Lecture Notes in Mathematics

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J. A. Green

Polynomial Representations of GL_n

Second Edition

With an Appendix on
Schensted Correspondence and Littelmann Paths

By K. Erdmann, J. A. Green and M. Schocker

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Preface to the second edition

This second edition of "Polynomial representations of $\mathsf{GL}_n(K)$" consists of two parts. The first part is a corrected version of the original text, formatted in LaTeX, and retaining the original numbering of sections, equations, etc. The second is an Appendix, which is largely independent of the first part, but which leads to an algebra $L(n,r)$, defined by P. Littelmann, which is analogous to the Schur algebra $S(n,r)$. It is hoped that, in the future, there will be a structure theory of $L(n,r)$ rather like that which underlies the construction of Kac-Moody Lie algebras.

We use two operators which act on "words". The first of these is due to C. Schensted (1961). The second is due to Littelmann, and goes back to a 1938 paper by G. de B. Robinson on the representations of a finite symmetric group. Littelmann's operators form the basis of his elegant and powerful "path model" of the representation theory of classical groups. In our Appendix we use Littelmann's theory only in its simplest case, i.e. for GL_n.

Essential to my plan was to establish two basic facts connecting the operations of Schensted and Littelmann. To these "facts", or rather conjectures, I gave the names Theorem A and Proposition B. Many examples suggested that these conjectures are true, and not particularly deep. But I could not prove either of them.

This work was therefore stalled, until I sought the help of my colleagues Karin Erdmann and Manfred Schocker. They accepted the challenge, and within a few weeks produced proofs of both conjectures. Their proofs constitute the heart of the Appendix, and make it possible to begin a comparison of the Littelmann algebra $L(n,r)$ with the Schur algebra $S(n,r)$. Karin and Manfred have made this Appendix possible, and have written large parts of the text. It has been a happy experience for me to work with them.

A few weeks before the final manuscript of the Appendix was ready, we heard that A. Lascoux, B. Leclerc and J.-Y. Thibon have published a work on "The plactic monoid", which contains results equivalent to Theorem A and Proposition B. Their methods are rather different from ours, and they prove also many important facts which do not come into our Appendix. We give a brief summary of this work in §D.11.

Oxford, August 2006 Sandy (J. A.) Green

Contents

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Introduction

Issai Schur determined the polynomial representations of the complex general linear group $\mathsf{GL}_n(\mathbb{C})$ in his doctoral dissertation [47], published in 1901. This remarkable work contained many very original ideas, developed with superb algebraic skill. Schur showed that these representations are completely reducible, that each irreducible one is "homogeneous" of some degree $r \geq 0$ (see 2.2), and that the equivalence types of irreducible polynomial representations of $\mathsf{GL}_n(\mathbb{C})$, of fixed homogeneous degree r, are in one-one correspondence with the partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of r into not more than n parts. Moreover Schur showed that the character of an irreducible representation of type λ is given by a certain symmetric function S_λ in n variables (since described as "Schur function"; see 3.5). An essential part of Schur's technique was to set up a correspondence between representations of $\mathsf{GL}_n(\mathbb{C})$ of fixed homogeneous degree r, and representations of the finite symmetric group $G(r)$ on r symbols, and through this correspondence to apply G. Frobenius' discovery of the characters of $G(r)$ (see [17]).

This pioneering achievement of Schur was one of the main inspirations for Hermann Weyl's monumental researches on the representation theory of semi-simple Lie groups [54]. Of course Weyl's methods, based on the representation theory of the Lie algebra of the Lie group Γ, and the possibility of integrating over a compact form of Γ, were very different from the purely algebraic methods of Schur's dissertation; in particular Weyl's general theory contained nothing to correspond to the symmetric group $G(r)$. In 1927 Schur published another paper [48] on $\mathsf{GL}_n(\mathbb{C})$, which has deservedly become a classic. In this he exploited the "dual" actions of $\mathsf{GL}_n(\mathbb{C})$ and $G(r)$ on r^{th} tensor space $E^{\otimes r}$ (see 2.6) to rederive all the results of his 1901 dissertation in a new and very economical way. Weyl publicized the method of Schur's 1927 paper, with its attractive use of the "double centralizer property", in his influential book "The Classical Groups" [55]. In fact the exposition in Chapters 3B and 4 of that book has become a standard treatment of polynomial representations of $\mathsf{GL}_n(\mathbb{C})$ (and, incidentally, of Alfred Young's representation theory of the symmetric group $G(r)$), and perhaps this explains the comparative neglect of

Schur's work of 1901. I think this neglect is a pity, because the methods of this earlier work are in some ways very much in keeping with the present-day ideas on representations of algebraic groups. It is the purpose of these lectures to give some accounts, in part based on the ideas of Schur's 1901 dissertation, of the polynomial representations of the general linear groups $\mathsf{GL}_n(K)$, where K is an infinite field of arbitrary characteristic.

