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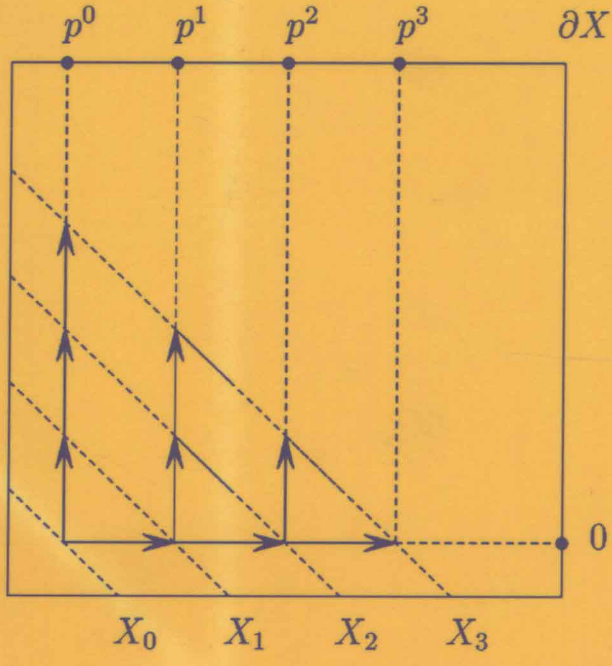
Shai M. J. Haran

Arithmetical Investigations

Representation Theory, Orthogonal Polynomials, and Quantum Interpolations



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Representation Theory, Orthogonal Polynomials, and Quantum Interpolations

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To Yedidya, Antonia, Elisha, Yehonadav,
Amiad & Yoad.

Preface

This book grew out of lectures given at Kyushu University under the support of the Twenty-first Century COE Program "Development of Dynamical Mathematics with High Functionality" (Program Leader: Prof. Mitsuhiro Nakao). They were meant to serve as a primer to my book [Har5]. Indeed that book is very condense, and hard to read. We included however many new themes, such as the higher rank generalization of [Har5], and the fundamental semigroup. Since the audience consisted mainly of representation theorists, the focus shifted more into representation theory (hence less into geometry). We kept the lecture flair, sometimes explaining basic material in more detail, and sometimes only giving brief descriptions.

This book would have never come to life without the many efforts of Professor Masato Wakayama. The author thanks him also for his incredible hospitality. Thanks are also due to Yoshinori Yamasaki, who did an excellent job of writing down and typing the lectures into LaTeX.

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Contents

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Introduction: Motivations from Geometry

Summary. In chap. 0 we begin with geometrical motivations and introduction. We recall the analogies between geometry (curve X over a finite field $\mathbb{F}_q$) and arithmetic (number field K), and the two basic problems of arithmetic: the problem of the real primes and the problem of non-existence of a surface $\mathrm{Spec}\mathcal{O}_K \times \mathrm{Spec}\mathcal{O}_K$ (analogues to $X \times_{\mathbb{F}_q} X$). We then give the "Weil philosophy": the explicit sums of arithmetic are the intersection number of Frobenius divisors on the (non-existing, but see [Har6]) surface. This was never made explicit by Weil (and only was spelled out in [Har2]). The proof of the functional equation and the Riemann–Roch in arithmetic give the "Tate philosophy": we are studying the action of the idele-class $\mathbb{A}_K^*/K^*$ on the problematic space $\mathbb{A}/K^*$. The important part of the ergodic action of K^* on the Adele $\mathbb{A}_K$ is encoded in the action of K^* on $\mathbb{A}_K/K$. We then recall the author formula that connects these two philosophies ([Har2], [Har1]), giving the explicit sums in terms of the Fourier transform of the degree $\log |x|_{\mathfrak{p}}^{-1}$.

0.1 Introduction

The main subject of this course is arithmetic. There are many different reasons for which people are attracted to arithmetic. Simple formulation of complicated problems is one of them. Such problems are the Fermat last theorem, the twin primes problem, Goldbach's conjecture and so on. These are very easy to state, however, they are very hard to solve.

