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ANALYTIC TOPOLOGY

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PREFACE

The material here presented represents an elaboration on my Colloquium Lectures delivered before the American Mathematical Society at its September, 1940 meeting at Dartmouth College. The results of some of my own efforts together with a selection of those of other mathematicians relative to the subject chosen are presented in what is intended to be a coherent and approximately self-contained exposition, framed in the familiar topology of separable metric spaces. No attempt is made at comprehensive coverage either of the known work embraced by the title of the book or of the work of others, or even myself, which may be closely related to that included.

I wish to express my appreciation to the American Mathematical Society for the opportunity of delivering the lectures and publishing in its Colloquium Series. Thanks are due also to the Waverly Press for its careful and sympathetic handling of the manuscript.

On the personal side, it has been my privilege and good fortune to stand in the middle ground between distinguished teachers on the one hand and a group of distinguished students and associates on the other and receive stimulus and inspiration from both. Of the former, the influence of R. L. Moore will be apparent and his invaluable contribution in this way is gratefully though inadequately acknowledged. Of the other group, too numerous to mention, Hubert A. Arnold, M. Garcia and Paul A. White have helped directly by reading and correcting parts of the manuscript and proof. Finally, I wish to acknowledge the generous assistance rendered by my wife, Lucille Whyburn, who contributed materially to the content and organization of the lectures and manuscript and assisted greatly in the preparation of both and whose unfailing encouragement transcends all attempts at evaluation.

G. T. WHYBURN

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INTRODUCTION

As used here the term "Analytic Topology" is meant to cover those phases of topology which are being developed advantageously by methods in which continuous transformations play the essential role. In the process of evolving, coming of age and assuming more stable form, topology, through interaction with other branches of mathematics, not only is leaving its mark on them but is itself adopting more and more the language and symbolism of the older fields. Thus, for example, we have not only a topological function theory giving the results of analysis which are essentially topological in character, but also a function-theoretic topology dealing with topological situations with the aid and principal use of some of the basic tools of analysis. Without drawing the lines too sharply or giving too clear cut a definition, let us say in a general way that analytic topology deals with topological situations with the aid of analytical language and tools, and to some extent conversely, just as analytic geometry handles geometric situations by analytic methods. I hope this concept will be made clearer as the treatment progresses and actual examples are given illustrating the type of relationship which has been so vaguely defined.

The major questions to be dealt with are, first, the existence of transformations of various sorts from a space A to the same or another space B and, second, the analysis of the action of these transformations on A to produce B. Since thus we are dealing with the transition from A to itself or to something else possibly quite different topologically, our subject exhibits kinship with earlier work on dynamics in the Colloquium Series. This is especially true of the final chapter on periodicity which connects directly with many of the concepts of this subject as discussed by G. D. Birkhoff. However, the even closer kinship with other purely topological treatises, notably that of R. L. Moore in the Colloquium Series and that of K. Menger on "Kurventheorie", will be too obvious to require comment.

The book divides roughly into two parts, corresponding to the first six and last six chapters, respectively. In the first part there is developed the necessary topological machinery and framework for the latter part, which is devoted to pure analytic topology. Even in the second chapter, however, notably in §§3, 4, there emerge some of the fruits of the application of analytic or transformation methods to topological situations. For here a variety of results, some classic and others quite recent, are brought together in what seems their proper relationship and derived in a simple and novel way from one central mapping theorem.

The book is meant to be largely self-contained, at least in so far as topological developments are concerned. In the later stages some use is made of a few notions of combinatorial topology and of the theory of groups without any attempt at adequate introduction. Since these appear largely in end-results and applications, there seems little need or justification for taking the space to develop them here.

At the beginning we assume once for all a set of axioms sufficient to make all spaces considered separable and metrizable. Once the metric is introduced, however, attention is no longer focused on the axioms, but rather on the (equivalent) standpoint that we are operating always in a given separable metric space.

Cross references are given in brackets, with the roman numeral for the chapter followed by the section and number of the theorem, lemma, or corollary referred to, e.g., [IV, (3.2)] refers to result (3.2) of §3 in Chapter IV, which would be the *second* main result in this section. If only the number in parenthesis is given, as (3.2) for example, the reference is to the result of that number in the *present chapter*, i.e., the one being read at the time. To assist in locating results referred to, the chapter number and section number appear at the heading of each double page.

