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SERIES EDITOR'S STATEMENT

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive change of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the nonspecialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

Gian-Carlo Rota

PREFACE

This book is the second in a three-volume series, the first of which is *Theory of Matroids*, and the third of which will be called *Combinatorial Geometries: Advanced Theory*. The three volumes together will constitute a fairly complete survey of the current knowledge of matroids and their closely related cousins, combinatorial geometries. As in the first volume, clear exposition of our subject has been one of our main goals, so that the series will be useful not only as a reference for specialists, but also as a textbook for graduate students and a first introduction to the subject for all who are interested in using matroid theory in their work.

This volume begins with three chapters on coordinatization or vector representation, by Fournier and White. They include a general chapter on 'Coordinatizations,' and two chapters on the important special cases of 'Binary Matroids' and 'Unimodular Matroids' (also known as regular matroids). These are followed by two chapters by Brualdi, titled 'Introduction to Matching Theory' and 'Transversal Matroids,' and a chapter on 'Simplicial Matroids' by Cordovil and Lindström. These six chapters, together with Oxley's 'Graphs and Series-Parallel Networks' from the first volume, constitute a survey of the major special types of matroids, namely, graphic matroids, vector matroids, transversal matroids, and simplicial matroids. We follow with two chapters on the important matroids invariants, 'The Möbius Function and the Characteristic Polynomial' by Zaslavsky and 'Whitney Numbers' by Aigner. We conclude with a chapter on the aspect of matroid theory that is primarily responsible for an explosion of interest in the subject in recent years, 'Matroids in Combinatorial Optimization' by Faigle.

My deepest thanks are due to the contributors to this volume, and to all others who have helped, including chapter referees. I am particularly indebted to Henry Crapo for continued support in securing the graphics work for all three of these volumes. Richard Brualdi thanks the National Science Foundation for their partial support of his work under grant DMS-8320189.

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CONTENTS

1

Coordinatizations

NEIL WHITE

1.1. Introduction and Basic Definitions

The purpose of this chapter is to provide background and general results concerning coordinatizations, while the more specialized subtopics of binary and unimodular matroids are covered in later chapters. The first section of this chapter is devoted to definitions and notational conventions. The second section concerns linear and projective equivalence of coordinatizations. Although they are not usually explicitly considered in other expositions of matroid coordinatization, these equivalence relations are very useful in working with examples of coordinatizations, as well as theoretically useful as in Proposition 1.2.5. Section 1.3 involves the preservation of coordinatizability under certain standard matroid operations, including duality and minors. The next section presents some well-known counterexamples, and Section 1.5 considers characterizations of coordinatizability, especially characterizations by excluded minors. The final five sections are somewhat more technical in nature, and may be omitted by the reader who desires only an introductory survey. Section 1.6 concerns the bracket conditions, another general characterization of coordinatizability. Section 1.7 presents techniques for construction of a matroid requiring a root of any prescribed polynomial in a field over which we wish to coordinatize it. These techniques are extremely useful in the construction of examples and counterexamples, yet are not readily available in other works, except Greene (1971). The last three sections concern characteristic sets, the use of transcendentals in coordinatizations, and algebraic representation (i.e., modeling matroid dependence by algebraic dependence). Some additional topics which could have been considered here, such as chain groups, are omitted because they are well-covered in other readily available sources, such as Welsh (1976).

Since the prototypical example of a matroid is an arbitrary subset of a finite dimensional vector space, that is, a vector matroid, and since many matroid operations have analogs for vector spaces, which are algebraic and therefore easier to employ, a natural and important problem is to determine which matroids are isomorphic to vector matroids. This leads directly to the concept of coordinatization. In this chapter we assume that matroids are finite.

A *coordinatization* of a matroid $M(S)$ in a vector space V is a mapping $\zeta:S\to V$ such that for any $A\subseteq S$, A is independent in $M\Leftrightarrow\zeta|_A$ is injective (one-to-one) and $\zeta(A)$ is linearly independent in V.

Thus we note that a dependent set in M may either be mapped to a linearly dependent set in V or mapped non-injectively.