There are many similar points between arithmetic and geometry. André Weil says in [We7] that the situation between arithmetic and geometry is like the "Rosetta Stone". It is a big ancient-Egyptian stone in which the same thing was written in three different languages; hieroglyphic, demotic and Greek. Hieroglyphic was used by ancient Egyptians and it had not been known yet. Demotic was used by Arabs including modern Egyptians. Greek was used by Greeks, and other eastern Europeans. Since the last two languages had been well known, we also understood the mysterious first language, Hieroglyphic. As for arithmetic and geometry, correspondings to the above languages are the number fields and the function fields over a finite field $\mathbb{F}_q$, and over the complex numbers $\mathbb{C}$, (that is, compact Riemann surfaces) which are the one-dimensional objects of Geometry. We have tried to understand number fields from the analogies between arithmetic and geometry. Our position is a bit different from the Weil's point of view. Note that on the Rosetta stone there were three different language talking about the same things, but in our case there is one language talking about three different things. We believe that the language we are using is wrong, and there is a "new language" that will unite Arithmetic and Geometry.

The Rosetta Stone
Hieroglyphic
Demotic
Greek

Global Field
Number field
Function field $/\mathbb{F}_q$
Function field $/\mathbb{C}$

0.2 Analogies Between Arithmetic and Geometry

Let us begin by reviewing the analogy between arithmetic and geometry. In arithmetic, we start from the ring of integers $\mathbb{Z}$. The ring $\mathbb{Z}$ is included in its fraction field, the field of the rational numbers $\mathbb{Q} = \mathrm{Frac}(\mathbb{Z})$. In geometric, the basic object is the ring of polynomials $k[x]$ in one variable over a field k included in the field of rational functions $k(x) = \mathrm{Frac}(k[x])$. These rings $\mathbb{Z}$ and $k[x]$ have many common properties. For example, they have the division with remainder principal, are PID and are UFD, that is, every element can be uniquely written as a product of irreducible elements. Take an irreducible polynomial $f \in k[x]$. Then $k[x]$ embeds in the local ring $k_f[[f]] := \varprojlim k[x]/(f^n)$, the ring of formal power series in f. Here $k_f := k[x]/(f)$ is the residue field. Namely every element in $k[x]$ can be written as a power series in f. Also $k_f[[f]] \subset k_f((f))$ where $k_f((f))$ is the field of formal Laurent series in f. For example if we take $k = \mathbb{C}$ (or any algebraically closed field), an irreducible polynomial can be written as $f(x) = x - \alpha$ for some $\alpha \in \mathbb{C}$. Since $\mathbb{C}_f = \mathbb{C}$, every rational function can be expressed as a Laurent series in $(x - \alpha)$. Let p be a prime. In arithmetic side, the ring of p-adic integers corresponds to $k_f[[f]]$. Every rational integer is represented as a power series in p and we obtain $\mathbb{Z}_p := \varprojlim \mathbb{Z}/(p^n)$. Similarly $\mathbb{Q} \subset \mathbb{Q}_p$, the field of p-adic numbers.

In geometry we have two types of geometry; affine and projective geometry. If $k = \mathbb{C}$, every rational function is written as a Laurent series of $\frac{1}{x}$, that is, $k(x) \subset k((\frac{1}{x}))$. Projectively speaking, a rational function can be expanded at the "infinite point" ∞. Then ∞ clearly corresponds to the ring of formal power series $k[[\frac{1}{x}]]$ in $\frac{1}{x}$. In arithmetic this resembles to the inclusion of $\mathbb{Q}$ into the completion $\mathbb{R} = \mathbb{Q}_\eta$ of $\mathbb{Q}$ at the "real prime η". Then a problem occurs; what is $\mathbb{Z}_\eta$? For a finite prime $p \neq \eta$, the p-adic integers $\mathbb{Z}_p$ is given by $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \,|\, |x|_p \leq 1\}$ where $|\cdot|_p$ is the p-adic absolute value. From this point of view, $\mathbb{Z}_\eta$ is considered as the interval $[-1, 1]$, however, it is not closed under addition and is not given by any inverse limit neither.

Function fields (Geometry)		Number fields (Arithmetic)	
$k[x] \subset k(x)$		$\mathbb{Z} \subset \mathbb{Q}$	
$k_f[[f]] \subset k_f((f))$	$k[[\frac{1}{x}]] \subset k((\frac{1}{x}))$	$\mathbb{Z}_p \subset \mathbb{Q}_p$	"$\mathbb{Z}_\eta$" $\subset \mathbb{Q}_\eta = \mathbb{R}$

We further remark that there is an obvious difference between number fields and function fields from the point of the tensor product. In geometry one can take a product of given geometrical object and obtain a new geometrical object. For example, the product of the affine line $\mathbb{A}^1$ with itself is the plane. This correspond to the tensor product of two polynomial rings $k[x_1]$ and $k[x_2]$. The product $k[x_1] \otimes_k k[x_2]$ in the category of k-algebra is equal to $k[x_1, x_2]$, ring of polynomials in two variables. On the other hand, if we consider the tensor product of two copies of $\mathbb{Z}$, taken in the category of commutative rings we obtain only $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$.