Our treatment will be "elementary" in the sense that we shall not use algebraic group theory in our main discussion. But it might be interesting to indicate here some general ideas from the representation theory of algebraic groups (or algebraic semigroups, since the group inverse is not important in this context), which are relevant to our work.

Let Γ be any semigroup (i.e. Γ is a set, equipped with an associative multiplication) with identity 1_Γ, and let K be any field. A *representation* τ of Γ on a K-space V (i.e. a vector space over K) is a map $\tau : \Gamma \to \mathsf{End}_K(V)$ which satisfies $\tau(gg') = \tau(g)\tau(g')$, $\tau(1_\Gamma) = \mathbb{I}_V$, for all $g, g' \in \Gamma$. (For any set V, we denote by $\mathbb{I}_V$ the identity map on V.) We can extend τ linearly to give a map of K-algebras $\tau : K\Gamma \to \mathsf{End}_K(V)$; here $K\Gamma$ is the *semigroup-algebra* of Γ over K, whose elements are all formal linear combinations

$$\kappa = \sum_{g \in \Gamma} \kappa_g g, \quad \kappa_g \in K,$$

whose support $\mathsf{supp}\,\kappa = \{\, g \in \Gamma : \kappa_g \neq 0 \,\}$ is finite. We can make $K\Gamma$ act on V by $\kappa v = \tau(\kappa)(v)$ ($\kappa \in K\Gamma$, $v \in V$), and thereby get a left $K\Gamma$-module, denoted (V, τ), or simply V. A *$K\Gamma$-map* between such $K\Gamma$-modules (V, τ), (V', τ') is, by definition, a K-map $f : V \to V'$ (i.e. f is a linear map) which satisfies $\tau'(g)f = f\tau(g)$ for all $g \in \Gamma$. A $K\Gamma$-map which is bijective is a $K\Gamma$-*isomorphism*, or an *equivalence* between the representations τ, τ'. One has analogous definitions for right $K\Gamma$-modules; a right $K\Gamma$-module can be regarded as a pair (V, τ) where $\tau : \Gamma \to \mathsf{End}_K(V)$ is an *anti-representation* of Γ on the K-space V, i.e. $\tau(gg') = \tau(g')\tau(g)$ for all $g, g' \in \Gamma$, $\tau(1_\Gamma) = \mathbb{I}_V$.

The set K^Γ of all maps $\Gamma \to K$ is a commutative K-algebra, with algebra operations defined "pointwise", e.g. ff' is defined to take $g \mapsto f(g)f'(g)$, for every element ("point") g of Γ. The identity element $\mathbf{1}$ of K^Γ takes each $g \in \Gamma$ to the identity element 1_K of K. If $s \in \Gamma$ and $f \in K^\Gamma$, then the *left* and *right translates* of f by s are defined to be the maps $L_s f, R_s f : \Gamma \to K$ given by

$$L_s f : g \mapsto f(sg), \quad R_s f : g \mapsto f(gs), \quad g \in \Gamma.$$

Each of the operators L_s, R_s maps K^Γ into itself and is a K-algebra map (i.e. K-algebra homomorphism) $K^\Gamma \to K^\Gamma$. In particular, L_s, R_s both belong to the space $\mathsf{End}_K(K^\Gamma)$. It is easy to check that $R : s \mapsto R_s$ gives a representation of Γ on K^Γ, while $L : s \mapsto L_s$ gives an anti-representation. Thus K^Γ can be made into a left $K\Gamma$-module (using R) and a right $K\Gamma$-module (using L). We denote both module actions by $\circ$, so that if $s \in \Gamma$ and $f \in K^\Gamma$ we write

$$s \circ f = R_s f \quad \text{and} \quad f \circ s = L_s f.$$

Notice that these actions commute: $(s \circ f) \circ t = s \circ (f \circ t)$ for all $s, t \in \Gamma$ and $f \in K^\Gamma$. There is a linear map $K^\Gamma \otimes K^\Gamma \to K^{\Gamma \times \Gamma}$ ($\otimes$ means $\otimes_K$) which takes $f \otimes f'$ ($f, f' \in K^\Gamma$) to the function mapping $\Gamma \times \Gamma \to K$ by $(s,t) \mapsto f(s)f'(t)$, for all $s, t \in \Gamma$. This linear map is injective, *and we use it to identify* $K^\Gamma \otimes K^\Gamma$ *with a subspace of* $K^{\Gamma \times \Gamma}$.

