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Karl-Goswin Grosse-Erdmann

# The Blocking Technique, Weighted Mean Operators and Hardy's Inequality

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*Für Klaus*

# Preface

The aim of these notes is to present a comprehensive treatment of the so-called blocking technique, together with applications to the study of sequence and function spaces, to the study of operators between such spaces, and to classical inequalities.

In these theories, and in other parts of Analysis, expressions of the form

$$\sum_{n=1}^{\infty}\left[a_n\left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p}\right]^q$$

play an important rôle, most prominently perhaps in connection with Hardy's inequality. The analysis of such an expression, which we shall briefly call a *norm in section form*, has turned out to be demanding.

In many cases a problem becomes more accessible under a suitable renorming. Now, throughout the last four decades expressions of the form

$$\sum_{\nu=0}^{\infty}\left[\frac{1}{2^{\nu\alpha}}\left(\sum_{k\in I_\nu}|x_k|^p\right)^{1/p}\right]^q$$

have been appearing quite naturally in various parts of Analysis, very often in connection with coefficient conditions on series expansions of functions. Here, the $I_\nu$ form a partition of $\mathbb{N}$ into disjoint intervals, the most common partition being that into the dyadic blocks $[2^\nu, 2^{\nu+1})$. An expression of the above type is called a *norm in block form*.

It has already been noted by several authors that certain norms in section form can be replaced equivalently by a norm in block form. Such a renorming, which is referred to as the *blocking technique*, is of great practical value, for the analysis of norms in block form is much simpler: in many respects they behave just like the familiar $l^p$-norms.

In these notes we show that, apart from some trivial cases, in fact every norm in section form can be transformed into block form and, what is perhaps even more surprising, every norm in block form can be re-translated into section form. In that sense the blocking technique is universal. Chapter I provides a dictionary of transformations between the two kinds of norms. The related problem of characterising when two given norms are equivalent is of less relevance to the applications in these notes and is treated in the Appendix. In Chapter II we apply the blocking technique to study the structure of sequence spaces defined by norms in section form, while Chapter III contains applications to (generalised) weighted mean operators in $l^p$ and to the weighted inequalities of Hardy and Copson.

It is more a matter of personal taste that we have chosen to concentrate our study on norms for sequences rather than on the corresponding integral norms for functions on the real line. In Chapter IV we indicate the integral analogues of our results.

Our research originated from a study of four papers by G. Bennett that revolve around the inequalities of Hardy and Copson. We have developed the blocking technique as a tool to attack some of his open problems. This has been successful; the solutions to three of his problems are contained in Sections 9, 10 and 17.

On the other hand, the results in Bennett's papers were instrumental in leading us to the appropriate transformations between section norms and block norms. Thus it is in two ways that these notes owe their existence to Grahame Bennett. I would therefore like to take this opportunity to express my sincere gratitude to him and my deep appreciation of the beauty of his work.

Hagen, October 1997 Karl-Goswin Grosse-Erdmann

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# Introduction

In four fundamental papers G. Bennett [12, 13, 14, 15] has undertaken a thorough investigation of the inequalities of Hardy and Copson and their weighted generalisations. Among other things he has completely solved the $l^p$-mapping problem for weighted mean matrices and, in [15], has introduced the new concept of factorisation of inequalities. This concept, for instance, allows Hardy's classical inequality to be seen in a new light, 75 years after its first appearance. Bennett has also formulated various open problems that were raised by his work. A new approach, the so-called blocking technique, has enabled us to solve three of his problems, and it turned out that this technique also serves to obtain a large part of Bennett's results in an elementary and unified way, as far as its *qualitative* aspect is concerned (we shall say more about this point below).

Since our investigation revolves around Bennett's four papers we shall refer to them throughout briefly as **BI, BII, BIII** and **BIV**. By means of Hardy's inequality we shall next illustrate what we mean by the blocking technique and how it comes into play.

## Hardy's inequality

This inequality, in its discrete form, asserts that for any $p > 1$ there is some constant $K > 0$ such that

$$(0.1) \qquad \sum_{n=1}^{N} \left( \frac{1}{n} \sum_{k=1}^{n} x_k \right)^p \leq K \sum_{n=1}^{N} x_n^p$$

holds for every $N \in \mathbb{N}$ and all non-negative numbers $x_1, \ldots, x_N$. Letting $N \to \infty$ we see that this immediately implies the inclusion

$$l^p \subset \operatorname{ces}(p) \tag{0.2}$$

between the space $l^p$ of $p$-summable (real or complex) sequences and the so-called *Cesàro sequence space*

$$\operatorname{ces}(p) = \left\{ x = (x_k) : \sum_n \left( \frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\}.$$

As a matter of fact, the inclusion (0.2) is the form in which Hardy [36] first announced his result. Since $\operatorname{ces}(p)$ is a Banach space under the norm

$$\|x\|_{\operatorname{ces}(p)} = \left( \sum_n \left( \frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p}, \tag{0.3}$$

an application of the closed graph theorem shows that (0.2) implies (0.1) so that the two are in fact equivalent.

The norm of $\operatorname{ces}(p)$, although at first sight a rather straightforward variation of the $l^p$-norm, defies simple analysis. This is seen most clearly in a result of A. A. Jagers [44]. Answering a *prijsvraag* of the Dutch Mathematical Society [101], Jagers determined the dual of $\operatorname{ces}(p)$ under its dual norm; it turned out to be more complicated than one would expect. One of Bennett's numerous surprising results is that in fact

$$\operatorname{ces}(p)^* \cong \left\{ x : \sum_n \sup_{k \geq n} |x_k|^{p^*} < \infty \right\}, \tag{0.4}$$

where $p^*$ is the conjugate exponent to $p$ (**BIV**, 12.17). The price one has to pay for this simple representation is that the norm on $\operatorname{ces}(p)^*$ implied by it is *not* the dual norm. Bennett rests the proof of (0.4) on a renorming of the Cesàro sequence space, the new norm being suggested by a factorisation result.

The main idea of these notes is to renorm $\operatorname{ces}(p)$ in another direction. What makes the analysis under (0.3) difficult is the fact that each term of the sequence $x$ appears in almost every expression $\frac{1}{n}\sum_{k=1}^{n} |x_k|$. It would considerably simplify matters if this norm could be replaced by one in which each $x_k$ only appeared once. This is indeed possible; we shall show that an equivalent norm on $\operatorname{ces}(p)$ is given by

$$\|x\| = \left( \sum_{\nu=0}^{\infty} 2^{\nu(1-p)} \left( \sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k| \right)^p \right)^{1/p}, \tag{0.5}$$

and that ces($p$) consists of all sequences $\boldsymbol{x}$ for which this new (extended) norm is finite, see Theorem 4.1. Since we can write

$$\|\boldsymbol{x}\|_{l^p} = \left(\sum_{\nu=0}^{\infty}\Big(\sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p\Big)^{p/p}\right)^{1/p},$$

a simple application of Hölder's inequality shows that (0.2) holds. We have thus obtained a new proof of Hardy's inequality.

## The spaces $l(p,q)$

We shall say that a norm like (0.5) is in *block form*, while (0.3) is in *section form*. In 1969, Hedlund [41] introduced the mixed norm spaces

$$l(p,q) = \left\{\boldsymbol{x} : \sum_{\nu=0}^{\infty}\Big(\sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p\Big)^{q/p} < \infty\right\},$$

see also Kellogg [51]. In many respects these spaces behave just like the familiar $l^p$-spaces. What we have found is that the Cesàro sequence space ces($p$) is a weighted $l(1,p)$-space. This also helps to locate the place of ces($p$) within the collection of classical and semi-classical Banach sequence spaces (cf. **BIV**, p. 7). We remark that the Besov sequence spaces $b^{\rho}_{p,q}$ introduced by Pietsch [82] in 1980 are weighted $l(p,q)$-spaces with, in particular, $b^0_{p,q} = l(p,q)$.

## The blocking technique: scope and limitations

The *blocking technique* consists in replacing norms in section form by norms in block form and vice versa. In our applications of this technique the rôle of the norms in block form is that of a catalyst. We start off with problems that are formulated in terms of section norms, translate these into block form, solve the new and usually much simpler problems, and re-translate the solution into section form. Thus, for example, questions on spaces like ces($p$) are reduced to questions on $l(p,q)$-spaces. The main difficulty in this programme consists in finding a suitably large number of transformations between section and block form that is flexible enough for differing purposes.

We shall see that our approach not only provides a new proof of Hardy's inequality (and of the related inequality of Copson, compare Section 10), but that its scope is much wider. Among other things it enables us to treat these inequalities in their weighted form. For this additional generality, however, one has to depart from the dyadic blocks $[2^\nu, 2^{\nu+1})$ and has to allow general blocks

$[m_\nu, m_{\nu+1})$. In addition there is an analogue of the blocking technique for integral norms of functions on the real line, so that everything that can be done for series can also be done for integrals.

In particular we shall here prove a conjecture of Bennett (**BIII**, pp. 160-161) and answer two of his open problems (**BII**, p. 393; **BIV**, p. 37). In addition, the blocking technique leads to elementary and unified proofs for a large part of Bennett's investigations in his four papers as far as its qualitative aspect is concerned. This is especially interesting at those points where Bennett uses deep functional analytic techniques, for example in **BIII**.

Another aspect of our work is more "philosophical". A striking feature of Bennett's papers is that many of the problems considered by him have surprisingly simple answers, for example, his solution of the ces($p$)-duality problem stated above; Bennett himself expresses his surprise at various places (**BIV**, pp. 2, 26, 68; see also [87, Introduction]). The present notes offer an explanation for this phenomenon: The spaces we are dealing with are, in reality, $l(p,q)$-spaces in disguise, and these spaces are rather well-behaved.

The blocking technique, however, also has its limitations, and it is important to be clear about this. By renorming the spaces involved we lose control over constants, for instance in inequalities, while the major and deeper part of Bennett's work is devoted to finding best-possible constants. Thus, for example, we can give a new proof of Hardy's inequality in its qualitative form (0.1), but we are not able to confirm Landau's result [55] that $K$ can be taken as $(\frac{p}{p-1})^p$, which is best-possible. In that respect our work is merely qualitative. And each of our results in turn poses a new problem: that of finding the hidden best-possible constants.

## The blocking technique in the literature

Norms in block form have been appearing in the literature for some time, and with it the blocking technique. The phrase "blocking technique" (or rather blocking method) was suggested by L. Leindler in a recent publication [63, Abstract] in a related context. It is also Leindler who has contributed a large number of equivalence results between section norms and block norms over the past decades, see [59, 65] and the literature cited therein.

Closest in spirit to our work is the use of the blocking technique in connection with spaces of strongly Cesàro summable sequences. These investigations, which were started by Taberski [92], Borwein [21] and Kuttner and Maddox [54] in the early 60's, were our main source of inspiration. For a recent survey see [71].

Norms in block form and the blocking technique also seem to come up naturally when the coefficients in series expansions of functions are studied. They play a major rôle, for instance, in connection with absolute summability of orthogonal series (first appearance in Tandori [96], 1960; for a recent contribution see [65]), multipliers between spaces of analytic functions (Hedlund [41], 1969; recently [17] and [3]), spaces of Fourier coefficients of $L^1$-functions (Fomin [32], 1978; recently [25] and [6]) and power series with positive coefficients (Mateljević and Pavlović [74], 1983; recently [64]). A very recent addition is the theory of wavelets (see, for example, [75, 6.10]). Thus in all of these areas our results are of relevance.

The present text is the first to apply the blocking technique systematically in the context of Hardy's inequality. The usefulness of the technique in this area was first observed in **BIV**, p. 81, but we shall see that we have to go beyond dyadic blocks in general. We also offer the first comprehensive treatment of the blocking technique itself. There is closely related work due to Totik and Vincze [97] and Leindler [65]. They characterise when two *given* norms, one in section form and one in block form, are equivalent (see also Section 4 and the Appendix). The problem comes to life again, however, if one is given a norm in one of the two forms and has to find an equivalent one in the other form, the transformation from block form into section form posing the main difficulty. This accounts for the fact that some authors have presented their results in block form and have failed to re-translate them into the more natural section form. These re-translations are not at all obvious. They were suggested to us by Bennett's various results.

There are analogues of the norms in block form for functions on the real line (or, more generally, on $\mathbb{R}^n$). