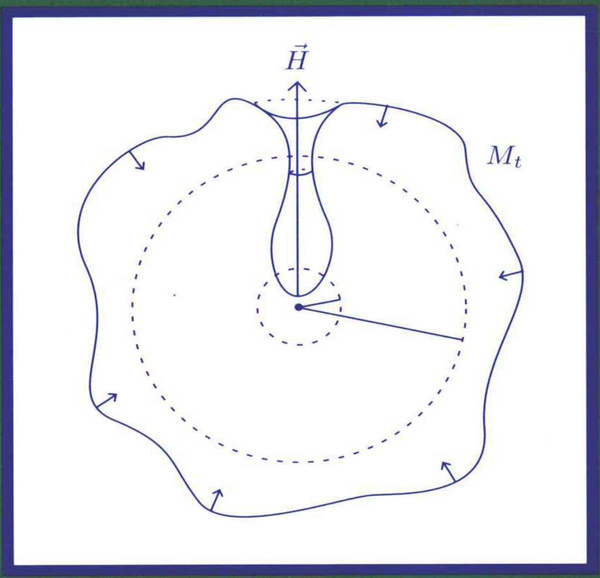
Progress in Nonlinear Differential Equations and Their Applications

Klaus Ecker

Regularity Theory for Mean Curvature Flow



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Preface

This book started as a set of informal notes on Brakke's regularity theory for mean curvature flow ([B]). These notes focussed on the special case where smooth solutions of mean curvature flow develop singularities for the first time, thus expressing the underlying ideas almost entirely in the language of differential geometry and partial differential equations. In particular, notation from geometric measure theory was kept to a minimum.

After I gave lectures on Brakke's work during 1994 in the Mathematics Departments at Stanford University and the University of Tübingen and at a workshop on Motion by Mean Curvature in Trento, I was encouraged by a number of colleagues to publish my notes.

Since that time, but particularly since 1978, when Brakke's work first appeared, there have been many new developments in mean curvature flow starting with Huisken's work in 1984 ([Hu1]). Some of these have resulted in significant simplifications of Brakke's original arguments as well as improvements of his results in special situations. This includes particularly the case of evolving hypersurfaces with positive mean curvature. Remarkably though, in the general case the estimate of the singular set provided in Brakke's main regularity theorem ([B], Theorem 6.12) has not been improved upon to date.

The bulk of the material in this text is based on lectures I gave in the Department of Mathematics at the Universität Freiburg, Germany, from November 2000 to February 2001 and at Monash University, Melbourne, Australia during the first half of 2001.

The central theme is the regularity theory for mean curvature flow leading up to a proof of Brakke's main regularity theorem ([B], Theorem 6.12) for the special case where smooth solutions develop singularities. In this self-contained presentation, I have replaced many of Brakke's original techniques by more recent methods wherever this led to a clear simplification of his arguments. Some of his original ideas, in only slightly modified form, have been included in an appendix.

Under additional assumptions such as symmetry conditions or dimensional restrictions on the solution or sign conditions on the mean curvature, improved estimates for the dimension of the singular set or refined descriptions of the behaviour of the solution near singularities can be established. I have, however, decided not to include a treatment of such results in this presentation. In particular, the book does not cover the following important contributions: Altschuler, Angenent and Giga's work on isolated singularities of surfaces of revolution ([AAG]), Angenent and Velazquez's construction of solutions exhibiting degenerate neckpinches ([AV]), Hamilton's influential Harnack inequality for convex solutions ([Ha4]), Huisken's classification of self-similar solutions with nonnegative mean curvature ([Hu3]), Huisken and Sinestrari's and White's asymptotic description of singularities in the mean convex case ([HS1], [W4]), Ilmanen's results on smooth blow-ups in two dimensions ([I1], [I2]) as well as White's dimension reduction argument ([W1]). The latter works without additional assumptions on the solution but so far implies improved (and optimal) estimates for the singular set only in the mean convex case ([W1], [W2], [W4]).

I also have chosen not to include important alternative approaches to mean curvature flow such as the level-set approach adopted by Evans and Spruck ([ES1]-[ES3]) and by Chen, Giga and Goto ([CGG], [GG1], [GG2]) as well as the work of Ilmanen ([I1]) which establishes a link between level-set flow and Brakke's varifold solution framework.

This project was supported by the Universität Freiburg, Monash University and the Australian Research Council. A shorter version of this exposition, which appeared in the preprint series of the Mathematics Department of the Universität Freiburg, was completed while I visited Gerhard Huisken at the Albert Einstein Institute in Golm in 2002. I would particularly like to thank Ernst Kuwert for the invitation to give these lectures and all my colleagues in the Mathematics Department at the Universität Freiburg for their hospitality.