The semigroup structure on Γ gives rise to two maps

$$\Delta : K^\Gamma \to K^{\Gamma \times \Gamma} \quad \text{and} \quad \varepsilon : K^\Gamma \to K,$$

as follows: if $f \in K^\Gamma$, then $\Delta f : (s,t) \mapsto f(st)$, and $\varepsilon(f) = f(1_\Gamma)$. Both Δ, ε are K-algebra maps. We shall say that an element $f \in K^\Gamma$ is *finitary*, or is a *representative function*, if it satisfies any one of the conditions F1, F2, F3 below: these three conditions are in fact equivalent (see e.g. [24, Chapter 2]).

F1. The left $K\Gamma$-submodule $K\Gamma \circ f$ generated by f is finite-dimensional.

F2. The right $K\Gamma$-submodule $f \circ K\Gamma$ generated by f is finite-dimensional.

F3. $\Delta f \in K^\Gamma \otimes K^\Gamma$. This means that there exist elements $f_h, f'_h \in K^\Gamma$ (where h runs over some *finite* index set) such that

(1a) $$\Delta f = \sum_h f_h \otimes f'_h.$$

This equation is equivalent to the system of equations

(1b) $$f(st) = \sum_h f_h(s) f'_h(t), \text{ all } s, t \in \Gamma.$$

It is also equivalent to each of the following systems

(1c) $$t \circ f = \sum_h f'_h(t) f_h, \text{ all } t \in \Gamma,$$

or

(1d) $$f \circ s = \sum_h f_h(s) f'_h, \text{ all } s \in \Gamma.$$

The set $F = F(K^\Gamma)$ of all finitary functions $f : \Gamma \to K$ is a K-bialgebra (see [51] for the definitions of coalgebras and bialgebras). It is a K-subalgebra of K^Γ, and is also closed to Δ in the sense that $\Delta F \subseteq F \otimes F$ (this means that if f is finitary, the functions f_h, f'_h in (1a) can be chosen to be themselves finitary). The K-space F, equipped with the maps $\Delta : F \to F \otimes F$, $\varepsilon : F \to K$, is a K-*coalgebra*; these two structures on F, of algebra and coalgebra, are linked by the fact that Δ and ε are both K-algebra maps (see [24, p. 15]).

Finitary functions on Γ appear as coefficient functions of finite-dimensional representations of Γ. Suppose τ is a representation of Γ on a finite-dimensional K-space V. If $\{ v_b : b \in B \}$ is a K-basis of V, we have equations

(1e) $$\tau(g) v_b = g v_b = \sum_{a \in B} r_{ab}(g) v_a, \text{ for } g \in \Gamma,\ b \in B;$$

here $r_{ab}(g) \in K$. The functions $r_{ab} : \Gamma \to K$ $(a, b \in B)$ are called *coefficient functions* of τ, or of the $K\Gamma$-module $V = (V, \tau)$. The K-span of these functions is a subspace of K^Γ called the *coefficient space*[1] of τ, or of the $K\Gamma$-module V. We denote this space by $\mathsf{cf}(V) = \sum_{a,b} K \cdot r_{ab}$; it is elementary to verify that it is independent of the choice of the basis $\{v_b\}$. The matrix $R = (r_{ab})$ gives a *matrix representation* of Γ, i.e. $R(gg') = R(g)R(g')$, $R(1_\Gamma) = (\delta_{ab})$ for all $g, g' \in \Gamma$ (δ_{ab} is the Kronecker delta). These conditions translate into conditions on the coefficients r_{ab}, viz.

$$\textbf{(1f)} \qquad \Delta r_{ab} = \sum_{c \in B} r_{ac} \otimes r_{cb}, \qquad \varepsilon(r_{ab}) = \delta_{ab}, \text{ all } a, b \in B.$$

The matrix $R = (r_{ab})$ is sometimes called an "invariant matrix" [20, p. 140]. From the first equations it follows that all the coefficient functions r_{ab} are finitary, hence that $\mathsf{cf}(V)$ is a subspace of $F = F(K^\Gamma)$. But (1f) also shows that $C = \mathsf{cf}(V)$ is a *subcoalgebra* of F, i.e. that $\Delta C \subseteq C \otimes C$. As a matter of fact, every finitary function $f : \Gamma \to K$ lies in the coefficient space of some finite-dimensional $K\Gamma$-module V; for this purpose we could take $V = K\Gamma \circ f$ (see F1). It is for this reason that finitary functions are sometimes called "representative functions".

If S is any K-algebra (possibly of infinite dimension as K-space), $\mathsf{mod}(S)$ shall denote the category of all *finite-dimensional* left S-modules. Similarly, $\mathsf{mod}'(S)$ is the category of all finite-dimensional right S-modules. An *algebraic representation theory* of Γ over K could be defined as follows: first choose a subcoalgebra A of $F(K^\Gamma)$, i.e. A is a K-subspace of $F(K^\Gamma)$ satisfying $\Delta A \subseteq A \otimes A$. Then "$A$-representation theory" of Γ, is defined to be the study of the full subcategory $\mathsf{mod}_A(K\Gamma)$ of $\mathsf{mod}(K\Gamma)$, whose objects are all finite-dimensional left $K\Gamma$-modules V such that $\mathsf{cf}(V) \subseteq A$. (The morphisms $f : V \to V'$ between two objects V, V' of this category are, by definition, just the $K\Gamma$-maps.) In some contexts we say that a $K\Gamma$-module V is "rational", or more precisely "A-rational", if $\mathsf{cf}(V) \subseteq A$; then $\mathsf{mod}_A(K\Gamma)$ is the category of finite-dimensional A-rational left $K\Gamma$-modules. It is clear that submodules, quotient-modules and finite direct sums of A-rational modules, are themselves A-rational. We can define the category $\mathsf{mod}'_A(K\Gamma)$ of finite-dimensional right $K\Gamma$-modules which are A-rational in the same way. The assumption $\Delta A \subseteq A \otimes A$ implies that if $f \in A$, then the functions f_h, f'_h appearing in (1a) can themselves be chosen to belong to A. Then from (1c), (1d) follows that A is a left and right $K\Gamma$-submodule of K^Γ; also by quite elementary calculations that any finite-dimensional left (or right) $K\Gamma$-submodule V of A belongs to the category $\mathsf{mod}_A(K\Gamma)$ (or $\mathsf{mod}'_A(K\Gamma)$).

Examples.

1. Let Γ be an affine algebraic group over an algebraically closed field K (see for example [24, p. 21]), and $A = K[\Gamma]$ the ring of regular functions on Γ

[1] In [24], this is called the "space of representative functions" of τ, or V.

(A is often called the *affine ring* of Γ). Then $\mathsf{mod}_A(K\Gamma)$ is the category of rational (finite-dimensional) $K\Gamma$-modules in the usual sense of algebraic group theory. In this case, A is not only a subcoalgebra of $F(K^\Gamma)$, but a subbialgebra (see [49, p. 46]). The same remarks apply when Γ is an affine algebraic semigroup.

2. Let Γ be a finite semigroup, then of course $F(K^\Gamma) = K^\Gamma$. If we take $A = K^\Gamma$, then $\mathsf{mod}_A(K\Gamma) = \mathsf{mod}(K\Gamma)$. The (left and right) $K\Gamma$-module structures on A, are dual to the (right and left) "regular" $K\Gamma$-module structures on K given by multiplication: we may identify K^Γ with the dual space $(K\Gamma)^* = \mathsf{Hom}_K(K\Gamma, K)$.

3. Let K be an infinite field, n a positive integer, and $\Gamma = \mathsf{GL}_n(K)$, the group of all non-singular $n \times n$ matrices with coefficients in K. We could take $A = A_K(n)$, the ring of all polynomial functions $f : \Gamma \to K$ (see 2.1). The objects (V, τ) in $\mathsf{mod}_A(K\Gamma)$ (we shall later denote this category by $M_K(n)$, see 2.2) are called *polynomial $K\Gamma$-modules*, and the associated representations (including the matrix representations $R = (r_{ab})$ obtained by using the K-bases $\{v_b\}$ of V) are called *polynomial representations of* Γ. The study of such representations is the subject of these lectures. We get another category (denoted by $M_K(n, r)$ in 2.2) by taking $A = A_K(n, r)$, the space of polynomial functions on Γ which are homogeneous of degree r in the n^2 coefficients of a general element $g \in \Gamma$ (see 2.1 for a precise formulation). Finally we might mention that $A_K(n)$ can also be regarded as the affine ring of the algebraic semigroup $M_n(K)$ of *all* $n \times n$ matrices (singular or not) over K, so that we may regard polynomial representations of $\mathsf{GL}_n(K)$, as rational representations of $M_n(K)$, and conversely.