References to the literature in the main are held to a minimum. For convenience these are made at the ends of the chapters in the form of author's name followed by numerals in brackets referring to his books or papers by that number in the bibliography at the close of the book. In some cases one or more authors' names have been used in connection with a theorem though by no means in all where this might well, or possibly should, be done. A considerable amount of the material in the first part is of such a classical nature and so well known that specific citations to sources in the literature are not made. Later on more attempt is made to cite the original author and source. In some cases, also, closely related material not actually covered is mentioned in the references.

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CHAPTER I

INTRODUCTORY TOPOLOGY

1. **Operations with sets.** We shall have occasion to use sets of points and sets or collections of point sets of various sorts. Capital letters A, B, C, $\cdots$ will be used to designate sets and, in general, small letters stand for points. $a \in A$ means "a is an element of the set A" or "a is a point of A" if A is a point set. a *non* $\in A$ means that a is not an element of A.

If A and B are sets,

$A = B$ means that every point in the set A is also a point in the set B, and conversely every point in B is also in A.

$A \subset B$—read "A is a subset of B" or "A is contained in B"—means that every point of A is a point of B.

$A \supset B$ means $B \subset A$, "A contains B." $A = B$ is equivalent to $A \subset B$ and $B \subset A$.

$A + B$ (sum of A and B) means the set of all points belonging either to A or to B. In general, if $[G]$ is a collection of sets, $\sum G$ is the set of all points x such that x belongs to at least one element (or set) of the collection $[G]$.

$A \cdot B$ (intersection or product) means the set of all points belonging to both A and B. For any collection of sets $[G]$, $\prod G$ is the set of all points x such that x belongs to every set of $[G]$.

If $B \subset A$, then $A - B$ is by definition the set of all points which belong to A but not to B.

If $[G]$ is a collection of sets, any collection of sets each of which is an element of the collection $[G]$ is called a *subcollection* of $[G]$.

Real and complex numbers and their properties will be used freely. A set or collection whose elements can be put into (1-1) correspondence with a subset of the set of all positive integers will be called *countable, or enumerable.* If such a correspondence is established and the elements arranged in order of ascending integers, e.g., a_1, a_2, a_3, $\cdots$, the resulting *arranged set* is called a sequence.

The empty or vacuous set is designated by 0. Two sets A and B are said to be *disjoint* if their intersection is empty, i.e., $A \cdot B = 0$.

2. **Topological spaces.** A class S of elements or points in which there is determined a class of subsets called neighborhoods satisfying the following four conditions or axioms is called a *topological space.*

(1) For any $x \in S$, any neighborhood of x is a set of points containing x.

(2) If U is a neighborhood of a point x and y is any point of U, then U is a neighborhood also of y.

(3) If U and V are neighborhoods* and $x \in U \cdot V$, then there exists a neighborhood W of x such that $W \subset U \cdot V$.

(4) If x and y are distinct points, there exists a neighborhood of x not containing y.

A point p in such a space is said to be a *limit point* of a point set M provided every neighborhood of p contains at least one point of M distinct from p.

If A is a set, the *closure* of A, written $\bar{A}$, is the set consisting of all points of A and all limit points of A.

We shall need also the two following additional axioms.

(5) If V is any neighborhood of a point x, there exists a neighborhood U_x of x with $\bar{U}_x \subset V$.

(6) There exists a countable (fundamental) sequence of neighborhoods $R_1, R_2, R_3, \cdots$ such that if p is any point and U_p is any neighborhood of p, then for some integer m we have $p \subset R_m \subset U_p$.

A topological space satisfying (5) is said to be *regular* and such a space satisfying (6) is said to be *perfectly separable*.

The following propositions are now readily provable in a regular, perfectly separable topological space S.

(2.1) *If p and q are distinct points, there exist neighborhoods U_p and U_q of p and q, respectively, with $U_p \cdot U_q = 0$.*

(2.2) *If p is a limit point of $A + B$, then p is a limit point either of A or of B (possibly of both).*

(2.3) *If p is a limit point of a set M, every neighborhood of p contains infinitely many distinct points of M.*

(2.4) *Given any point p which is in at least one neighborhood, there exists a monotone decreasing sequence of neighborhoods $Q_1 \supset Q_2 \supset Q_3 \supset \cdots$ closing down on p, i.e., such that p belongs to every Q_i and if U is any neighborhood of p, then for some n we have $p \subset Q_n \subset U$.*