I would like to thank Vadim Goutkovitch, Tim Hunt, Burkard Polster and particularly Kashif Rasul for advice and help in producing electronic versions of my hand sketches. Kashif also created the image of the catenoidal surface in Chapter 5 using Mathematica. Special thanks to Ann Björner who did a wonderful job of re-drawing all my handsketches using Adobe Illustrator and in particular produced the front cover image.

Thanks to Maria Athanassenas, Mark Aarons, Josh Bode, Kashif Rasul and Felix Schulze who pointed out several typographical and other errors and made valuable suggestions regarding the exposition.

I am indebted to my colleague Marty Ross who took the time to thoroughly read an earlier version of Chapter 2 and provided invaluable comments and advice on the general layout of this book.

I am particularly grateful to Craig Evans for encouraging me to publish my notes, continually urging me to complete this project and for his support throughout the production of this book.

Contents

Chapter 1

Introduction

Mean curvature flow evolves hypersurfaces in their normal direction with speed equal to the mean curvature at each point. It is the steepest descent flow for the area functional. In particular, minimal (zero mean curvature) hypersurfaces are stationary solutions.

Analytically, this process is described by a weakly parabolic system of partial differential equations for the local embedding map of the evolving hypersurfaces. At the curvature level it looks like a reaction-diffusion system. The reaction part, which is cubic in the curvatures, generally forces the formation of singularities (points near which the curvature blows up) in finite time. The diffusion part, involving the Laplace–Beltrami operator of the moving hypersurface, effects an infinitesimal separation of variables near the singularity. This means that the solution asymptotically moves by scaling or a rigid motion, with geometric shape determined by an elliptic system of partial differential equations.

Mean curvature flow is related to Hamilton's Ricci flow for metrics in many geometric and analytic aspects. The Ricci flow programme was developed by Hamilton with the aim of settling Thurston's geometrisation conjecture on the classification of all closed 3-manifolds (see [Ch] for a survey of Hamilton's work). The basic idea is to start with an arbitrary initial metric on such a manifold and canonically alter it using a combination of Ricci flow deformation and controlled topological surgery. Here, a careful analysis of the reaction-diffusion system satisfied by the curvature operator of the evolving metric plays a crucial role. Within this programme, there has recently been major progress by Perelman ([P1], [P2]). For mean curvature flow, applications to the classification of hypersurfaces with positive scalar curvature in four-dimensional Euclidean space were established in the work of Huisken and Sinestrari ([HS1]–[HS3]).

The label *regularity theory* generally refers to results regarding the set of singular times, the dimension and structure of the singular set at those times as well as the asymptotic behaviour of solutions near singularities.

Regularity results are established, for instance, by identifying scale invariant integral quantities which use up a certain portion of some appropriate total energy of the system (area in this case) at each singular point. By means of these scale invariant quantities one also proves the existence of asymptotic limits of solutions obtained by rescaling the original solution ever more closely about a singular point. Such rescaling limits turn out to be self-similar, that is, invariant under further scaling. Moreover, these limiting solutions need not be smooth so usually have to be described in the language of geometric measure theory.

In a subsequent step, one aims to classify all self-similar solutions of mean curvature flow as these describe the possible geometric shapes of the original solution near the singularity. One then linearises this solution about a self-similar limit and studies the spectrum of the linearised operator.

The goal of this analysis is a complete asymptotic description of the solution near the singularity, hopefully leading to a canonical way of continuing it beyond the singular time. It may also provide analytic insights for related nonlinear evolution equations such as reaction-diffusion equations, the harmonic map heat flow and the Ricci flow of metrics.

In this book we concentrate on the modest goal of estimating the size of the singular set and proving convergence of rescaled solutions to a self-similar one. The approach is analogous to the regularity theory for minimal hypersurfaces, where one now has an optimal estimate for the dimension of the singular set (see [S1] for a survey of the field) but to date no complete description of the structure of the singular set ([S2]).

In 1978, Brakke studied mean curvature flow in the framework of singular surfaces, so-called integral varifolds ([B]). In the special case where one considers smooth hypersurfaces which develop singularities for the first time, Brakke's main regularity theorem states that under certain additional assumptions (area continuity and unit density hypothesis) the hypersurfaces are still smooth at the singular time except for a lower dimensional set.