Now suppose once more that Γ is an arbitrary semigroup with identity 1_Γ, and that A is a subcoalgebra of the space $F(K^\Gamma)$ of all finitary functions on Γ. Then A is itself a coalgebra, relative to the maps $\Delta : A \to A \otimes A$ and $\varepsilon : A \to K$. So we may consider the category $\mathsf{com}(A)$ of all right A-comodules; an object V of $\mathsf{com}(A)$ is a finite-dimensional K-space, together with a "structure map" $\gamma : V \to V \otimes A$ which is K-linear and satisfies the identities $(\gamma \otimes \mathbb{I}_A)\gamma = (\mathbb{I}_V \otimes \Delta)\gamma$, $(\mathbb{I}_V \otimes \varepsilon)\gamma = \mathbb{I}_V$ (see [20, p. 138], where a right A-comodule is perversely referred to as a left A-comodule; better references are [24, p. 16], [49, p. 38] or [51, p. 30]). Our category $\mathsf{mod}_A(K\Gamma)$ is equivalent to $\mathsf{com}(A)$, as follows: if $V \in \mathsf{mod}_A(K\Gamma)$, take any K-basis $\{v_b\}$ of V and write down the equations (1e). Now define $\gamma : V \to V \otimes A$ to be the K-linear map given by equations

(1g) $$\gamma(v_b) = \sum_{a \in B} v_a \otimes r_{ab}, \text{ for } b \in B.$$

It is easy to check that γ is independent of the basis $\{v_b\}$. Moreover using (1f) we see that γ satisfies the comodule identities just given. Conversely given an A-comodule (V, γ), use equations (1g) to *define* the elements r_{ab} of A; the comodule identities now show that (1f) hold, so we may use (1e) to *define* the left $K\Gamma$-module $V = (V, \tau)$. It is evident that $\mathsf{cf}(V) \subseteq A$. So every A-rational, left $K\Gamma$-module can be regarded as a right A-comodule, and conversely. The definition of morphism $f : V \to V'$ in $\mathsf{com}(A)$ (see the references cited) is such that these morphisms are the same as $K\Gamma$-maps in $\mathsf{mod}_A(K\Gamma)$.

This formal transition from $K\Gamma$-modules to A-comodules is rather trivial, but it is nevertheless worth making, from several points of view. To begin with, the basic representation theory of arbitrary A-comodules (we should here work in the category $\mathsf{Com}(A)$, whose objects $V = (V, \gamma)$ are possibly infinite-dimensional) follows to a surprising extent the pattern discovered by R. Brauer and C. Nesbitt for finite-dimensional algebras (see [5, 20]). Included here is the possibility of a *modular theory*, which we shall discuss below.

Next, the A-comodule interpretation also permits us to profit by an important fact, namely that every right A-comodule can be regarded as a left module for the K-algebra $A^* = \mathsf{Hom}_K(A, K)$. The algebra structure in A^* is the dual of the coalgebra structure on A, i.e. if $\xi, \eta \in A^*$, we define the product[2] $\xi\eta$ to be the map of A into K which takes the element $f \in A$ to

(1h) $$\xi\eta(f) = \sum_h \xi(f_h)\eta(f'_h),$$

see (1a). The identity element of A^* is $\varepsilon : A \to K$. If $V = (V, \gamma)$ belongs to $\mathsf{com}(A)$, we make V into an A^*-module by the rule $\xi v = (\mathbb{I}_V \otimes \xi)(\gamma(v))$, for $\xi \in A^*$, $v \in V$. Working in terms of a basis $\{v_b\}$ of V, this rule becomes (see (1g))

(1i) $$\xi v_b = \sum_{a \in B} \xi(r_{ab}) v_a, \text{ for } b \in B.$$

Therefore we have three kinds of matrix representation associated with our original $K\Gamma$-module $V = (V, \tau)$, relative to the basis $\{v_b\}$:

(i) the representation $g \mapsto (r_{ab}(g))$ of Γ;

(ii) the matrix $R = (r_{ab})$ whose elements are functions on Γ, satisfying equations (1f), and which can be thought of as a kind of representation of the coalgebra A;

(iii) the representation $\xi \mapsto (\xi(r_{ab}))$ of the algebra A^*, given by equations (1i).

We can recover (i) from (iii) very easily: for each $g \in \Gamma$ let $e_g : A \to K$ be "evaluation at g", i.e. $e_g(f) = f(g)$, for all $f \in A$. Then $e_g \in A^*$, and the map $e : \Gamma \to A^*$ satisfies $e_g e_{g'} = e_{gg'}$, $e_{1_\Gamma} = \varepsilon$, for $g, g' \in \Gamma$. So e may be extended linearly to a K-algebra map $e : K\Gamma \to A^*$, and if we compose the representation (iii) with e, we recover (i).

If A is finite-dimensional, then it is quite elementary to show that the two categories $\mathsf{mod}_A(K\Gamma)$ and $\mathsf{mod}(A^*)$ are equivalent; this amounts to showing that every finite-dimensional left A^*-module V yields a module in $\mathsf{mod}_A(K\Gamma)$ by composition with the map e. Schur exploited this fact in

[2] This product is often called "convolution".

the case $A = A_K(n,r)$, and could thereby work with the finite-dimensional algebra $A_K(n,r)^* = S_K(n,r)$ (which I have called the "Schur algebra" in these lectures, see 2.3, 2.4), instead of with the infinite-dimensional and irrelevantly complicated group algebra $K\Gamma$.

If A is infinite-dimensional, it is useful in many cases to regard modules $V \in \mathsf{mod}_A(K\Gamma)$ as modules over some "dense" subalgebra S of A^* (S is dense in A^* if, for every $0 \neq a \in A$, there is some $\xi \in S$ such that $\xi(a) \neq 0$). When $A = K[\Gamma]$ is the affine ring of a connected algebraic group Γ over an algebraically closed field K, one may take S the "hyperalgebra" $\mathsf{hy}(\Gamma)$ of Γ (see [9, §6]). In case Γ is simply-connected and semisimple, the correspondence between $\mathsf{mod}_A(K\Gamma)$ and $\mathsf{mod}(S)$ sets up an equivalence of categories (J. Sullivan; see [9, 6.8]). Moreover in that case $\mathsf{hy}(\Gamma)$ can be identified with an algebra U_K constructed out of the complex semisimple Lie algebra associated with the root system of Γ (W. Haboush; [9, 6.5, 6.6] or [22, 1.3]). This algebra U_K (which is sometimes *defined* to be the hyperalgebra of Γ) has an explicit basis with sufficiently good multiplicative properties to make it immensely valuable in studying the rational representations of Γ. In an important paper [6] R. Carter and G. Lusztig have used the hyperalgebra—rather than the Schur algebra—to investigate the polynomial representations of $\Gamma = \mathsf{GL}_n(K)$.

Carter-Lusztig use the idea, which is derived from C. Chevalley's fundamental paper [7] on split semisimple algebraic groups, that the family of all groups $\mathsf{GL}_n(K)$ (n fixed, K varying over some class $\mathcal{K}$ of commutative rings) is "defined over $\mathbb{Z}$". This makes possible a "modular theory" for the polynomial representations of these groups, which in its essentials corresponds to R. Brauer's modular representation theory for finite groups. We can give a sufficiently general setting for such a theory as follows. Suppose we have a family $\{\Gamma_K, A_K\}$, where for each K in the class $\mathcal{K}$ of all infinite fields, Γ_K is a semigroup and A_K is a K-subcoalgebra of $F(K^{\Gamma_K})$. Suppose also that the following two conditions are satisfied. ($\mathbb{Q}$ denotes the rational field.)

Z1. The $\mathbb{Q}$-coalgebra $A_\mathbb{Q} = (A_\mathbb{Q}, \Delta_\mathbb{Q}, \varepsilon_\mathbb{Q})$ contains a $\mathbb{Z}$-*form* $A_\mathbb{Z}$, i.e. (a) $A_\mathbb{Z}$ is a lattice in $A_\mathbb{Q}$, which means $A_\mathbb{Z} = \sum_\nu \mathbb{Z}a_\nu$ for some $\mathbb{Q}$-basis $\{a_\nu\}$ of $A_\mathbb{Q}$, and (b) $\Delta_\mathbb{Q}(A_\mathbb{Z}) \subseteq A_\mathbb{Z} \otimes A_\mathbb{Z}$, $\varepsilon_\mathbb{Q}(A_\mathbb{Z}) \subseteq \mathbb{Z}$.

Z2. For each $K \in \mathcal{K}$ there is a K-coalgebra isomorphism $\alpha_K : A_\mathbb{Z} \otimes K \to A_K$ (here $\otimes$ means $\otimes_\mathbb{Z}$, and $A_\mathbb{Z} \otimes K$ is made into a K-coalgebra by "extension of scalars").

In this case we say that the family $\{\Gamma_K, A_K\}$ is *defined over* $\mathbb{Z}$ by means of $A_\mathbb{Z}$.

Examples.