The proofs of (2.1), (2.2), (2.3) are left as exercises. To prove (2.4), let $[R_{n_i}]$ be the collection of all neighborhoods of the fundamental sequence in Axiom (6) which contain the point p. Set $Q_1 = R_{n_1}$. By Axiom (3), since $p \in Q_1 \cdot R_{n_2}$, there exists a neighborhood Q_2 such that $p \subset Q_2 \subset Q_1 \cdot R_{n_2}$. Similarly, there exists a neighborhood Q_3 satisfying $p \subset Q_3 \subset Q_2 \cdot R_{n_3} \subset R_{n_1} \cdot R_{n_2} \cdot R_{n_3}$. Continuing this process indefinitely we obtain a sequence

$$Q_1 \supset Q_2 \supset Q_3 \supset \cdots \supset p$$

such that for each i,

$$Q_i \subset R_{n_1} \cdot R_{n_2} \cdots R_{n_i}.$$

* Note that in view of Axiom 2, it is no longer essential to associate a neighborhood with any particular one of its points.

Now if U is any neighborhood of p, there exists an integer m such that $p \subset R_m \subset U$. For some i, $n_i = m$. Whence, $p \subset Q_i \subset R_{n_i} \subset U$, as was to be shown.

DEFINITION. *A sequence of points $p_1, p_2, p_3, \cdots$ (not necessarily distinct) is said to converge to a point p, written* $\begin{Bmatrix} p_n \to p, \text{ or} \\ \lim p_n = p \end{Bmatrix}$, *provided every neighborhood of p contains almost all (i.e., all but a finite number) of the points in the sequence.*

(2.5) *If p is a limit point of a set M and p is in some neighborhood, M contains an infinite sequence of distinct points converging to p.*

To prove this, let $Q_1 \supset Q_2 \supset Q_3 \supset \cdots$ be a sequence of neighborhoods closing down on p [see (2.4)]. Since p is a limit point of M, for each n, Q_n contains a point p_n of M distinct from p. For any n, there exists a neighborhood U of p not containing p_n and an m such that $Q_m \subset U$. Hence $p_n \neq p_{m+i}$ ($i = 1, 2, \cdots$). Thus any given point p_n occurs in the sequence $[p_n]$ only a finite number of times. Accordingly there are infinitely many distinct points p_n. Any neighborhood of p contains almost all the sets $Q_1, Q_2, \cdots$ and hence almost all the points p_n. Accordingly, $p_n \to p$; and if we eliminate duplicates we retain an infinite sequence of distinct points $p_{n_i} \to p$.

3. Open and closed sets. Compact sets. Separability. A set of points G is said to be *open* provided that for every point $p \in G$ there exists at least one neighborhood V_p of p which is contained wholly in G.

A set of points F is said to be *closed* provided F contains all of its limit points.

The *complement* of a set G is the aggregate of points not contained in G.

(3.1) *The closure of any set is closed, i.e.,*

$$\bar{\bar{X}} = \bar{X}.$$

(3.2) (a) *If a set is open, its complement is closed.*
(b) *If a set is closed, its complement is open.*

(3.3) (a) *The sum of any collection of open sets is open.*
(b) *The product of any collection of closed sets is closed.*

(3.4) (a) *The product of any finite number of open sets is open.*
(b) *The sum of any finite number of closed sets is closed.*

These propositions are valid in any topological space. The proofs are left as exercises. In the remaining theorems of this section, we assume a space satisfying all of the Axioms (1)–(6).

A set K is said to be *compact* provided every infinite subset of K has at least one limit point which belongs to K.

(3.5) *Every compact set is closed.*

For suppose K is compact but not closed. Then there exists a limit point p of K which does not belong to K. By (2.5), K contains an infinite sequence $[p_n]$ of distinct points converging to p. By the compactness of K, $[p_n]$ must have a limit point $q \subset K$. But if U_p and U_q are disjoint neighborhoods of p and q, respectively, almost all the points p_n must be in U_p, thus only a finite number can be in U_q contrary to (2.3).

(3.6) *If $K_1 \supset K_2 \supset K_3 \supset \cdots$ is a monotone decreasing sequence of nonvacuous compact sets, $\prod_1^\infty K_i$ is nonvacuous.*

For each n, let p_n be a point of K_n. If some p_n, say p_k, is chosen infinitely many times, we have $p_k \epsilon \prod_1^\infty K_i$. If not, the set $[p_n]$ is an infinite subset of K_1. Hence it has a limit point, say p. Then for each n, p is a limit point of the subset $\sum_n^\infty p_i$ of K_n. Since each K_n is closed, by (3.5), this gives $p \epsilon K_n$ for each n. Whence, $p \epsilon \prod_1^\infty K_i$ so that $\prod_1^\infty K_i \neq 0$.