We note that $\zeta(s)=0$ if and only if s is a loop. Moreover for non-loops s and t, $\zeta(s)$ is a non-zero scalar multiple of $\zeta(t)$ if and only if $\{s,t\}$ is a circuit (i.e., s and t are parallel). Thus $\zeta(s)=\zeta(t)$ only if $\{s,t\}$ is a circuit, and we see that non-injective coordinatizations exist only for matroids which are not combinatorial geometries. Furthermore, we also see that coordinatizing a matroid is essentially equivalent to coordinatizing its associated combinatorial geometry.

If B is any basis of $M(S)$, then let W be the span of $\zeta(B)$ in V. Then dim $W=$ rk M and $\zeta(S)\subseteq W$. Thus we may restrict the range of ζ to W, and thus, without loss of generality, all coordinatizations will be assumed to be in a vector space of dimension equal to the rank of the matroid. If n is the rank of $M(S)$, then for a given field K there is, up to isomorphism, a unique vector space V of dimension n over K. Thus we may also speak of a *coordinatization of M over K*, meaning a coordinatization in V.

Let $GF(q)$ denote the finite field of order q. A matroid which has a coordinatization over $GF(2)$, or $GF(3)$, is called *binary*, or *ternary*, respectively. A matroid which may be coordinatized over every field is called *unimodular* (or *regular*). Further characterizations of these classes of matroids will be given later in this chapter and in the following chapters.

It is often convenient to represent a coordinatization in matrix form. If $\zeta:S\to V$ is a coordinatization of $M(S)$ of rank n, and E a basis of V, let $A_{\zeta,E}$ be the matrix with n rows and with columns indexed by S whose a-th column, for $a\in S$, is the vector $\zeta(a)$ represented with respect to E. Since the matrix $A_{\zeta,E}$ also determines the coordinatization ζ if we are given E, we often simply say $A_{\zeta,E}$ is a coordinatization of $M(S)$.

1.2. Equivalence of Coordinatizations and Canonical Forms

If $\phi:V\to V$ is a non-singular linear transformation and $\zeta:S\to V$ is a coordinatization of $M(S)$, then $\phi\circ\zeta:S\to V$ is also a coordinatization. If Q is the non-singular $n\times n$ matrix representing ϕ with respect to the basis E of V, then $A_{\phi\circ\zeta,E}=QA_{\zeta,E}$. On the other hand, we may easily check that

$A_{\phi\circ\zeta,E} = A_{\zeta,\phi^{-1}E}$, so multiplying $A_{\zeta,E}$ on the left by Q may also be regarded as simply a change of basis for the coordinatization ζ.

We recall from elementary linear algebra that multiplying $A_{\zeta,E}$ on the left by a non-singular matrix Q is equivalent to performing a sequence of elementary row operations on $A_{\zeta,E}$, and that any such sequence of elementary row operations on $A_{\zeta,E}$ may be realized by an appropriate choice of Q. We will say $A_{\zeta,E}$ and $QA_{\zeta,E}$ are *linearly equivalent* (where Q is non-singular), and any matrix linearly equivalent to $A_{\zeta,E}$ may be regarded as representing the same coordinatization ζ of the same matroid with respect to a new basis of V.

Conversely, given a coordinatization matrix $A_{\zeta,E}$, we may choose any new basis E' of V, and $A_{\zeta,E'}$ is linearly equivalent to $A_{\zeta,E}$. As a special case of this, we pick $E' = \zeta(B)$, where B is a fixed basis of the matroid $M(S)$.

Then, by reordering the elements of S so that the first n elements are the elements of B, we have a matrix $A_{\zeta,E'}$ in echelon form

$$A_{\zeta,E'} = \overset{\quad B \quad S-B}{(I_n| \quad L)}$$

where I_n is the $n \times n$ identity matrix, with columns indexed by B, and L is an $n \times (N-n)$ matrix with columns indexed by $S - B$, where $N = |S|$.