4. Let $\pi : \mathcal{G}_\mathbb{C} \to \mathsf{End}_\mathbb{C} E$ be a faithful representation of a complex semisimple Lie algebra $\mathcal{G}_\mathbb{C}$ over a complex vector space E of finite dimension n, and let $E_\mathbb{Z}$ be an "admissible lattice" in E (see [4, p. A-5] or [50, p. 17]). For each $K \in \mathcal{K}$ let Γ_K be the Chevalley group over K defined by π, $E_{\mathbb{Z}}$; its elements can be regarded as matrices $g = (g_{\mu\nu})$ in $\mathsf{SL}_n(K)$. For each pair (μ, ν) define the coefficient function $c^K_{\mu\nu} : g \mapsto g_{\mu\nu}$. From the equations $\Delta c^K_{\mu\nu} = \sum_\lambda c^K_{\mu\lambda} \otimes c^K_{\lambda\nu}$, we deduce that the K-subalgebra generated by all the $c^K_{\mu\nu}$ is a K-subcoalgebra (hence even a K-subbialgebra) of $F(K^{\Gamma_K})$; we take this to be A_K. Chevalley showed in [7] (see also [4, §4]) that the family $\{\Gamma_K, A_K\}$ is defined over $\mathbb{Z}$. The relevant $\mathbb{Z}$-form $A_{\mathbb{Z}}$ of $A_{\mathbb{Q}}$ is just the subring of $A_{\mathbb{Q}}$ generated by the $c^{\mathbb{Q}}_{\mu\nu}$; the maps α_K are K-algebra (as well as K-coalgebra) isomorphisms, and take $c^{\mathbb{Q}}_{\mu\nu} \otimes 1_K \mapsto c^K_{\mu\nu}$ for all μ, ν. (From the standpoint of algebraic group theory, each pair (Γ_K, A_K) is an affine algebraic group defined over K, and the family $\{\Gamma_K, A_K\}$ is an "affine group scheme over $\mathbb{Z}$", defined by the $\mathbb{Z}$-bialgebra $A_{\mathbb{Z}}$. See [49, p. 46].)

5. Fix a positive integer n, and let $\Gamma_K = \mathsf{GL}_n(K)$ for each $K \in \mathcal{K}$. For A_K we may take either $A_K(n)$, or $A_K(n, r)$ for some fixed $r \geq 0$ (see 2.1). It is completely elementary to verify that in each case the family $\{\Gamma_K, A_K\}$ is defined over $\mathbb{Z}$; the relevant $\mathbb{Z}$-forms $A_{\mathbb{Z}}(n)$, $A_{\mathbb{Z}}(n, r)$ are described in 2.5. In these lectures, we study the family $\{\Gamma_K, A_K(n, r)\}$.

The first essential of the *modular representation theory* of any family $\{\Gamma_K, A_K\}$ which is defined over $\mathbb{Z}$, is the process of *modular reduction*. We shall write M_K for the category $\mathsf{mod}_{A_K}(K\Gamma_K)$, for any $K \in \mathcal{K}$. Then an object $V_{\mathbb{Q}}$ in $M_{\mathbb{Q}}$ is a finite-dimensional $\mathbb{Q}$-space on which $\Gamma_{\mathbb{Q}}$ acts. If $\{ v_{b,\mathbb{Q}} : b \in B \}$ is a $\mathbb{Q}$-basis of $V_{\mathbb{Q}}$, we have equations like (1e)

(1j) $$g v_{b,\mathbb{Q}} = \sum_{a \in B} r^{\mathbb{Q}}_{ab}(g) v_{a,\mathbb{Q}}, \text{ for } g \in \Gamma_{\mathbb{Q}},\ b \in B.$$

Here the functions $r^{\mathbb{Q}}_{ab}$ belong to $A_{\mathbb{Q}}$, and satisfy equations like (1f). We make the following definition: a subset $V_{\mathbb{Z}}$ of $V_{\mathbb{Q}}$ is called a *$\mathbb{Z}$-form* (or *admissible lattice*) of $V_{\mathbb{Q}}$ if

(a) $V_{\mathbb{Z}}$ is a lattice in $V_{\mathbb{Q}}$, which means $V_{\mathbb{Z}} = \sum_b \mathbb{Z} v_{b,\mathbb{Q}}$ for some $\mathbb{Q}$-basis $\{v_{b,\mathbb{Q}}\}$ of $V_{\mathbb{Q}}$, and

(b) All the coefficient functions $r^{\mathbb{Q}}_{ab}$, relative to this basis, lie in $A_{\mathbb{Z}}$.

Another way of expressing condition (b) is to convert $V_{\mathbb{Q}}$ into an $A_{\mathbb{Q}}$-comodule by means of the map $\gamma_{\mathbb{Q}} : V_{\mathbb{Q}} \to V_{\mathbb{Q}} \otimes A_{\mathbb{Q}}$, using equations like (1g). Then (b) is equivalent to

(b') $\gamma_{\mathbb{Q}}(V_{\mathbb{Z}}) \subseteq V_{\mathbb{Z}} \otimes A_{\mathbb{Z}}$.