0.3 Zeta Function for Curves

Let X be a (smooth, projective) curve of genus g defined over the finite field $k = \mathbb{F}_q$. Let $K = k(X)$ be the field of k-rational functions and $\mathfrak{p}$ be a maximal ideal of the coordinate ring $k[X]$. We denote by $K_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ the completion of K with respect to $\mathfrak{p}$ and the ring of integers of $K_\mathfrak{p}$, respectively. Let $K_\mathfrak{p}^* = K_\mathfrak{p}\backslash\{0\}$ and $\mathcal{O}_\mathfrak{p}^*$ be the unit group of $\mathcal{O}_\mathfrak{p}$. Let $\phi_\mathfrak{p}$ be the characteristic function of $\mathcal{O}_\mathfrak{p}$ and $dx_\mathfrak{p}$ (resp. $d^*x_\mathfrak{p}$) be the additive (resp. multiplicative) Haar measure on $K_\mathfrak{p}$ (resp. $K_\mathfrak{p}^*$) normalized by $dx_\mathfrak{p}(\mathcal{O}_\mathfrak{p}) = 1$ (resp. $d^*x_\mathfrak{p}(\mathcal{O}_\mathfrak{p}^*) = 1$). We denote by $\mathbb{A}$ (resp. $\mathbb{A}^*$) the adele ring (resp. idele group) of K. Put $\mathcal{O}_\mathbb{A} := \prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}$, $\mathcal{O}_\mathbb{A}^* := \prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}^*$, $dx := \bigotimes_\mathfrak{p} dx_\mathfrak{p}$ and $d^*x := \bigotimes_\mathfrak{p} d^*x_\mathfrak{p}$. Then the zeta function for X is defined by

(i) $$\zeta_X(s) := \exp\Big(\sum_{n\geq 1} \#X(k_n)\frac{q^{-sn}}{n}\Big),$$

where $k_n := \mathbb{F}_{q^n}$. Let $k(\mathfrak{p}) := \mathcal{O}_\mathfrak{p}/\mathfrak{p}$ be the residue field and $\mathbb{N}\mathfrak{p} := \#k(\mathfrak{p}) = q^{\deg\mathfrak{p}}$ with $\deg\mathfrak{p} := [k(\mathfrak{p}) : k]$. Then we have the following calculations:

(ii) $$\zeta_X(s) = \prod_\mathfrak{p} \left(1 - \mathbb{N}\mathfrak{p}^{-s}\right)^{-1}$$

(iii) $$= \sum_{\mathfrak{a}\geq 0} \mathbb{N}\mathfrak{a}^{-s}$$

$$= \int_{\mathbb{A}^*/K^*\mathcal{O}_\mathbb{A}^*} \Big(\sum_{\gamma\in K^*/k^*} \phi_\mathbb{A}(\gamma\mathfrak{a})\Big) |\mathfrak{a}|_\mathbb{A}^s d^*\mathfrak{a} = \int_{\mathbb{A}^*/\mathcal{O}_\mathbb{A}^*} \phi_\mathbb{A}(\mathfrak{a})|\mathfrak{a}|_\mathbb{A}^s d^*\mathfrak{a}$$

(iv) $$= \prod_\mathfrak{p} \int_{K_\mathfrak{p}^*/\mathcal{O}_\mathfrak{p}^*} \phi_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})|\mathfrak{a}_\mathfrak{p}|_\mathfrak{p}^s d^*\mathfrak{a}_\mathfrak{p} \quad .$$

$$\text{(v)} \qquad = \sum_{\mathfrak{a}\in \mathrm{Pic}_K} \frac{q^{h^0(\mathfrak{a})}-1}{q-1} q^{-s\cdot \deg(\mathfrak{a})}$$

$$\text{(vi)} \qquad = \frac{\prod_{i=1}^{2g}(1-\lambda_i q^{-s})}{(1-q^{-s})(1-q^{1-s})}.$$

