Generalized Concavity

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Preface

Concavity plays a central role in mathematical economics, engineering, management science, and optimization theory. The reason is that concavity of functions is used as a hypothesis in most of the important theorems concerning extremum problems. In other words, concavity is usually a sufficient condition for satisfying the underlying assumptions of these theorems, but concavity is definitely not a necessary condition. In fact, there are large families of functions that are nonconcave and yet have properties similar to those of concave functions. Such functions are called generalized concave functions, and this book is about the various generalizations of concavity, mainly in the context of economics and optimization.

Although hundreds of articles dealing with generalized concavity have appeared in scientific journals, numerous textbooks have specific chapters on this subject, and scientific meetings devoted to generalized concavity have been held and their proceedings published, this book is the first attempt to present generalized concavity in a unified framework. We have collected results dealing with this subject mainly from the economics and optimization literature, and we hope that the material presented here will be useful in applications and will stimulate further research.

The writing of this book constituted a unique experience for the authors in international scientific cooperation—cooperation that extended over many years and at times spanned three continents. It was an extremely fruitful and enjoyable experience, which we will never forget.

We are indebted to our respective home universities—the Technion–Israel Institute of Technology, the University of British Columbia, the University of Alberta, and Tel Aviv University—for including the other authors in their exchange programs and for the technical assistance we received. Thanks are also due to the Center for Operations Research and Econometrics, Université Catholique de Louvain, for the hospitality extended to one of the authors.

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Introduction

In this introductory chapter, we provide a brief and mostly nonmathematical description of the contents of this book on generalized concavity. Formal mathematical definitions of the various types of concavity may be found in subsequent chapters.

The first question we must attempt to answer in this chapter is: why do concave functions occupy such an important position in economics, engineering, management science, and applied optimization theory in general? A real-valued function of n variables defined over a convex subset of Euclidean n-dimensional space is *concave* iff (if and only if) the line segment joining any two points on the graph of the function lies on or below the graphs; a set is convex iff, given any two points in the domain of definition of the set, the line segment joining the two points also belongs to the set.

Returning to the question raised above, we suggest that the importance of concave functions perhaps rests on the following two properties: (i) a local maximizer for a concave function is also a global maximizer, and (ii) the usual first-order necessary conditions for maximizing a differentiable function f of n variables over an open set [i.e., x^* is a point such that the gradient vector of f vanishes so that $\nabla f(x^*) = 0$] are also sufficient to imply that x^* globally maximizes f if f is a concave function defined over a convex set. Various generalizations of concavity (studied in Chapter 3) preserve properties (i) and (ii), respectively. In Chapter 2, we also study two classes of functions that are more restrictive than the class of concave functions: *strictly concave* and *strongly concave* functions. Strictly concave functions have the following useful property, which strengthens property (i) above: (iii) A local maximizer for a strictly concave function is also the unique global maximizer. A function is *strictly concave* iff the line segment joining any two distinct points on the graph of the function lies below the graph of the function (with the obvious exception of the end points of the line segment). A function f is *strongly concave* iff it is equal to the sum of a concave function and a negative definite quadratic form, i.e., $f(x) = g(x) - \alpha x^T x$ for every x belonging to the convex domain of definition set, where g is a concave function, $\alpha > 0$ is a positive scalar, and $x^T x = \sum_{i=1}^{n} x_i^2$. Strongly concave functions also have a property (iii) above, and, in addition, in a neighborhood of a local maximizer, a strongly concave twice continuously differentiable function will have the curvature of a negative definite quadratic form. This property is useful in proving convergence of certain optimization algorithms, and it is also useful in enabling one to prove comparative statics theorems in economics; see Section 4.9 in Chapter 4.

We now describe the contents of each chapter.

Chapter 2 deals with concave functions and the two classes of functions that are stronger than concavity, namely, strictly and strongly concave functions. The first three sections of Chapter 2 develop alternative characterizations of concave functions. In addition to the definition of a concave function, there are three additional very useful characterizations of concavity: (i) the hypograph of the function (the graph of the function and the set in $(n + 1)$-dimensional space lying below the graph) is a convex set; (ii) the first-order Taylor series approximation to the function around any point in the domain of definition lies on or above the graph of the function (this characterization requires the existence of first-order partial derivatives of the function); (iii) the Hessian matrix of second-order partial derivatives of the function evaluated at each point in the domain of definition is a negative semidefinite matrix (this characterization requires the existence of continuous second-order partial derivatives of the function).