As yet another way of viewing linear equivalence, let W_ζ be the subspace spanned by the rows of $A_{\zeta,E'}$ in an N-dimensional vector space U. What we have seen is that W_ζ is independent of E', and that indeed the choice of E' actually amounts to a choice of a basis for W_ζ. Thus every linear equivalence class of $n \times N$ matrices coordinatizing $M(S)$ corresponds to an n-dimensional subspace of U. Conversely, every n-dimensional subspace of U corresponds to a coordinatization of some rank n matroid on S, which is a weak-map image of $M(S)$.

Remark. Algebraic geometers regard the collection of all n-dimensional subspaces of an N-dimensional vector space as a Grassmann manifold, and the coordinatizations of $M(S)$ correspond to a certain submanifold.

Besides row operations, another operation on $A_{\zeta,E}$ which leaves invariant the matroid coordinatized by $A_{\zeta,E}$ is non-zero scalar multiplication of columns. This may be accomplished by multiplying $A_{\zeta,E}$ on the right by an $N \times N$ diagonal matrix with non-zero diagonal entries. Combining this with the previous operations, we say that two $n \times N$ matrices A and A' are *projectively equivalent* if there exist Q, an $n \times n$ non-singular matrix, and D, an $N \times N$ non-singular diagonal matrix, such that $A' = QAD$.

Let us recall that projective $n-1$ dimensional space P is obtained from V by identifying the non-zero vectors of each one-dimensional subspace of V to give a point of P. Let $\pi: V \to P \cup \{0\}$ be the resulting map, where 0 is an element adjoined to P which is the image of $0 \in V$. Then if $\zeta: S \to V$ is a coordinatization,

$\pi \circ \zeta$ is an embedding of $M(S)$ into $P \cup \{0\}$, except that parallel elements become identified in $P \cup \{0\}$. If $T': V \to V$ is a linear transformation, let $T = \pi \circ T' \circ \pi^{-1}$, which is well-defined since T' preserves scalar multiples. Then we call T a linear transformation of $P \cup \{0\}$. Since non-zero scalar multiples in V are identified in $P \cup \{0\}$, we immediately have the following:

1.2.1. Proposition. *Let J and L be $n \times N$ matrices over the field K. Then if J coordinatizes $M(S)$ and J is projectively equivalent to L, then L also coordinatizes $M(S)$. J and L are projectively equivalent if and only if their corresponding coordinatizations ζ_J and ζ_L determine the same projective embedding up to change of basis in $P \cup \{0\}$, i.e., $\pi \circ \zeta_J = T \circ \pi \circ \zeta_L$, where T is a non-singular linear transformation of $P \cup \{0\}$.*

We next ask whether there exists a canonical form for a projective equivalence class of coordinatizations, as echelon form was for a linear equivalence class. For a given coordinatization

$$A = (I_n | L)$$

in echelon form with respect to a basis B, let L^+ be the matrix obtained by replacing each non-zero entry of L by 1. In fact, L^+ is just the incidence matrix of the elements of B with the basic circuits of the elements of $S - B$, so it is independent of the particular coordinatization. Now let Γ be the bipartite graph whose adjacency matrix is L^+. Thus each entry of 1 in L^+ corresponds to an edge of Γ. Let T be a basis (i.e., spanning tree) of Γ.

1.2.2. Proposition. (*Brylawski and Lucas, 1973*) *A is projectively equivalent to a matrix A' which is in echelon form with respect to B, and which has 1 for each entry corresponding to an edge of T.*

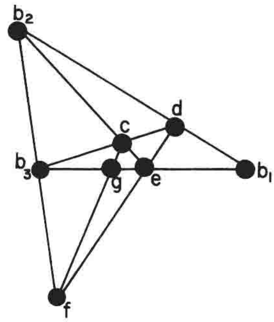
Proof. This may be accomplished by non-zero scalar multiplication of rows and columns, and is left as an exercise. □

The matrix A' of the preceding proposition is said to be in (B, T)-*canonical form*, or when B and T are understood, *canonical projective form*. The simplest canonical projective form and most useful version of this canonical form occurs when $M(S)$ has a spanning circuit C. Then by choosing B to be $C - \{c\}$ for some $c \in C$, the column corresponding to c in L has no zeros, hence we may pick T to correspond to the n entries of column c, together with the first non-zero entry in every other column of L.