Now suppose that $K \in \mathcal{K}$. We can make the K-space $V_K = V_{\mathbb{Z}} \otimes K$ (here $\otimes$ means $\otimes_{\mathbb{Z}}$) into an object of M_K, as follows. Define $r^K_{ab} = \alpha_K(r^{\mathbb{Q}}_{ab} \otimes 1_K) \in A_K$, using the K-coalgebra isomorphism $\alpha_K : A_{\mathbb{Z}} \otimes K \to A_K$ postulated in Z2. These r^K_{ab} satisfy equations like (1f). So we may define an action of Γ_K on V_K by equations

(1k) $$gv_{b,K} = \sum_{a\in B} r^K_{ab}(g)v_{a,K}, \text{ for } g \in \Gamma_K,\ b \in B.$$

Here $v_{b,K} = v_{b,\mathbb{Q}} \otimes 1_K$, for $b \in B$. The process by which $V_\mathbb{Q}$ is converted, via the $\mathbb{Z}$-form $V_\mathbb{Z}$, into V_K is called modular reduction. A general theorem guarantees that each $V_\mathbb{Q} \in M_\mathbb{Q}$ possesses at least one $\mathbb{Z}$-form $V_\mathbb{Z}$ (see [49, Lemma 2, p. 43] or [20, (2.2d), p. 159]). Different $\mathbb{Z}$-forms $V_\mathbb{Z}, V'_\mathbb{Z}, \ldots$ of the same $V_\mathbb{Q}$ may give non-isomorphic $V_K = V_\mathbb{Z} \otimes K, V'_K = V'_\mathbb{Z} \otimes K, \ldots$ in M_K, but another general theorem (due in its original form to Brauer and Nesbitt) says that all these modules $V_K, V'_K, \ldots$ have the same composition factor multiplicities; from this the notion of *decomposition numbers* can be defined (see [49, p. 44] or [20, (2.5a), p. 162]).

In these lectures we take $\Gamma = \mathsf{GL}_n(K)$, where K is an infinite field, and study $K\Gamma$-modules $V = (V, \tau)$, which belong to the category $M_K(n, r)$, for a fixed homogeneity degree r (see Example 3, above). In chapter 2 the Schur algebra $S_K(n, r)$ is defined, and it is shown how $K\Gamma$-modules in $M_K(n, r)$ can be regarded as left $S_K(n, r)$-modules, and conversely. An alternative description of $S_K(n, r)$ is that it is the endomorphism algebra of the r^{th} tensor space $E^{\otimes r}$, when the latter is given its natural structure as a module for the symmetric group $G(r)$. This has as corollary Schur's theorem (2.6e): if $\mathsf{char}\, K = 0$, then every module V in $M_K(n, r)$ is completely reducible.

Schur's multiplication rule for $S_K(n, r)$ (see (2.3b)) provides an effective method for calculating with modules in $M_K(n, r)$. For example, the "weight spaces" of such a module V are easily expressed in terms of certain idempotent elements ξ_α in $S_K(n, r)$. Weights and characters are discussed in chapter 3. By definition, the character of V is a symmetric polynomial over $\mathbb{Z}$, which is homogeneous of degree r in a set of n variables $X_1, \ldots, X_n$. In 3.5 is reproduced the argument by which Schur showed that the isomorphism classes of irreducible modules in $M_K(n, r)$ are in one-one correspondence with the partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of r into not more than n parts. Of course Schur considered only the case $K = \mathbb{C}$, but his argument requires only minor modification for an arbitrary infinite field K. The character of an irreducible module of type λ depends only on the characteristic p of K; we write this $\phi_{\lambda,p}$. For $p \neq 0$ these characters have not yet been determined except in special cases. For $p = 0$, Schur showed in [47] that they are the symmetric functions now known as "Schur functions". A proof of this is given at then end of 3.5—our proof uses some identities involving symmetric functions which can be found, for example, in I. G. Macdonald's recent book [39].

In chapters 4 and 5, I have departed widely from Schur's dissertation. These chapters are concerned with the construction, for each λ and for each K, of two modules $D_{\lambda,K}$ and $V_{\lambda,K}$ in our category $M_K(n, r)$. They are "explicit" in the sense that a basis can be given for each. They are dual to each other, in the sense of the "contravariant" duality described in 2.7. $V_{\lambda,K}$ has a unique irreducible factor module; this is denoted $F_{\lambda,K}$. $D_{\lambda,K}$ has a unique minimal submodule, which is isomorphic to $F_{\lambda,K}$. The set $\{F_{\lambda,K}\}$, as λ ranges over all partitions λ of r into not more than n parts, gives a full set of