Section 2.3 of Chapter 2 also develops some composition rules for concave functions; e.g., a nonnegative sum of concave functions is a concave function or the pointwise minimum of a family of concave functions is a concave function, and so on. Additional composition rules are developed in Chapter 5. Section 2.3 also provides characterizations for strictly and strongly concave functions.

Section 2.4 derives the local-global maximizer properties of concave functions referred to earlier. As we stated before, these properties are probably the main reason for the importance of the concavity concept in applied optimization theory.

Section 2.5 deals with another extremely important topic from the viewpoint of applications, namely, concave mathematical programming problems. A *concave program* is a constrained maximization problem, where (i) the objective function being maximized is a concave function; (ii) the functions used to define equal to or greater than zero inequality constraints are concave functions; and (iii) the functions used to define any equality constraints are linear (or affine). If we have a concave program with once-differentiable objective and constraint functions, then it turns out that certain conditions due to Karush (1939) and Kuhn and Tucker (1951) involving the gradient vectors of the objective and constraint functions evaluated at a point x^* as well as certain (Lagrange) multipliers are *sufficient* to imply that x^* solves the concave programming problem; see Theorem 2.30. In addition, if a relatively mild constraint qualification condition is satisfied, then these same Karush-Kuhn-Tucker conditions are also *necessary* for x^* to solve the constrained maximization problem; see Theorem 2.29. The multipliers that appear in the Karush-Kuhn-Tucker conditions can often be given physical or economic interpretations: the multiplier (if unique) that corresponds to a particular constraint can be interpreted as the incremental change in the optimized objective function due to an incremental relaxation in the constraint. For further details and rigorous statements of this result, see Samuelson (1947, p. 132), Armacost and Fiacco (1974), and Diewert (1984). Another result in Section 2.5, Theorem 2.28, shows that a concave programming problem has a solution iff a certain Lagrangian saddle point problem (which is a maximization problem in the primal variables and a minimization in the dual multiplier variables) has a solution. This theorem, due originally to Uzawa (1958) and Karlin (1959), does not involve any differentiability conditions; some economic applications of it are pursued in the last section of Chapter 4.

Chapter 3 deals with generalized concave functions; i.e., functions that have some of the properties of concave functions but not all.

Section 3.1 defines the weakest class of generalized concave functions, namely, the class of quasiconcave functions. A function (defined over a convex subset of Euclidean n-dimensional space—throughout the book we make this domain assumption) is *quasiconcave* iff the values of the function along the line segment joining any two points in the domain of definition of the function are equal to or greater than the minimum of the function values at the end points of the line segment. Comparing the definition of a quasiconcave function with the definition of a concave function, it can be seen that a concave function is quasiconcave (but not vice versa). Recall that concave functions played a central role in optimization theory because of their extremum properties. Quasiconcave functions also have a useful extremum property, namely: every strict local maximizer of a quasiconcave function is a global maximizer (see Proposition 3.3). Quasiconcave functions also play an important role in the *generalized concave mathematical programming problem* (see Section 3.6), where the concave inequality constraints that occurred in the concave programming problem of Section 2.5 become quasiconcave inequality constraints. Finally, quasiconcave functions play a central role in economic theory since the utility functions of consumers and the production functions of producers are usually assumed to be quasiconcave functions (see Chapter 4 below).

Sections 3.1 and 3.2 provide various alternative characterizations of quasiconcavity in a manner that is analogous to the alternative characterizations of concavity that were developed in the opening sections of Chapter 2. Three alternative characterizations of quasiconcavity are as follows: (i) the upper level sets of the function are convex sets for each level (Definition 3.1); (ii) if the directional derivative of the function in any feasible direction is negative, then function values in that direction must be less than the value of the function evaluated at the initial point (this is the contrapositive to Theorem 3.11); and (iii) the Hessian matrix of second-order partial derivatives of the function evaluated at each point in the domain of definition is negative semidefinite in the subspace orthogonal to the gradient vector of the function evaluated at the same point in the domain of definition (Corollary 3.20). Characterization (ii) above requires once differentiability of the function, while characterization (iii) requires twice continuous differentiability over an open convex set *and* the existence of a nonzero gradient vector at each point in the domain of definition. The restriction that the gradient vector be nonzero can be dropped (see Theorem 3.22), but the resulting theorem requires an additional concept that probably will not be familiar, namely, the concept of a *semistrict local minimum*, explained in Definition 3.3. This concept is also needed to provide a characterization of semistrictly quasiconcave functions in the twice-differentiable case; see Theorem 3.22. On the other hand, the familiar concept of a *local minimum* is used to provide a characterization of strictly quasiconcave functions in the twice-differentiable case; see Theorem 3.26. In fact, all of the different types of generalized concave functions can be characterized by their local minimum or maximum behavior along line segments; see Diewert, Avriel, and Zang (1981) for the details.