A major use of this projective canonical form is in actual computation with coordinates and in presenting examples.

1.2.3. Example. Let $M(S)$ be the 8-point rank 3 geometry whose affine diagram appears in Figure 1.1. If we choose the standard basis $B = \{b_1, b_2, b_3\}$

Figure 1.1. An 8-point rank 3 geometry.



and spanning circuit $C=\{b_1,b_2,b_3,c\}$, we may coordinatize M over $\mathbb{Q}$ by the following matrix in canonical projective form:

$$\begin{pmatrix} b_1 & b_2 & b_3 & c & d & e & f & g \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & -1 & 2 \end{pmatrix}.$$

1.2.4. Example. Let $M(S)$ be the 4-point line, that is, $U_{2,4}$, the uniform geometry of cardinality 4 and rank 2, whose bases are all of the subsets of S of cardinality 2, where $|S|=4$. Then any coordinatization of $M(S)$ over any field K may be put in the following projective echelon form:

$$\begin{matrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \alpha \end{matrix}$$

where $\alpha \in K-\{0,1\}$. Thus we can say that up to projective equivalence, there is a one-parameter family of coordinatizations of $U_{2,4}$. We note that this parameter α is equivalent to the classical cross-ratio of four collinear points in projective geometry.

Since $U_{2,4}$ is the simplest non-binary matroid, one might be led to surmise the following, first proved by White (1971, Proposition 5.2.5), and later by Brylawski and Lucas (1973) using more elementary techniques. The proof is omitted here, because of its fairly technical nature.

1.2.5. Proposition. *Let $M(S)$ be a binary matroid and K a field over which M has a coordinatization. Then any two coordinatizations of M over K are projectively equivalent.*

Brylawski and Lucas (1973) have investigated the question of which matroids have, over a particular field K, any two coordinatizations projectively equivalent. Such matroids are said to be *uniquely coordinatizable over K*, and among their findings is that ternary matroids are uniquely coordinatizable over $GF(3)$ (although not over an arbitrary field, as the example of $U_{4,2}$ shows).

1.2.6. Example. We return to Example 1.2.3. This example is, in fact, a ternary matroid, which is uniquely coordinatizable not only over $GF(3)$, but over every field K such that char $K \neq 2$. To see this, we first note that the matrix given over $\mathbb{Q}$ may be regarded as a coordinatization of M over every field K such that char $K \neq 2$. If we take an arbitrary coordinatization of M over any such field K and put that coordinatization in canonical projective form with respect to B and C, the elements b_1, b_2, b_3, and c are assigned the vectors shown, and then the vector for d is determined since d is on the intersection of the two lines b_1b_2 and b_3c. Likewise $e \in b_1b_3 \cap b_2c$, $f \in b_2b_3 \cap de$, and $g \in b_1b_3 \cap cf$.

1.3. Matroid Operations

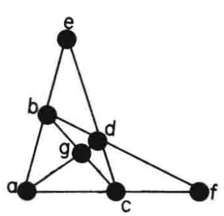
We now note that coordinatizability is preserved under various matroid operations, including duality, minors, direct sums, and, in a restricted sense, truncation. This material is also found scattered through Chapter 7 of White (1986), and is collected here for convenience.

1.3.1. Proposition. *Let $A_{\zeta,E}$ coordinatize $M(S)$ over a field K, and let W_ζ be the row-space of $A_{\zeta,E}$ in U, a vector space of dimension $N = |S|$ over K. Then if $M^*(S)$ denotes the dual matroid of M, the subspace $W_\zeta^\perp$ orthogonal to W_ζ is the subspace of U corresponding to a coordinatization of M^*. Thus M is coordinatizable over K if and only if M^* is.*

Furthermore, if $A_{\zeta,E}$ is in echelon form, $A_{\zeta,E} = (I_n, L)$, then $A^* = (-L^t, I_{N-n})$ is a coordinatization of M^*, where t denotes transpose.