It is unclear whether Brakke's additional assumptions are automatically satisfied as a consequence of information about the initial hypersurface. In the special case where the evolving hypersurfaces have positive mean curvature (which holds automatically by the maximum principle if it does for the initial hypersurface) it follows from work of White ([W1], [W2] and [W4]) that the maximum dimension of the singular set is 1 below the dimension of the hypersurface. This result is optimal in view of the behaviour of some special solutions discussed in Chapter 2.

There are also other approaches to mean curvature flow with corresponding regularity theories, notably the level-set flow covered in ([ES1]–[ES3]) and ([CGG], [GG1],[GG2]). The regularity results within these frameworks are comparable to the one in Brakke's setting only to a limited extent (see [I1]). In particular, Brakke's original result, though not generally considered optimal, has not been improved upon to date except in special situations.

This book is structured as follows: Chapter 2 grew out of survey talks on mean curvature flow and is therefore of an expository nature. It introduces the concept of mean curvature flow and illustrates several important examples and special solutions. Homothetic solutions, which play a central role in the regularity theory, are covered in significant detail. We also state several global results which describe long-term existence and asymptotic behaviour in special situations. In the cases of convex initial data ([GH] and [Hu1]) and of embedded curves ([Gr1]), no singularities form before the solution disappears. Entire graph solutions never develop singularities ([EH1], [EH2]).

In Chapter 3, we derive local point-wise estimates on geometric quantities for smooth hypersurfaces moving by mean curvatures. Most of this material appeared in [EH2] and [Ec1], but the presentation of the interior estimates from [EH2] has been streamlined here. We first establish control on the position vector of the moving surfaces. This leads in particular to conditions on an initial hypersurface that guarantee the formation of singularities before the solution disappears. We continue by proving local gradient estimates. Instead of then using standard techniques for uniformly parabolic equations to obtain higher derivative estimates, we establish local bounds for the curvature and its derivatives directly from the evolution equations of these quantities and the weak maximum principle. This is technically easier and has the additional advantage of making the geometric dependence of the constants more explicit, as well as improving them.

In Chapter 4, we study the behaviour of integral quantities, starting with an integrated version of mean curvature flow (see Proposition 4.4), which also serves as the basis for Brakke's weak solution concept. The main result is Huisken's monotonicity formula (Theorem 4.11), first proved in [Hu2], for which we derive a new localised version (Proposition 4.17). Consequences of these include upper and lower bounds on the area ratio inside balls (the latter implying a version of Brakke's clearing out lemma ([B], Theorem 6.3)), as well as local mean value inequalities. Further, we explain how homothetically shrinking solutions arise as limits of rescaled solutions. Although all results are formulated in the smooth case, most of the proofs given here are easily adapted to Brakke solutions. This chapter draws heavily on ideas from [B], [I1], [I2], [W1] and [Ec1], [Ec2].

Chapter 5 contains the actual regularity theory. After introducing a number of concepts from geometric measure theory, we state Brakke's main regularity theorem (Theorem 5.3) and discuss the central hypothesis of his result. In particular, we refer to more recent developments where Brakke's theorem has been improved in special situations ([W1], [W2], [I1], [I2]). For other solution frameworks such as the level-set flow and its relation to Brakke flow we refer to [ES1]–[ES3], [CGG], [GG1], [GG2] and [I1]. The technical part of this chapter begins with two local regularity results (Theorems 5.6 and 5.7). The first version is due to White and uses the monotonicity formula as the essential tool. The second version is a new result. It employs a form of L^2-height deviation from a hyperplane and is closer in spirit to Brakke's local regularity theorem in [B], Theorem 6.11. The proof of Theorem 5.3 given here is quite different from Brakke's original one. In order to streamline the exposition, we have split the argument into a string of smaller components (following Lemma 5.10). Chapters 4 and 5 contain a number of ideas and results which have not appeared elsewhere in this form. However, the bulk of the material has been adapted from work in [B], [W3] and [Ec1], [Ec2].

For the convenience of the reader we have added an appendix: The first three parts (Appendices A–C) list in more detail the definitions and facts for hypersurfaces in Euclidean space used throughout the text, give a derivation of the basic evolution equations for mean curvature flow and provide some background on the quite limited amount of geometric measure theory used throughout the book.

The remaining parts (Appendices D–F) present material which is related to but not essential for the main part of the book: Brakke's main regularity theorem ([B], Theorem 6.12) may be regarded as an analogue of Allard's and de Giorgi's regularity theorem for stationary varifolds with multiplicity 1 almost everywhere ([All], [S1], [DG]). In fact, many ideas from minimal surface theory can be adapted to mean curvature flow. Appendix D provides an account of various fundamental techniques in minimal surface theory such as monotonicity and mean value formulas (all in the setting of smooth hypersurfaces) as well as minimal surface versions of the regularity proofs in Chapter 5. These should be slightly easier to read than the relevant proofs for mean curvature flow but convey the central ideas all the same. We therefore recommend reading this chapter before Chapter 5. For related material we refer to the book by Colding and Minicozzi ([CM]).