Section 3.4 deals with the properties and uses of the class of semistrictly quasiconcave functions. A function is *semistrictly quasiconcave* iff for every two points in the domain of definition such that the function has unequal values at those two points, then the value of the function along the interior of the line segment joining the two points is greater than the minimum of the two end-point function values; see Definition 3.11. If the function is continuous (or merely upper semicontinuous so that its upper level sets are closed), then a semistrictly quasiconcave function is also quasiconcave (Proposition 3.30). It is easy to verify that a concave function is also semistrictly quasiconcave. Hence, in the continuous (or upper semicontinuous) case, the class of semistrictly quasiconcave functions lies between the concave and quasiconcave classes. An alternative characterization of the concept of semistrict quasiconcavity for continuous functions in terms of level set properties is given by Proposition 3.35: the family of upper level sets must be convex and each nonmaximal level set must be contained in the boundary of the corresponding upper level set (a maximal level set obviously must coincide with the corresponding upper level set). Semistrictly quasiconcave functions have the same extremum property that concave functions had, namely: any local maximizer for a semistrictly quasiconcave function is a global maximizer (Theorem 3.37). Semistrictly quasiconcave functions also play a role in consumer theory; see Section 4.5.

A more restrictive form of generalized concavity than semistrict quasiconcavity is strict quasiconcavity, discussed in Section 3.3 A function is *strictly quasiconcave* iff for every two distinct points in the domain of definition of the function the value of the function along the interior of the line segment joining the two points is greater than the minimum of the two end-point function values; see Definition 3.8. It is easy to verify that a strictly concave function is strictly quasiconcave and that a strictly quasiconcave function is semistrictly quasiconcave and quasiconcave. Strictly quasiconcave functions have the same extremely useful extremum property that strictly concave functions had: any local maximum is the unique global maximum. Strictly quasiconcave functions also play an important role in economics; see Section 4.6. Continuous strictly quasiconcave functions have strictly convex upper level sets (Proposition 3.28).

Section 3.5 deals with three new classes of generalized concave functions: (i) pseudoconcave, (ii) strictly pseudoconcave, and (iii) strongly pseudoconcave. These classes of functions are generalizations of the class of concave, strictly concave, and strongly concave functions, respectively. The three new classes of functions are usually defined only in the differentiable case (although nondifferentiable definitions exist in the literature and are referred to in the text).

A *pseudoconcave function* may be defined by the following property (the contrapositive to Definition 3.13): if the directional derivative of the function in any feasible direction is nonpositive, then function values in that direction must be less than or equal to the value of the function evaluated at the initial point. Pseudoconcave functions have the same important extremum property that concave functions had: if the gradient vector of a function is zero at a point, then that point is a global maximizer for the function (Theorem 3.39). A characterization of pseudoconcave functions in the twice continuously differentiable case is provided by Theorem 3.43.

A *strictly pseudoconcave function* may be defined by the following property (the contrapositive to Definition 3.13): if the directional derivative of the function in any feasible direction is nonpositive, then the function values in that direction must be less than the value of the function evaluated at the initial point. Strictly pseudoconcave functions have the same important extremum property that strictly concave functions had: if the gradient vector of a function is zero at a point, then that point is the unique global maximizer for the function (Theorem 3.39). A characterization of strictly pseudoconcave functions in the twice continuously differentiable case is provided by Theorem 3.43.