Proof. Let B be a basis of $M(S)$ and we may assume $A_{\zeta,E}$ is in echelon form with respect to B, since W_ζ is invariant under linear equivalence. Thus $A_{\zeta,E} = (I_n, L)$, and we note that $A^* = (-L^t, I_{N-n})$ has each of its rows orthogonal to each row of $A_{\zeta,E}$, hence the rows of A^* are a basis of $W_\zeta^\perp$. Let $M'(S)$ be the matroid coordinatized by the columns of A^*. Since $S - B$ corresponds to the columns

Figure 1.2. A 7-point rank 3 matroid M.



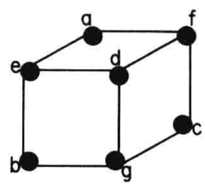
of I_{N-n} in A^*, we see that $S - B$ is a basis of M'. Conversely, if B' is any basis of M', $S - B'$ is a basis of M by a similar argument. Since B was an arbitrary basis of M, $M' = M^*$ and the theorem follows. □

1.3.2. Example. Let $M(S)$ be the 7-point rank 3 matroid shown in Figure 1.2, along with a coordinatization A over $\mathbb{R}$ given below. Then M^*, a rank 4 matroid which is shown in Figure 1.3, has the coordinatization A^* over $\mathbb{R}$ as in the preceding proposition.

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}, \quad (\text{columns } a\ b\ c\ d\ e\ f\ g)$$

$$A^* = \begin{pmatrix} -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{columns } a\ b\ c\ d\ e\ f\ g)$$

Figure 1.3. M^*, the dual of the matroid M in Figure 1.2, where *abfg*, *aceg*, *bcef* are coplanar sets.



1.3.3. Proposition. *Let $M(S)$ be a matroid.*

(1) *If M is coordinatizable over a field K, then so is every minor of M.*
(2) *If $M = M_1 \oplus M_2$, then M is coordinatizable over K if and only if both M_1 and M_2 are coordinatizable over K.*
(3) *If K is sufficiently large and M is coordinatizable over K, then the truncation $T(M)$ is coordinatizable over K.*

Proof. (1) If $A_{\zeta,E}$ coordinatizes M, then any submatroid $M - X$ is coordinatized by deleting the columns of $A_{\zeta,E}$ corresponding to X. Since contraction is the dual operation to deletion, (1) follows from the preceding proposition. For a direct construction of a coordinatization of a contraction, see the following remark and example.

(2) If $A^{(1)}$ and $A^{(2)}$ are matrices coordinatizing M_1 and M_2 respectively, then

the matrix direct sum

$$\begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix}$$

is a coordinatization of $M = M_1 \oplus M_2$. The converse follows from (1).

(3) The construction of truncation (to rank $n-1$, say) described in Section 7.4 of White (1986) may be carried out within the vector space V provided only that the field is sufficiently large to guarantee the existence of a free extension (by one point) within V. □

1.3.4. Remark. To construct the coordinatization of a contraction $M(S)/X$ from a coordinatization $A_{\zeta,E}$ of M, we first choose a basis I of the set X. By row operations on $A_{\zeta,E}$ we may make the first $n-k$ entries 0 in each column corresponding to I, where $k = |I|$. Then delete the columns corresponding to X, as well as the last k rows.

This construction really amounts to simply taking a linear transformation T from V, the vector space in which M is coordinatized, to a vector space of dimension $n-k$, such that the kernel of T is precisely span (ζX).

1.3.5. Example. Let M be the matroid shown in Figure 1.4, with coordinatization A over $\mathbb{Q}$. Let $X = \{e,f\}$. Then row operations on A lead to the matrix A', and deletion of the appropriate rows and columns gives A'', a coordinatization of M/X, which is put into canonical projective form A'''. The matroid M/X is shown in Figure 1.5.

$$A = \begin{array}{c} \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 7 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & -5 \end{pmatrix} \end{array},$$

$$A' = \begin{array}{c} \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 2 & -5 \\ -3 & 1 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & -2 & 7 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array},$$

$$A'' = \begin{array}{c} \begin{matrix} a & b & c & d & g & h \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 2 & -5 \\ -3 & 1 & 0 & 0 & -3 & 0 \end{pmatrix} \end{array},$$

$$A''' = \begin{array}{c} \begin{matrix} d & b & g & a & c & h \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}.$$