Here $\phi_{\mathbb{A}}$ is the characteristic function of $\mathcal{O}_{\mathbb{A}}$, $|\mathfrak{a}|_{\mathbb{A}} := \prod_{\mathfrak{p}} |\mathfrak{a}_{\mathfrak{p}}|_{\mathfrak{p}}$ and $|\cdot|_{\mathfrak{p}}$ is the normalized absolute value on $K_{\mathfrak{p}}$ as $d(ax_{\mathfrak{p}}) = |a|_{\mathfrak{p}} \cdot dx_{\mathfrak{p}}$ and $|\pi|^s = \mathbb{N}\mathfrak{p}^{-s}$ with $\mathfrak{p} = (\pi)$. It is easy to check these equalities. In fact, one obtains (0.3) (i) $\Longleftrightarrow$ (0.3) (ii) by taking $d\log$ and the fact

$$\#X(k_n) = \sum_{\substack{\mathfrak{p} \\ \deg\mathfrak{p}|n}} \deg\mathfrak{p} = \sum_{k(\mathfrak{p})\subseteq k_n} \deg\mathfrak{p}$$

since each $\mathfrak{p}$ with $k(\mathfrak{p}) \subseteq k_n$ gives $\deg\mathfrak{p}$ points in $X(k_n)$. By the unique factorization, we have (0.3) (ii) $\Longleftrightarrow$ (0.3) (iii). To show the equalities (0.3) (iii) $\Longleftrightarrow$ (0.3) (iv) $\Longleftrightarrow$ (0.3) (v), recall that

$$\mathrm{Div}_K = \mathbb{A}^*/\mathcal{O}_{\mathbb{A}}^*, \qquad \mathrm{Pic}_K = \mathbb{A}^*/\mathcal{O}_{\mathbb{A}}^* K^* \xrightarrow{|\cdot|_{\mathbb{A}}} q^{\mathbb{Z}}$$

and the kernel $\mathbb{A}^{(1)}/\mathcal{O}_{\mathbb{A}}^* K^* = \mathrm{Pic}_K^{(1)}$ is finite. Let $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_n\}$ be the representative of $\mathrm{Pic}_K^{(1)}$. Let $\mathfrak{c} \in \mathrm{Pic}_K$ of degree 1. Then for any $\mathfrak{a} \in \mathrm{Div}_K$, we can write $\mathfrak{a} = f \cdot \mathfrak{a}_i \cdot \mathfrak{c}^n$ with some $f \in K^*/k^*$ and $n = \deg\mathfrak{a}$. Then we have

$$\mathfrak{a} \geq 0 \Longleftrightarrow (f) \geq -\mathfrak{a}_i - n\mathfrak{c} \Longleftrightarrow f \in H^0(X, \mathcal{O}_{\mathbb{A}}(\mathfrak{a}_i + n\mathfrak{c})) \Longleftrightarrow \phi_{\mathbb{A}}(f\cdot\mathfrak{a}_i\cdot\mathfrak{c}^n) = 1.$$

Hence the number of such $\mathfrak{a}$ is equal to

$$\frac{q^{h^0(\mathfrak{a}_i+n\mathfrak{c})}-1}{q-1} = \sum_{f\in K^*/k^*} \phi_{\mathbb{A}}(f\cdot\mathfrak{a}_i\cdot\mathfrak{c}^n),$$

where

$$h^0(\mathfrak{a}) := \dim H^0(X, \mathcal{O}_{\mathbb{A}}(\mathfrak{a})) = \frac{1}{\log q} \log \sum_{\gamma\in K} \phi_{\mathbb{A}}(\gamma\mathfrak{a}).$$

Therefore, using the formula $|\mathfrak{a}|_{\mathbb{A}} = q^{-\deg\mathfrak{a}}$ (or $\frac{\log|\mathfrak{a}|_{\mathbb{A}}^{-1}}{\log q} = \deg\mathfrak{a}$), we obtain the desired equalities. The shape (0.3) (vi) of $\zeta_X(s)$ follows from the Riemann–Roch theorem. Comparing the formula (0.3) (i) and (0.3) (vi) and taking $d\log$, we have

$$\#X(k_n) = \underbrace{1+q^n}_{\#\mathbb{P}^1(k_n)} - \sum_{i=1}^{2g} \lambda_i^n \qquad (n \geq 1).$$