Appendix E gives a proof of a stronger version of Brakke's clearing out lemma than the one we use in our proof of his main regularity Theorem 5.3. This proof is closer to Brakke's original one in that it employs some version of the isoperimetric inequality. To simplify the exposition, we have restricted ourselves to the two-dimensional case.

Appendix F gives the derivation of a new local monotonicity formula due to the author ([Ec3]) which is analogous to the monotonicity formula for minimal hypersurfaces. In particular, in this formula, space and time are combined in a geometrically natural fashion, reminiscent of the recent work of Perelman for Ricci flow ([P1]). This formula provides an alternative to the localised version of Huisken's monotonicity formula in Proposition 4.17 on which much of the proof of Brakke's theorem in Chapter 5 is based.

Chapter 2

Special Solutions and Global Behaviour

Definition 2.1 (Mean Curvature Flow) A family of smoothly embedded hypersurfaces $(M_t)_{t \in I}$ in $\mathbb{R}^{n+1}$ *moves by mean curvature* if

$$\frac{\partial x}{\partial t} = \vec{H}(x) \tag{2.1}$$

for $x \in M_t$ and $t \in I$, $I \subset \mathbb{R}$ an open interval. Here $\vec{H}(x)$ is the mean curvature vector at $x \in M_t$.

Remark 2.2 (1) *Family of Embeddings.* Consider the family of smooth embeddings $F_t = F(\cdot, t) : M^n \to \mathbb{R}^{n+1}$ with $M_t = F_t(M^n)$ where M^n is an n-dimensional manifold. Setting $x = F(p, t)$, (2.1) is then interpreted as

$$\frac{\partial F}{\partial t}(p, t) = \vec{H}(F(p, t)) \tag{2.2}$$

for $p \in M^n$ and $t \in I$.

(2) *A Nonlinear Heat Equation.* We use the convention

$$\vec{H} = -H\nu$$

where ν is a choice of unit normal field and H is the mean curvature of M_t (see Appendix A for a definition). In view of the identity

$$\Delta_{M_t} x = \vec{H}$$

involving the Laplace–Beltrami operator on M_t (see (A.1)) (2.1) can also be expressed in the form

$$\frac{\partial x}{\partial t} = \Delta_{M_t} x \tag{2.3}$$

which formally resembles the heat equation.

(3) *Normal Motion and Tangential Diffeomorphisms.* We will often consider smoothly embedded hypersurfaces M_t satisfying

$$\left(\frac{\partial x}{\partial t}\right)^{\perp} = \vec{H}(x)$$

($\perp$ denotes the projection onto the normal space of M_t). This equation is equivalent to (2.1) up to diffeomorphisms tangent to M_t. Indeed, let $\tilde{F}_t = \tilde{F}(\cdot, t) : M^n \to \mathbb{R}^{n+1}$ with $M_t = \tilde{F}_t(M^n)$ be a family of embeddings satisfying the equation

$$\left(\frac{\partial \tilde{F}}{\partial t}(q, t)\right)^{\perp} = \vec{H}(\tilde{F}(q, t))$$

for $q \in M^n$ (here $\perp$ denotes the projection onto the normal space of $\tilde{F}_t(M^n)$). Let $\phi_t = \phi(\cdot, t)$ be a family of diffeomorphisms of M^n satisfying

$$D_q \tilde{F}(\phi(p, t), t)\left(\frac{\partial \phi}{\partial t}(p, t)\right) = -\left(\frac{\partial \tilde{F}}{\partial t}(\phi(p, t), t)\right)^{T}$$

(here the superscript T denotes projection onto the tangent space of $\tilde{F}_t(M^n)$). The local existence of such a family is guaranteed by the assumptions on $\tilde{F}$. If we set

$$F_t(p) = F(p, t) = \tilde{F}(\phi(p, t), t) = \tilde{F}(\phi_t(p), t) \tag{2.4}$$

then $M_t = F_t(M^n) = \tilde{F}_t(M^n)$, and one easily checks that

$$\frac{\partial F}{\partial t}(p, t) = \vec{H}(F(p, t)).$$

Two natural examples will be given below.

