

This paper consists of a revised version of the first part of the author's doctoral dissertation written at the University of Essen under the supervision of Professor R. Göbel; part of the research was also done in 1981 in Bowling Green (Ohio, U.S.A.). Several sections have been rewritten during the preparation of this memoir. The author wishes to express his appreciation to both Professor R. Göbel (Essen) and Professor W.C. Holland (Bowling Green) for their encouragement and continuous support. He would also like to thank Professor A.M.W. Glass for stimulating discussions in Bowling Green.

This paper is dedicated to my parents.

TABLE OF CONTENTS

§1 INTRODUCTION

In this paper let $(\Omega,\leq)$ always be an infinite partially ordered set and $A(\Omega) = \mathrm{Aut}(\Omega,\leq)$ the group of all order-automorphisms of $(\Omega,\leq)$. If $(\Omega,\leq)$ is linearly ordered (a chain) and $k \in \mathbb{N}$, $(\Omega,A(\Omega))$ or simply Ω is called k-homogeneous if the following condition holds:

(+) Whenever $A,B \subseteq \Omega$ are subsets of Ω with $|A| = |B| = k$, there exists $\alpha \in A(\Omega)$ with $A^{\alpha} = B$.

Doubly homogeneous chains $(\Omega,\leq)$ and their automorphism groups $A(\Omega)$ have been extensively studied. These groups can be used e.g. for the construction of certain infinite simple groups (Higman [20]). In a natural way they also become important examples of lattice-ordered groups, and any lattice-ordered group can be embedded into the automorphism group of some doubly homogeneous chain (Holland [23]). The interplay between the structure of such chains and the normal subgroup lattices of their automorphism groups was studied in [1,2,10-12]. Obviously, $\mathbb{Q}$, $\mathbb{R}$, and more generally all linearly ordered fields are examples of 2-homogeneous chains. For a variety of further results see Glass [17].

In this memoir we examine the structure of partially ordered sets (p.o. sets) $(\Omega,\leq)$ under similar transitivity conditions as (+). A study of this kind was already proposed by H. Wielandt [32]. Answering another question of Wielandt, in [9] we showed that the assumption (+) for $(\Omega,\leq)$, even if k is only assumed to be an arbitrary cardinal with $2 \leq k \leq |\Omega|$, is very strong: Then either the order on Ω is trivial, or $(\Omega,\leq)$ is linearly ordered and k is finite. Almost the same conclusion is true, as shown here in §3, even if we just suppose that any two subsets of Ω of cardinality k are elementarily equivalent in the first order language of

---

Received by the editors September 12, 1983 and, in revised form March 1, 1985.

predicate calculus for partially ordered sets. Therefore, the following weakening of (+) was suggested by W.C. Holland.

Let $(\Omega,\leq)$ be a p.o. set and $k \in \mathbb{N}$. We call $A(\Omega)$ *k-transitive (k-homogeneous)*, if whenever $A,B \subseteq \Omega$ each have k elements and $\varphi: A \to B$ is an isomorphism, then there exists $\alpha \in A(\Omega)$ with $A^{\alpha} = B$ $(\alpha|_A = \varphi)$, respectively. $A(\Omega)$ is *ω-transitive (ω-homogeneous)*, if $A(\Omega)$ is k-transitive (k-homogeneous) for each $k \in \mathbb{N}$, respectively.

For chains $(\Omega,\leq)$ condition (+), k-transitivity, and k-homogeneity of $A(\Omega)$ coincide $(k \in \mathbb{N})$. Obviously, k-transitivity is always implied by k-homogeneity and ω-transitivity by ω-homogeneity. Henson [18,19] showed that there are $2^{\aleph_0}$ non-isomorphic countable binary relational structures with ω-homogeneous automorphism group; precisely countably many of these are graphs (Lachlan and Woodrow [26]), for related results see [14-16,25, 28]. Schmerl [30] characterized all countable p.o. sets $(\Omega,\leq)$ with ω-homogeneous automorphism group. We will obtain his result as a consequence of our considerations. Lattices with certain homogeneity properties were investigated in [5,6,13]. For further related work, see [3,4,8,21,22, 27].