Strongly pseudoconcave functions are strictly pseudoconcave functions with the following additional property: if the directional derivative of the function in any feasible direction is zero, then the function diminishes (locally at least) at a quadratic rate in that direction. Recall that in the twice differentiable case, a strongly concave function could be characterized by having a negative definite Hessian matrix of second-order partial derivatives at each point in its domain of definition. In the twice differentiable case, a function is strongly pseudoconcave iff its Hessian matrix is negative definite in the subspace orthogonal to the gradient vector at each point in the domain of definition (Proposition 3.45). The property of strong pseudoconcavity is sometimes called *strong quasiconcavity* in the economics literature, and some economic applications of this concept are developed in Section 4.7.

It should be noted that all of our concavity and quasiconcavity concepts have *convex* and *quasiconvex* counterparts: a function f is convex (quasiconvex) iff $-f$ is concave (quasiconcave).

Chapter 3 is concluded by Section 3.6, which deals with generalizations of the concave programs studied in Section 2.5. An example shows that the Karlin–Uzawa Saddle Point Theorem for (not necessarily differentiable) concave programming problems cannot be readily generalized. However, for differentiable programs, the sufficiency of the Karush–Kuhn–Tucker conditions for concave problems can be generalized to programming problems involving objective and constraint functions that satisfy some type of generalized concavity property: Theorem 3.48 shows that the objective function need only be pseudoconcave, the equal to or greater than inequality constraint functions need only be quasiconcave, and the equality constraint functions need only be quasimonotonic. A function is *quasimonotonic* iff it is both quasiconcave and quasiconvex (inequality 3.35). Thus these pseudoconcavity, quasiconcavity, and quasimonotonic properties replace the earlier concavity and linearity properties that occurred in Theorem 2.30.

Chapter 4 deals with economic applications. We consider four models of economic behavior: (i) a producer's cost minimization problem, (ii) a consumer's utility maximization problem, (iii) a producer's profit maximization problem, and (iv) a model of national product maximization for an economy that faces world prices for the outputs it produces and is constrained by domestic resource availabilities. In the context of the above four models, we show how each of the types of generalized concavity studied in Chapters 2 and 3 arises in a natural way.

Chapter 4 also proves some economics *duality theorems*. Many problems in economics involve maximizing or minimizing a function subject to another functional constraint. If either the objective function or the constraint function is linear (or affine), then the optimized objective function may be regarded as a function of the parameters or coefficients (these are usually prices) of the linear function involved in the primal optimization problem. This optimized objective function, regarded as a function of the prices appearing in the primal problem, is called the *dual function.* Under certain conditions, this dual function may be used to reconstruct the nonlinear function that appeared in the primal optimization problem. The regularity conditions always involve some kind of generalized concavity restrictions on the nonlinear primal function. Some applications of these economics duality theorems are provided in Chapter 4.

Chapters 5 and 6 deal with the following important question: how can we recognize whether a given function has a generalized concavity property?

In the first part of Chapter 5, *composition rules* for the various types of generalized concave functions are derived. Suppose we know that certain functions have a generalized concavity property (or are even concave). Then under what conditions will an increasing or decreasing function of the original function or functions have a generalized concavity property?

In the second part of Chapter 5, we apply these composition rules to derive conditions under which a *product* or *ratio* of two or more functions has a generalized concavity property, provided that the original functions are concave or convex. Special attention is given to the case of products and ratios of only two functions. The material in this chapter draws heavily on the work of Schaible (1971, 1972).

Chapter 6 deals with the generalized concavity properties of an important class of functions, namely, the class of *quadratic* functions. It turns out that restricting ourselves to the class of quadratic functions simplifies life somewhat: quasiconcave and semistrictly quasiconcave quadratic functions cannot be distinguished. Furthermore, strictly and strongly pseudoconcave quadratic functions cannot be distinguished. However, even with these simplifications, the characterization of the generalized concavity properties of quadratic functions proves to be a rather complex task. Chapter 6 develops all known results using a unified framework (based on the composite function criteria developed in Chapter 5) on the generalized concavity properties of quadratic functions. Furthermore, many of the criteria are expressed in alternative ways using eigenvalues and eigenvectors or determinantal conditions. The material in this chapter summarizes and extends the work of Schaible (1981a, b).