DEFINITION. *A set M is said to be separable provided there exists a countable subset P of M such that $\bar{P} \supset M$, i.e., every point of M is either a point or a limit point of P.*

(3.7) *Every set is separable.*

To prove this, let M be any set and let $[R_{n_i}]$ be the collection of all neighborhoods in the fundamental sequence (Axiom (6)) which intersect M. For each i, let $p_i \epsilon M \cdot R_{n_i}$ and set $P = \sum p_i$. Then if x is any point of M and x non ϵP, we must show that x is a limit point of P. To this end, let U be any neighborhood of x. There exists a neighborhood R_{n_k} of the fundamental sequence such that $x \subset R_{n_k} \subset U$. This gives $p_k \subset R_{n_k} \subset U$ and $p_k \neq x$. Thus x is a limit point of P.

DEFINITION. *A point p is said to be a condensation point of a set M provided every neighborhood of p contains uncountably many points of M.*

(3.8) *For any set M, all save possibly a countable number of points of M are condensation points of M.*

For let R be the sum of all those sets R_i of the fundamental sequence such that $M \cdot R_i$ is countable. Then $M \cdot R$ is countable and every point of $M - M \cdot R$ is a condensation point of M since any neighborhood R_n of the fundamental sequence which contains a point of $M - M \cdot R$ contains uncountably many points of M.

4. **Covering theorems.** A point set M is said to be *covered* by a collection of open sets G provided that each point of M lies in at least one set of the collection $[G]$.

An immediate consequence of the perfect separability of our space is the following proposition, known as the

(4.1) LINDELÖF THEOREM. *If $[G]$ is any collection of open sets covering a point set M, some countable subcollection $[G_i]$ of $[G]$ covers M.*

For, referring to the fundamental sequence of neighborhoods $R_1, R_2, \cdots$, it follows that for each point p of M there exists an integer δ so that we have (i) $p \subset R_\delta \subset G \epsilon [G]$, where G is some set of the collection $[G]$ containing p. Let $[R_{n_i}]$ be the collection of those neighborhoods R_{n_i} of the fundamental sequence having the property that some $G \epsilon [G]$ exists such that $R_{n_i} \subset G$. For each i, select one such set G and call it G_i. Then $[G_i]$ is a countable subcollection of $[G]$ and by virtue of (i) it follows that $[R_{n_i}]$ and thus also $[G_i]$ covers M.

If restrictions are placed on the set M, we can prove the stronger covering property embodied in the

(4.2) BOREL THEOREM. *If $[G]$ is any collection of open sets covering a compact set M, then some finite subcollection of $[G]$ covers M.*

Proof. By the Lindelöf Theorem, some countable subcollection $[G_i]$ of $[G]$ covers M. Suppose, contrary to our theorem, that no finite subcollection of $[G_i]$ covers M. Then if for each n, we set $N_n = M - M \cdot \sum_1^n G_i$, it follows that for each n, N_n is nonvacuous, closed and compact and contains N_{n+1}. Hence $N = \prod_1^\infty N_n \neq 0$. But this is impossible because $N \subset M$ and $N \cdot \sum_1^\infty G_i = 0$ whereas $[G_i]$ covers M. Thus the Borel Theorem is demonstrated.

DEFINITION. *A set M is said to be conditionally compact provided every infinite subset of M has a limit point (which may or may not belong to M). Obviously any subset of a compact set is conditionally compact.*

As an application of the Lindelöf Theorem we prove the following converse proposition.

(4.3) *If M is conditionally compact, $\bar{M}$ is compact.*

Suppose, on the contrary, that $\bar{M}$ contains an infinite subset X which has no limit point in $\bar{M}$. Since $\bar{M}$ is closed, X can have no limit point. Thus with the aid of Axiom (5) it follows that for each point p of our space, there exists a neighborhood V_p of p such that $\bar{V}_p$ contains at most one point of X. By the Lindelöf Theorem there exists a countable sequence $V_1, V_2, \cdots$ of the neighborhoods V_p covering the space. Now since for each n, $\sum_1^n \bar{V}_i$ contains at most n points of X, there exists a point p_n of M not belonging to $\sum_1^n \bar{V}_i$. Since the set $[p_n]$ is infinite and M is conditionally compact, there exists a limit point p of this set $[p_n]$; but for some k, $p \epsilon V_k$; and this is impossible since V_k can contain at most k of the points p_n.

5. Metric spaces. Metrization Theorem. By a *metric space* is meant a class of elements, or points, in which a *distance function* or *metric* is defined, i.e., to each pair of elements x, y of S a non-negative real number $\rho(x, y)$ is associated satisfying the conditions:

(1) $\rho(x, y) = 0$ if and only if $x = y$,

(2) $\rho(x, y) = \rho(y, x)$ (symmetry),

(3) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ (triangle inequality).

The following examples of metric spaces are of fundamental importance.

(i) The *real number system* R in which the distance function is defined as

$$\rho(x, y) = |x - y|, \qquad x, y \epsilon R.$$

(ii) *Euclidean n-space* R^n with the ordinary distance function

$$\rho(x, y) = \sqrt{\sum_1^n (x_i - y_i)^2}, \qquad x = (x_1, x_2, \cdots, x_n), \qquad y = (y_1, y_2, \cdots, y_n),$$
$$x_i, y_i \in R.$$

(iii) *Hilbert space* H consisting of all sequences of real numbers $x = (x_1, x_2, \cdots)$ with $\sum_1^\infty x_i^2$ convergent as points and with the metric

$$\rho(x, y) = \sqrt{\sum_1^\infty (x_i - y_i)^2}.$$

(iv) The *Hilbert fundamental parallelotope* Q_ω consisting of all sequences $x = (x_1, x_2, \cdots)$ of real numbers satisfying $0 \leqq x_i \leqq 1/i$ as points and the same metric as in (iii). Clearly $Q_\omega \subset H$.

(v) The *space* Q'_ω consisting of all sequences of real numbers $x = (x_1, x_2, \cdots)$ with $0 \leqq x_i \leqq 1$ as points and with the metric

$$\rho(x, y) = \sum_1^\infty 2^{-i} | x_i - y_i |.$$

(vi) The *topological* (or *cartesian*) *product* $X \times Y$ of two metric spaces X and Y, consisting of all ordered pairs (x, y) where $x \in X$, $y \in Y$ and where the distance $\rho(p_1, p_2)$ between two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is defined by

$$\rho(p_1, p_2) = \sqrt{\rho(x_1, x_2)^2 + \rho(y_1, y_2)^2}$$

or by

$$\rho(p_1, p_2) = \max [\rho(x_1, x_2), \rho(y_1, y_2)].$$

It results that either of these is a distance function and, no matter which is used, it follows that a sequence $p_1, p_2, p_3, \cdots$ in $X \times Y$ will converge to a point p in $X \times Y$ (i.e., $\lim \rho(p_i, p) = 0$) if and only if both $\lim \rho(x_i, x) = 0$ and $\lim \rho(y_i, y) = 0$, where $p_i = (x_i, y_i)$ and $p = (x, y)$..

A topological space is said to be *metrizable* provided it is possible to define a distance function satisfying conditions (1), (2), and (3) and such that a point p will be a limit point of a set M if and only if there exist points of M distinct from p as close to p as we please in the sense of this distance. Our principal object in this section will be to prove that *any regular, perfectly separable topological space is metrizable.*

To this end we first establish two lemmas.

(5.1) (Tychonoff) *Any regular perfectly separable topological space is normal,* i.e., any two disjoint closed sets are contained in disjoint open sets.

Proof. Let A and B be any two disjoint closed sets lying in our space. By the regularity property it follows that for each $x \in A$ there exists a neighborhood U of x such that $\bar{U} \cdot B = 0$. Applying the Lindelöf Theorem to the collection $[U]$ we obtain a countable sequence of neighborhoods $U_1, U_2, \cdots$ covering A and such that $\bar{U}_i \cdot B = 0$ $(i = 1, 2, \cdots)$. Similarly there exists a countable collection of neighborhoods $V_1, V_2, \cdots$ covering B such that $\bar{V}_i \cdot A = 0$ $(i = 1, 2, \cdots)$. Now let

$$\begin{array}{ll} U_1^* = U_1, & V_1^* = V_1 - V_1 \cdot \bar{U}_1, \\ U_2^* = U_2 - U_2 \cdot \bar{V}_1, & V_2^* = V_2 - V_2(\bar{U}_1 + \bar{U}_2), \\ \cdots\cdots\cdots & \cdots\cdots\cdots, \\ U_n^* = U_n - U_n \cdot \sum_1^{n-1} \bar{V}_i, & V_n^* = V_n - V_n \cdot \sum_1^{n} \bar{U}_i \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{array}$$

Then for each n, U_n^* and V_n^* are open and $U_n^* \cdot A = U_n \cdot A$ and $V_n^* \cdot B = V_n \cdot B$. Thus if $U^* = \sum_1^\infty U_n^*$ and $V^* = \sum_1^\infty V_n^*$, U^* and V^* are open sets containing A and B, respectively. Also $U^* \cdot V^* = 0$. For if not we would have an m and an n such that $U_m^* \cdot V_n^* \neq 0$. But if $m > n$, the definition of $U_m^* = U_m - U_m \cdot \sum_1^{m-1} \bar{V}_i$ gives $U_m^* \cdot V_n^* = 0$ since $V_n^* \subset V_n$; and if $m \leqq n$, the definition of $V_n^* = V_n - V_n \cdot \sum_1^n \bar{U}_i$ gives $U_m^* \cdot V_n^* = 0$ since $U_m^* \subset U_m$.

(5.2) *If A and B are disjoint closed subsets of a perfectly separable, regular, topological space S, then there exists a continuous real function $f(x)$ defined for all $x \in S$ and such that:*

(i) $0 \leqq f(x) \leqq 1$ *on S.*
(ii) $f(x) = 0$ *on A.*
(iii) $f(x) = 1$ *on B.*

Proof. For each fraction of the form $r = m/2^n$ $(m = 0, 1, 2, \cdots, 2^n)$ we shall define an open set $G(r)$ such that (a) $A \subset G(0)$, $G(1) = S - B$, (b) $r_1 < r_2$ always implies $\overline{G(r_1)} \subset G(r_2)$. By (5.1) there exists an open set $G(0)$ such that $A \subset G(0) \subset \overline{G(0)} \subset G(1) = S - B$. Thus we have the sets $G(r)$ defined and conditions (a) and (b) satisfied for $n = 0$. Let us suppose this has been done for $n = k - 1$ and proceed to do it for $n = k$. Thus we have to define $G(i/2^k)$ for odd integers $i < 2^k$, since for i even, $i/2^k$ reduces to the form $(i/2)/2^{k-1}$. Now by (b) we have $\overline{G[(i-1)/2^k]} \subset G[(i+1)/2^k]$. Thus applying (5.1) with $A = \overline{G[(i-1)/2^k]}$, $B = S - G[(i+1)/2^k]$, we obtain an open set $G(i/2^k)$ such that $\overline{G[(i-1)/2^k]} \subset G(i/2^k) \subset \overline{G(i/2^k)} \subset G[(i+1)/2^k]$. Thus we obtain the sets $G(r)$ by complete induction for all numbers r of the form $m/2^n$.

Now define $f(x) = 0$, for $x \in G(0)$, and $f(x) =$ the least upper bound of the numbers r such that $x \in [S - G(r)]$, for x non $\in G(0)$. Then $f(x) = 0$ on A, $f(x) = 1$ on B, $0 \leqq f(x) \leqq 1$ on S. It remains to show that f is continuous. Let $p \in S$, and $\epsilon > 0$. If $0 < f(p) < 1$, let r_1 and r_2 be fractions of the form $m/2^n$ such that $f(p) - \epsilon/2 < r_1 < f(p) < r_2 < f(p) + \epsilon/2$. Thus in the open set $G(r_2) - \overline{G(r_1)}$ we have $r_1 \leqq f(x) \leqq r_2$ which gives $|f(x) - f(p)| < \epsilon$. If $f(p) = 0$ or 1, a similar argument proves the continuity of $f(x)$ using only r_2 when $f(p) = 0$ and only r_1 when $f(p) = 1$.

(5.3) METRIZATION THEOREM. *Any perfectly separable, regular, topological space S is metrizable.*

Proof. Let $R_1, R_2, \cdots$ be a fundamental sequence of neighborhoods in S. Order all pairs R_i, R_j such that $\bar{R}_i \subset R_j$ into a sequence $P_1, P_2, \cdots$. For each such pair $P_n = R_i, R_j$, we take $\bar{R}_i = A$ and $S - R_j = B$ and, applying (5.2), obtain a continuous function $f_n(x)$ defined on S and satisfying (i), (ii), and (iii) of (5.2).

For each pair of points $x, y \in S$ we now define

$$\rho(x, y) = \sum_1^\infty 2^{-n} | f_n(x) - f_n(y) |,$$

and proceed to show that this function $\rho(x, y)$ is a distance function effecting the desired metrization.

We first show that conditions (1), (2), and (3) on a distance function are satisfied by ρ. Now if $x = y$, we have $f_n(x) = f_n(y)$ for all n, and hence $\rho(x, y) = 0$. On the other hand if $x \neq y$, then for some n and $P_n = R_i, R_j$, we have $x \in R_i$, $y \in (S - R_j)$. Whence $f_n(x) = 0$, $f_n(y) = 1$ and thus $\rho(x, y) \geqq 2^{-n}$. This proves (1). Now (2) is obvious and (3) results at once from the following

$$\rho(x, y) + \rho(y, z) = \sum_1^\infty 2^{-n}[| f_n(x) - f_n(y) | + | f_n(y) - f_n(z) |]$$

$$\geqq \sum_1^\infty 2^{-n} | f_n(x) - f_n(z) | = \rho(x, z).$$

To complete the proof we have to show that $p \in S$ is a limit point of a point set M if and only if there are points of M distinct from p but arbitrarily near p. Let p be a limit point of M and let $\epsilon > 0$ be arbitrary. Taking N so large that $2^{-N} < \epsilon/2$, we can find a neighborhood U of p throughout which the oscillation of $\sum_1^N 2^{-n} | f_n(x) - f_n(y) |$ is less than $\epsilon/2$. Then if $q \in U \cdot M$, we have $\rho(p, q) < \epsilon/2 + \sum_{N+1}^\infty 2^{-n} < \epsilon$. On the other hand suppose that a point p is not a limit point of a set M. Then there exists a pair $P_n = R_i, R_j$ such that $p \in R_i$, $M - p \cdot M \subset S - R_j$. This gives $f_n(p) = 0$, $f_n(x) = 2^{-n}$ on $M - p \cdot M$. Whence $\rho(p, x) \geqq 2^{-n}$ for all $x \in M - p \cdot M$. Thus there are no points of M different from p at a distance less than 2^{-n} from p, and the proof is complete.

Now if in any metric space S we define neighborhoods to be sets of the form $V_r(x)$, where x is a point of S, r is a positive number and $V_r(x)$ is the set of all points y of S with $\rho(x, y) < r$ clearly Axioms (1)–(5) of §2 are satisfied. If in addition we suppose S separable and take a countable set $P = \sum p_i$ such that $\bar{P} = S$, it is readily seen that the countable set of neighborhoods $[V_r(p_i)]$, r rational, satisfies Axiom (6). Accordingly:

(5.4) *Any separable metric space is a regular perfectly separable topological space, and conversely any regular perfectly separable topological space is metrizable.*

Henceforth we shall assume all our spaces separable and metric. Further, the terms "neighborhood" and "open set" will be used synonomously since they give equivalent topologies to our spaces.

Note. In the proof of the Metrization Theorem we have really assigned to each point x of S the point $f(x) = [f_1(x), f_2(x), \cdots]$ of the space Q'_ω (see example (v), p. 6) in such a way that this correspondence is (1-1) (i.e., for each x, $f(x)$ is uniquely determined and $f(x) = f(y)$ implies $x = y$). Furthermore we also showed that this correspondence and its "inverse" are continuous, i.e., $x_i \to x$ implies $f(x_i) \to f(x)$, and conversely. Such a correspondence or transformation is said to be *topological* or to be a *homeomorphism* (see Chapter II below). Thus we have shown that *any space S satisfying Axioms* (1)–(6) *is homeomorphic with a subset of* Q'_ω.

Also it is seen at once that if to each point $x = (x_1, x_2, \cdots)$ of Q'_ω we make correspond the point $y = g(x) = (x_1, x_2/2, x_3/3, \cdots)$ of Q_ω, each point $y = (y_1, y_2, \cdots)$ of Q_ω will correspond uniquely to the point $x = (y_1, 2y_2, 3y_3, \cdots)$ of Q'_ω and this correspondence is also topological. Hence Q_ω and Q'_ω are homeomorphic. Thus *any space S as above is homeomorphic with a subset of* Q_ω (and hence of H), because clearly two sets each homeomorphic with a given third set are homeomorphic with each other.

Finally, *the spaces* Q_ω *and* Q'_ω *are both compact.* For in either space clearly "$p_n \to p$" is equivalent to "for every i, the ith coordinates of p_n converge to the ith coordinate of p." Thus if $[p_n]$ is an infinite sequence of distinct points of Q'_ω (or Q_ω), we choose an infinite subsequence (p_n^1) whose first coordinates converge to a number p^1. Then from (p_n^1) choose an infinite sequence (p_n^2) whose second coordinates converge to p^2, and so on. Thus clearly the sequence (p_i^i) will converge to the point $(p^1, p^2, \cdots)$.

6. **Diameters and distances.** As an immediate consequence of the definition we have

(6.1) *Any distance function* $\rho(x, y)$ *is continuous.*

That is, for any two points a and b and any $\epsilon > 0$ there exist neighborhoods U_a and U_b of a and b, respectively, such that for $x \in U_a$, $y \in U_b$

(i) $|\rho(a, b) - \rho(x, y)| < \epsilon$.