We will examine infinite p.o. sets $(\Omega,\leq)$ of arbitrary cardinality under the assumption that $A(\Omega)$ is k-transitive for some $2 \leq k \in \mathbb{N}$. Under this assumption instead of (+), various different and complicated structures $(\Omega,\leq)$ are possible. We now give a summary of our results. We derive a classification of these partial orders in §4, and in this and subsequent sections we obtain in almost all cases a characterization of the condition that $A(\Omega)$ is k-transitive for some $k \geq 2$ by the structure of the p.o. set $(\Omega,\leq)$. In many of these cases, we either give an explicit description of the order on Ω or we reduce it to the structure of doubly homogeneous chains, which we consider as basic in our study.

Sections 5 and 6 deal with the special case that $(\Omega,\leq)$ is a "tree"; then, in particular, Ω is not a chain, for any two elements $a,b \in \Omega$ there exists $c \in \Omega$ with $c < a$ and $c < b$, and for each $a \in \Omega$ the set $\{x \in \Omega;\ x < a\}$ is a dense chain. We prove in §5 that in this case $A(\Omega)$ can be k-transitive or m-homogeneous only for $k \in \{1,2,3\}$ and

$m \in \{1,2\}$, respectively. We show that 3-transitivity of $A(\Omega)$ implies 2-homogeneity (which trivially implies 2-transitivity), and we characterize by the structure of $(\Omega,\leq)$ when these implications can be reversed. As a consequence, we obtain that if $A(\Omega)$ is 2-homogeneous, then it also satisfies certain other kinds of higher homogeneity properties. For instance, if $(\Omega,\leq)$ is, in addition, assumed to be a meet-semilattice, then any isomorphism between two maximal subchains of Ω extends to an automorphism of Ω; for countable trees the converse is also true, as shown in §6.

In section 6 we first construct a large class of trees of arbitrary cardinality with 2-homogeneous or 3-transitive automorphism groups and various additional properties. This provides many examples for the results of §5 and also a solution of a problem of Fraïssé [14], since for these partial orders $A(\Omega)$ is 1- and 2-homogeneous, but, as noted above, clearly not ω-homogeneous. Then we give an explicit characterization and construction of all countable trees $(\Omega,\leq)$ with 2-transitive automorphism groups; it follows that then $A(\Omega)$ is also 2-homogeneous. Up to isomorphism, there are precisely countably many such countable trees, of which countably many are meet-semilattices and equally many not. Moreover, there is a unique countable tree with 3-transitive automorphism group.

Another important class of p.o. sets with 2-transitive automorphism groups consists of those sets $(\Omega,\leq)$ in which any two elements have a lower and an upper bound and which contain a subset $\{a,b,c\} \subseteq \Omega$ such that $a < b$ and c is incomparable with both a and b. Here we have for any $n \in \mathbb{N}$ and all sufficiently large $k > n$ the following implication: If $A(\Omega)$ is k-transitive, then $A(\Omega)$ is n-homogeneous and any finite p.o. set $(P,\leq)$ with at most n elements can be embedded into $(\Omega,\leq)$. As a special case we see that if $A(\Omega)$ is ω-homogeneous, any finite p.o. set $(P,\leq)$ can be embedded into $(\Omega,\leq)$; for countable sets Ω this was proved by Schmerl [30].

Finally, in section 8 we study the relationship between k-transitivity and k-homogeneity of $A(\Omega)$ and various properties of the order on Ω. For instance, there exist precisely two countable lattices $(\Omega,\leq)$ for which $A(\Omega)$ is k-transitive for some $k \geq 3$. If $(\Omega,\leq)$ is any p.o. set and $A(\Omega)$ k-transitive for some $k \geq 2$, the maximal chains (antichains) in Ω are either all infinite or all have the same finite cardinality, respectively; as shown by example, here it can happen that two infinite maximal subchains of Ω have different cardinality.