(4) *Minimal Hypersurfaces.* Setting $M_t = M$ for all $t \in \mathbb{R}$ where $M \subset \mathbb{R}^{n+1}$ is a minimal ($H \equiv 0$) hypersurface gives a solution of (2.1). These arise as

stationary limits in the case of complete M or in boundary value problems. We shall see later that (2.1) decreases the area element (and thus decreases the area of compact hypersurfaces). It is the steepest descent flow (gradient flow) for the area functional.

Examples 2.3 (1) *Spheres.* Let (M_t) be a family of concentric n-spheres in $\mathbb{R}^{n+1}$, i.e.,

$$M_t = \partial B^{n+1}_{r(t)}.$$

By the invariance of mean curvature under isometries of $\mathbb{R}^{n+1}$, (2.1) reduces to an ordinary differential equation for the radius function $r(t)$ given by

$$\dot{r} = -\frac{n}{r}.$$

If we require $r(0) = \rho$, that is, $M_0 = \partial B_\rho$ then

$$r(t) = \sqrt{\rho^2 - 2nt}$$

so that this solution of (2.1) exists for $t \in (-\infty,\ \rho^2/2n)$.

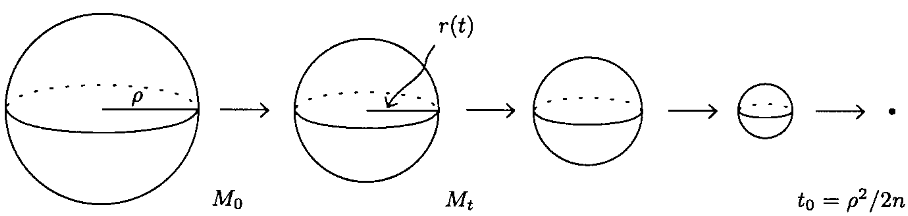

Figure 2.1: Shrinking sphere

(2) *Cylinders.* If M_t is a spherical cylinder, i.e.,

$$M_t = \partial B^{n+1-k}_{r(t)} \times \mathbb{R}^k$$

for $0 \leq k \leq n$ (this includes the previous example when $k = 0$), then

$$\dot{r} = -\frac{(n-k)}{r}$$

so that with $r(0) = \rho$ we obtain

$$r(t) = \sqrt{\rho^2 - 2(n-k)t}$$

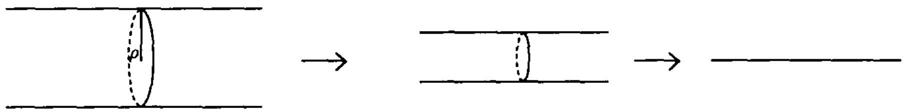

Figure 2.2: Shrinking cylinder

and the solution exists for $t \in (-\infty,\ \rho^2/2(n-k))$.

(3) *Circular Torus.* Let $M_0 \subset \mathbb{R}^3$ be the circular torus defined as the set of points at distance ρ from a unit circle. For $\rho < 1/2$, the mean curvature of M_0 is positive.

Let Ω_t be the region enclosed by M_t. Since $H > 0$ throughout the evolution (in fact, by the maximum principle stated in Chapter 3, the minimum of H on M_t is an increasing function of t) we have $\Omega_t \subset \Omega_s$ for $t > s$.

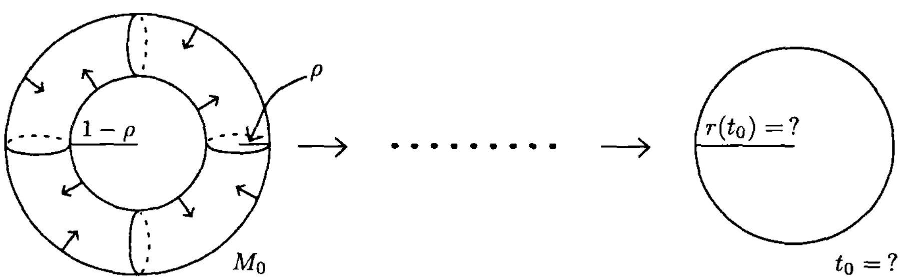

Figure 2.3: Torus contracting to a circle

It is easily checked that the evolving torus will not maintain its circular cross-section but it will remain a surface of revolution. It takes a little more thought to see that it collapses onto a circle in finite time.

(4) *Homothetic Solutions.* The simplest example of a homothetic solution of (2.1) is given by the shrinking spheres discussed in (1). For $\rho = 1$, these satisfy

$$M_t = \sqrt{1-2nt}\, M_0$$

for all $t \in (-\infty,\ 1/2n)$. More generally (following [B], Appendix C), we consider solutions of (2.1) of the form

$$M_t = \lambda(t)\, M_{t_1} \tag{2.5}$$