Chapter 7 provides a brief survey of *fractional programming* and indicates how generalized concavity concepts play a role in this important applied area. A *fractional program* is a constrained maximization problem where the objective function is a ratio of two functions, say $f(x)/g(x)$, and the decision variables x are restricted to belong to a closed convex set S in finite-dimensional Euclidean space. In a *linear fractional program*, the functions f and g are both restricted to be linear or affine. In a *concave fractional program*, the numerator function f is restricted to be nonnegative and concave and the denominator function is restricted to be convex and positive over the constraint set S. In a *generalized fractional program*, we maximize a sum of ratios or we maximize the minimum of a finite number of ratios.

In Section 7.1, we show that the objective function in a concave fractional programming problem is semistrictly concave. Hence, a local maximum for the problem is a global maximum. If, in addition, the objective function in a concave fractional program is differentiable, we show that the objective function is pseudoconcave. In this latter case, the Karush–Kuhn–Tucker conditions are sufficient (and necessary if a constraint qualification condition is satisfied) to characterize the solution to the fractional programming problem.

Section 7.2 surveys a number of applications of fractional programming.

Business and economics applications of fractional programming include the following:

1. *Maximization of productivity.* The productivity of a firm, enterprise, or economy is usually defined as a function of outputs produced divided by a function of the inputs utilized by the firm.
2. *Maximization of the rate of return on investments.*
3. *Minimization of cost per unit of time.*
4. *Maximization of an economy's growth rate.* This problem originates in von Neumann's (1945) model of an expanding economy. The overall growth rate in the economy is the smallest of certain sectoral growth rates. Maximizing the minimum of the sectoral growth rates leads to a generalized fractional programming problem.
5. *Portfolio selection problems in finance.* Here we attempt to maximize the expected return of a portfolio of investments divided by the risk of the portfolio.

Applied mathematics applications of fractional programming include the following:

1. *Finding the maximal eigenvalue.* The maximal eigenvalue λ of a symmetric matrix A can be found by maximizing the ratio of two quadratic forms, i.e., $\lambda = \max_x \{x^T Ax/x^T x: x \neq 0\}$.
2. *Approximation theory.* Some problems in numerical approximation theory generate generalized fractional programs.
3. *Solution of large-scale linear programs.* Using decomposition methods, the solution to a large linear program can be reduced to

the solution of a finite number of subproblems. These subproblems are linear fractional programs.

4. *Solving stochastic programs.* Certain stochastic linear programming problems lead to fractional programming problems. This class of applications includes the portfolio selection problem mentioned above.

The above applications of fractional programming (and additional ones) are discussed in Section 7.2 and references to the literature are provided there.

In Section 7.3, we indicate how a concave fractional program may be transformed into a family of ordinary concave programs using a separation of variable technique. However, an even more convenient transformation is available. Propositions 7.2 and 7.3 show how concave fractional programs can be transformed into ordinary concave programs using a certain change of variables transformation. We also derive the (saddle point) dual programming problems for a concave fractional program in this section.

Section 7.4 concludes Chapter 7 by outlining some possible algorithmic approaches to the solution of concave fractional programs.

The material in Chapter 7 draws heavily on Schaible (1978, 1981c).

Chapter 8 introduces two new classes of generalized concave functions: transconcave functions and (h, ϕ)-concave functions.

A function f defined over a convex subset C of Euclidean n-dimensional space is *transconcave* (or *G-concave*) iff it can be transformed into a concave function by means of a monotonically increasing function of one variable G; hence f is G-concave iff $h(x) \equiv G[f(x)]$ is a concave function over C.

Transconcave functions are used in at least two important areas of application. The first use is in numerical algorithms for maximizing functions of n variables; if the objective function f in the nonlinear programming problem can be transformed into a concave function $G[f(x)]$ by means of an increasing function of one variable G, then the original objective function $f(x)$ may be replaced by the concave objective function $G[f(x)]$ and one of many concave programming algorithms may be used to solve the problem. A second use is in the computation of general equilibria in economic models where the number of consumers in the model is smaller than the number of commodities. In order to compute a general equilibrium (see Debreu, 1959, for a formal definition and references to the literature), an algorithm is required that will compute a fixed point under the hypotheses of the Kakutani (1941) Fixed Point theorem. Scarf (1967) has constructed such an algorithm, but it is not efficient if the number of commodities exceeds 50. However, if the preferences of all consumers in the general equilibrium