It is well-known that chains $(\Omega,\leq)$ have the following property: If $A(\Omega)$ is k-transitive for some $k \geq 2$, then $A(\Omega)$ is ω-transitive. Chains are not the only p.o. sets with this property, but for arbitrary p.o. sets $(\Omega,\leq)$ this implication fails by two reasons. First, as already mentioned, trees are examples of p.o. sets Ω for which $A(\Omega)$ can be k-transitive or k-homogeneous only for small values of $k \in \mathbb{N}$. Secondly, we characterize all p.o. sets $(\Omega,\leq)$ for which $A(\Omega)$ is k-transitive or k-homogeneous for some large, but not for smaller values of $k \in \mathbb{N}$; there exist precisely countably many such countable sets. However, we show that if $A(\Omega)$ is k-transitive for some $k \geq 2$, then $A(\Omega)$ is always also n-transitive either for each $n \leq k$ or for each $n \geq k$. Moreover, k-transitivity ($k \geq 2$) is inherited not only by either all smaller or all larger values, but also by intermediate values: If $2 \leq m < n < k$ and $A(\Omega)$ is both m- and k-transitive, then again $A(\Omega)$ is also n-transitive. These results remain true if "transitive" is replaced by "homogeneous". Furthermore, we show that in general for each $k \geq 2$ the assumption of k-transitivity of $A(\Omega)$ is strictly weaker than that of k-homogeneity. Hence the following result is quite surprising.

Let $(\Omega,\leq)$ be any infinite p.o. set. Then $A(\Omega)$ is ω-transitive iff $A(\Omega)$ is ω-homogeneous.

Here the question arises for which other binary relations than partial order an analogous equivalence holds.

We conclude with a list of open problems in §9.

§2 NOTATION

For the convenience of the reader, we summarize our notation here. Background information on linearly ordered sets may be found in Glass [17] or Rosenstein [29].

<u>*SETS*</u>. Let $A \dot{\cup} B$, $\dot{\cup} A_i$ denote <u>disjoint</u> unions. Let $\mathbb{N} = \{1,2,3,...\}$ denote the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \dot{\cup} \{0\}$, and $\mathbb{N}_\infty = \mathbb{N} \dot{\cup} \{\aleph_0\}$. For a mapping f let $f|_A$ designate its restriction to A, a^f its value at a, and $A^f = \{a^f;\ a \in A\}$; the composition of mappings is from left to right. Let $S(M)$ denote the symmetric group of all permutations of a set M and id_M (or id, if there is no ambiguity) the identity map of M. If A_i $(i \in I)$ are pairwise disjoint sets and $\alpha_i : A_i \to M_i$ maps, we denote by $\alpha = \bigoplus_{i \in I} \alpha_i$ the map from $\bigcup_{i \in I} A_i$ into $\bigcup_{i \in I} M_i$ defined by $\alpha|_{A_i} = \alpha_i$ $(i \in I)$.

<u>*PARTIALLY ORDERED SETS*</u>. A set Ω with a reflexive, antisymmetric, transitive relation $\leq$ defined on it is called a *partially ordered set*, or *p.o. set*. We always let subsets of p.o. sets carry the induced partial order. Let $A \subseteq (\Omega, \leq)$. Then A is called *linearly ordered* or a *chain*, if $a \leq b$ or $b \leq a$ whenever $a,b \in A$, and A is *trivially ordered* or an *antichain*, if $a \leq b$ implies $a = b$ for any $a,b \in A$. We say that A is *dense in* Ω if whenever $a,b \in \Omega$ with $a < b$, there is some $c \in A$ with $a < c < b$. We call A *bounded above (below) in* Ω if there exists $x \in \Omega$ with $a \leq x$ $(x \leq a)$ for all $a \in A$; A is *unbounded above (below) in* Ω if it is not bounded above (below) in Ω; A is *unbounded (bounded) in* Ω if it is unbounded (bounded) both above and below in Ω, respectively. The set $(A, \leq)$ is called *dense (bounded above, etc.)* if $(A, \leq)$ is dense (bounded above, etc.) in itself. In particular, a chain $(A, \leq)$ is unbounded iff A contains neither a greatest nor a smallest element.

For elements $a,b \in \Omega$ we write $a \parallel b$ if a and b are incomparable, i.e. if neither $a \leq b$ nor $b \leq a$ (in particular, $a \neq b$), and $a \nleq b$

($a \nleq b$, $a \nparallel b$), if not $a \leq b$ ($a < b$, $a \parallel b$), respectively. For subsets $A,B \subseteq \Omega$ let $A \parallel B$ ($A < B$, $A \leq B$) denote that $a \parallel b$ ($a < b$, $a \leq b$), respectively, for all $a \in A$, $b \in B$. In particular, $A \parallel B$ and $A < B$ each imply that A and B are disjoint. We also write $a \parallel B$ ($a < B$, $a \leq B$) for $\{a\} \parallel B$ ($\{a\} < B$, $\{a\} \leq B$).