To prove this we have only to take U_a and U_b so that for $x \in U_a$, $y \in U_b$

(ii) $\rho(a, x) < \epsilon/2$, $\rho(b, y) < \epsilon/2$.

This gives by the triangle inequality

$$\rho(x, y) \leq \rho(x, a) + \rho(a, b) + \rho(b, y) < \rho(a, b) + \epsilon,$$

$$\rho(a, b) \leq \rho(a, x) + \rho(x, y) + \rho(y, b) < \rho(x, y) + \epsilon,$$

or

$$\rho(a, b) - \epsilon < \rho(x, y) < \rho(a, b) + \epsilon.$$

which is equivalent to (i).

Definitions. *By the diameter* $\delta(N)$ *of any set* N *is meant the least upper bound, finite or infinite, of the aggregate* $[\rho(x, y)]$ *where* $x, y \in N$. *By the distance* $\rho(X, Y)$ *between the two sets* X *and* Y *is meant the greatest lower bound of the aggregate* $[\rho(x, y)]$ *for* $x \in X$, $y \in Y$.

Obviously $\delta(N) = \delta(\bar{N})$ and $\rho(X, Y) = \rho(\bar{X}, \bar{Y})$ for any sets N, X, Y.

(6.2) *If N is compact, there exist points $x, y \in N$ such that $\rho(x, y) = \delta(N) < \infty$.*

For let $(x_1, y_1), (x_2, y_2), \cdots$ be a sequence of pairs of points of N such that $\lim \rho(x_n, y_n) = \delta(N)$, finite or infinite. Since N is compact, the sequence $x_1, x_2, x_3, \cdots$ contains a subsequence which converges, in the sense of §2, to a point x. We may suppose the notation adjusted so that $x_n \to x$. Similarly the sequence $[y_n]$, after the adjustment for $[x_n]$ has been made, contains a subsequence converging to a point y. Again we can adjust the notation so that $x_n \to x, y_n \to y$. Since $\lim \rho(x_n, y_n) = \delta(N)$, it results from (6.1) that $\rho(x, y) = \delta(N) < \infty$.

(6.3) *If X and Y are disjoint compact sets, there exist points $x \in X$ and $y \in Y$ such that*

$$\rho(x, y) = \rho(X, Y) > 0.$$

To prove this we choose a sequence of pairs of points $[(x_n, y_n)]$ as in (6.2) so that $x_n \in X$, $y_n \in Y$, $x_n \to x \in X$, $y_n \to y \in Y$, and $\lim \rho(x_n, y_n) = \rho(X, Y)$. The continuity of ρ gives $\rho(X, Y) = \rho(x, y)$ and since $x \neq y$, $\rho(x, y) > 0$.

7. **Superior and inferior limits. Convergence.** Let G be any infinite collection of point sets, not necessarily different. The set of all points x of our space S such that every neighborhood of x contains points of infinitely many sets of G is called the *superior limit* or *limit superior* of G and is written lim sup G. The set of all points y such that every neighborhood of y contains points of all but a finite number of the sets of G is called the *inferior limit* or *limit inferior* of G and is written lim inf G. If for a given system G, lim sup G = lim inf G, then the system (collection, or sequence) G is said to be *convergent* and we write lim G = lim sup G = lim inf G. Under these conditions we say that G *converges* to the limit lim G.

For example, let G be the collection of all positive integers. Then lim sup G = lim inf G = lim G = 0. Thus G is convergent and has a vacuous limit. Again, let G be the system of sets $[L_n]_{n=1}^{\infty}$, where L_n is the straight line interval joining the points $[(-1)^n(1 - (1/n)), 0]$ and $[(-1)^n(1 - (1/n)), 1]$. Then lim sup G is the sum of the interval from $(-1, 0)$ to $(-1, 1)$ and the one from $(1, 0)$ to $(1, 1)$, lim inf $G = 0$, and thus lim G does not exist.

From the definitions, we have at once for any system G

(i) $$\liminf G \subset \limsup G.$$

Furthermore, *lim inf G and lim sup G are always closed point sets.* For if x is a limit point of lim inf G, then any neighborhood V of x contains a point y of lim inf G; and since V is a neighborhood also of y, then V contains points of all save a finite number of the sets of G and thus x belongs to lim inf G. Similarly if x is a limit point of lim sup G, any neighborhood V of x contains a point z of lim sup G and thus V, a neighborhood of z, contains points of infinitely many of the sets of G. Therefore x belongs to lim sup G.