ISOMORPHISMS. A mapping $\varphi: (A,\leq) \to (B,\leq)$ is called an *embedding*, if for any $a,b \in A$ we have $a \leq b$ iff $a^\varphi \leq b^\varphi$. In particular, each embedding is injective. A bijective embedding is called an *isomorphism*, and an isomorphism from $(A,\leq)$ onto itself is an *automorphism* of $(A,\leq)$. If $(P,\leq)$ is a p.o. set, let $A(P)$ always be the group of all automorphisms of $(P,\leq)$.

CONVENTIONS ADOPTED. $(\Omega,\leq)$ always denotes a fixed infinite p.o. set. Capital roman letters $A,B,C,\ldots$ are used for subsets of Ω, small roman letters $a,b,c,\ldots,x,y,z$ for elements of Ω, and small Greek letters α, $\beta,\gamma,\ldots$ for automorphisms of Ω, i.e. elements of $A(\Omega)$. For mappings we use sometimes $\alpha,\beta,\gamma,\psi,\varphi,\pi$, sometimes f,g,h.

§3 TRANSITIVE AUTOMORPHISM GROUPS

<u>3.1. Chains</u>. In this section we start with our examination of the structure of infinite p.o. sets whose automorphism groups satisfy suitable transitivity assumptions.

In this paper, $(\Omega,\leq)$ *will always denote an infinite p.o. set.*

The following definition originated from a proposal by W.C. Holland.

<u>Definition 3.1.1</u>. *(a) Let* $k \in \mathbb{N}$. $A(\Omega)$ *is called* k*-transitive* *(*k*-homogeneous), if whenever* $A,B \subseteq \Omega$ *each have* k *elements and* $\varphi: (A,\leq) \to (B,\leq)$ *is an isomorphism, there exists* $\alpha \in A(\Omega)$ *with* $A^\alpha = B$ $(\alpha|_A = \varphi)$*, respectively.*

(b) $A(\Omega)$ *is called* ω*-transitive (*ω*-homogeneous), if* $A(\Omega)$ *is* k*-transitive (*k*-homogeneous) for all* $k \in \mathbb{N}$*, respectively.*

In other words, $A(\Omega)$ is k-transitive (k-homogeneous) if for any two isomorphic subsets A,B of Ω with k elements, some (any) isomorphism from A onto B extends to an automorphism of Ω. Trivially, 1-transitivity and 1-homogeneity of $A(\Omega)$ coincide, and if $A(\Omega)$ if k-homogeneous for some $k \in \mathbb{N}$ (ω-homogeneous), then it is also k-transitive (ω-transitive), respectively. Since the following well-known proposition is very basic and important for our setting, we include its proof:

<u>Proposition 3.1.2</u> (cf. Wielandt [32; Satz 6.18]). *Let* $(\Omega,\leq)$ *be an infinite chain and* $2 \leq k \in \mathbb{N}$*. The following are equivalent:*

(1) $A(\Omega)$ *is* k*-transitive.*

(2) $A(\Omega)$ *is* k*-homogeneous.*

(3) Whenever $A,B \subseteq \Omega$ *each have* k *elements, there exists* $\alpha \in A(\Omega)$ *with* $A^\alpha = B$.

(4) $A(\Omega)$ *is* ω*-transitive.*

(5) $(\Omega,\leq)$ *has neither a greatest nor a smallest element, and any two intervals* $[a,b]$, $[c,d]$ *of* Ω *are order-isomorphic.*

Here $[x,y] = \{z \in \Omega;\ x \leq z \leq y\}$ for any $x,y \in \Omega$ with $x < y$.

Proof. *(4) → (1) ↔ (2) ↔ (3):* Trivial.

(1) → (5): Suppose $a \in \Omega$ were the greatest element of Ω. Choose elements $a_1,\ldots,a_k \in \Omega$ such that $a_1 < a_2 < \ldots < a_k < a$. By assumption, there exists $\alpha \in A(\Omega)$ with $\{a_1,\ldots,a_k\}^\alpha = \{a_2,\ldots,a_k,a\}$. Then $a = a_k{}^\alpha < a^\alpha$, a contradiction. Similarly, Ω has no smallest element.

Now let $a,b,c,d \in \Omega$ with $a < b$, $c < d$. Since Ω has no smallest element, we may choose a subset $A \subseteq \Omega$ with $|A| = k-2$ and $A < \{a,c\}$. There exists $\alpha \in A(\Omega)$ with $A^\alpha = A$, $a^\alpha = c$, $b^\alpha = d$. Then $\alpha|_{[a,b]}$ is an isomorphism from $[a,b]$ onto $[c,d]$.

(5) → (4): Let $n \in \mathbb{N}$ and $A = \{a_1,\ldots,a_n\}$, $B = \{b_1,\ldots,b_n\} \subseteq \Omega$ such that $a_i < a_{i+1}$, $b_i < b_{i+1}$ for each $1 \leq i \leq n-1$. Choose $a_0, a_{n+1} \in \Omega$ such that $a_0 < \{a_1,b_1\}$ and $\{a_n,b_n\} < a_{n+1}$. Then, if we put $b_0 = a_0$, $b_{n+1} = a_{n+1}$, for each $i \in \{0,\ldots,n\}$ there exists an isomorphism φ_i from $[a_i,a_{i+1})$ onto $[b_i,b_{i+1})$. Let $Z = \{x \in \Omega;\ x < a_0 \text{ or } a_{n+1} \leq x\}$. Then $\alpha = id_Z \oplus \bigoplus_{i=0}^{n} \varphi_i \in A(\Omega)$ maps A onto B. Hence $A(\Omega)$ is n-transitive. The result follows.

As a consequence, we note:

Remark 3.1.3. *If $(\Omega,\leq)$ is an infinite chain with 2-transitive automorphism group, then $(\Omega,\leq)$ is dense and unbounded. Consequently, there exists up to isomorphism precisely one countable chain with 2-transitive automorphism group, namely $(\mathbb{Q},\leq)$.*

It is well-known that for each infinite cardinal $\aleph$ there exists a chain $(\Omega,\leq)$ of cardinality $\aleph$ with 2-transitive automorphism group; for instance, we can choose $(\Omega,\leq)$ to be a linearly ordered field of cardinality $\aleph$. A construction of chains Ω with 2-transitive automorphism groups having prescribed normal subgroup lattices is contained in Droste and Shelah [12].

In the situation of Proposition 3.1.2, the equivalence of conditions (1)-(3) is obvious. However, we will see that in general, if $(\Omega,\leq)$ is just supposed to be a p.o. set, these conditions are no longer equivalent.

In this section we examine condition (3) for p.o. sets which turns out to be very strong.

We say that two subsets $A,B \subseteq \Omega$ are elementarily equivalent (denoted by $A \equiv B$), if $(A,\leq)$ and $(B,\leq)$ satisfy the same first order sentences of predicate calculus in the language of partially ordered sets. Clearly, $A \cong B$ implies $A \equiv B$. If $(\Omega,\leq)$ is partially ordered, the inverse ordering $\leq'$ on Ω is defined by $a \leq' b$ iff $b \leq a$ $(a,b \in \Omega)$. Now we show:

<u>Theorem 3.1.4</u>. *Let* $(\Omega,\leq)$ *be an infinite p.o. set and* k *an arbitrary cardinal with* $2 \leq k \leq |\Omega|$. *Then the following are equivalent:*

(1) Any two subsets of Ω *of cardinality* k *are elementarily equivalent.*

(2) Any two subsets of Ω *of cardinality* k *are isomorphic.*

(3) $(\Omega,\leq)$ *and* k *satisfy one of the following three (mutually exclusive) conditions:*

(a) $(\Omega,\leq)$ *is trivially ordered.*

(b) $(\Omega,\leq)$ *is linearly ordered, and* $k \in \mathbb{N}$.

(c) We have $k = |\Omega|$, *and either* $(\Omega,\leq)$ *or* Ω *with the inverse ordering is well-ordered and, moreover, isomorphic to the least ordinal of cardinality* k.

<u>*Proof*</u>. <u>*(3) → (2)*</u>: Obvious, since in case of (3c), each subset of Ω of cardinality k is isomorphic to Ω.

<u>*(2) → (1)*</u>: Trivial.

<u>*(1) → (3)*</u>: Let us assume that the order on Ω is not trivial. We distinguish between two cases, $k \in \mathbb{N}$ and $k = \infty$.

<u>*Case I*</u>. *Assume* $k \in \mathbb{N}$.

We first show that if there exists a chain $A \subseteq \Omega$ with $|A| = n \in \mathbb{N}$ and $2 \leq n \leq k$, then there is also a chain $B \subseteq \Omega$ with $n+1$ elements. Indeed, choose $D,E \subseteq \Omega$ with $A \subseteq E$, $|E| = k$, and $|D| = k^2$. Split $D = \dot{\bigcup}_{i=1}^{k} C_i$ with $|C_i| = k$ for all $i = 1,\ldots,k$. Let $i \in \{1,\ldots,k\}$. Since $C_i \equiv E$ and $A \subseteq E$, there exists a chain $B_i \subseteq C_i$ with $|B_i| = |A_i| = n$. Let $a_i = \min B_i \in C_i$. Now put $C = \{a_i;\ i = 1,\ldots,k\}$. By $C \equiv E$, C again contains a chain with $n \geq 2$ elements. In particular, $a_i < a_j$ for some

$i,j \in \{1,\dots,k\}$. Hence $B = B_j \dot{\cup} \{a_i\}$ is a chain with $n+1$ elements.

Since by assumption there exist two elements $y,z \in \Omega$ with $y < z$, by induction we obtain a chain $L \subseteq \Omega$ with $|L| = k$. Now if $a,b \in \Omega$, choose $T \subseteq \Omega$ with $a,b \in T$ and $|T| = k$. Then $T \equiv L$ and either $a \leq b$ or $b < a$. Hence $(\Omega,\leq)$ is linearly ordered.

Case II. Let k be infinite.

We let k also denote the least ordinal of cardinality k. Choose $x \in \Omega$. Let $T = \{a \in \Omega;\ a \parallel x\}$. We first show $|T| < k$. Let $y,z \in \Omega$ with $y < z$. Choose $Y \subseteq \Omega$ with $y,z \in Y$ and $|Y| = k$, and decompose $Y = \dot{\bigcup}_{i \in I} Y_i$ with $|I| = |Y_i| = k$ for each $i \in I$. Since $Y_i \equiv Y$ for each $i \in I$, there are $y_i, z_i \in Y_i$ with $y_i < z_i$. Let $X = \{y_i, z_i;\ i \in I\}$. So $|X| = k$, and for each $c \in X$ there exists $d \in X$ with $c < d$ or $d < c$. Now if we had $|T| \geq k$, we could choose $S \subseteq T$ with $|S| = k$. Then $X \equiv S \dot{\cup} \{x\}$, in contradiction to $x \parallel S$. Hence $|T| < k$.

In particular, we have $|\{a \in \Omega;\ a > x\}| = |\Omega|$ or $|\{a \in \Omega;\ a < x\}| = |\Omega|$. W.l.o.g. we assume the first equality and then show that $(\Omega,\leq)$ is well-ordered. First we construct a well-ordered chain $A \subseteq \Omega$ with $A \cong k$. To do this, choose $Z \subseteq \{a \in \Omega;\ a \geq x\}$ with $x \in Z$ and $|Z| = k$. Each subset of Ω of cardinality k is elementarily equivalent to Z and thus contains a smallest element. Hence we can inductively choose elements $a_i \in Z$ $(i \in k)$ such that $\{a_j;\ j < i\} < a_i = \min(Z \setminus \{a_j;\ j < i\})$ for each $i \in k$. Then put $A = \{a_i;\ i \in k\}$.

Next we claim that $(\Omega,\leq)$ is linearly ordered. Indeed, let $a,b \in \Omega$. Choose $V \subseteq \Omega$ with $a,b \in V$ and $|V| = k$. Then $V \equiv A$, implying $a \leq b$ or $b < a$.

Now we show that there is no countable set $B \subseteq \Omega$ which is isomorphic to the set M of all negative integers with their natural ordering. Otherwise, let $m = \max B$. If $A < m$, $A \dot{\cup} \{m\}$ has a greatest element, but A does not, contradicting $A \equiv A \dot{\cup} \{m\}$. Hence $m < a_i$ for some $i \in k$. But now we obtain a (dual) contradiction by $A \equiv B \dot{\cup} \{a_j;\ i \leq j \in k\}$.

Consequently, $(\Omega,\leq)$ is well-ordered. Split $\Omega = C \dot{\cup} D$ such that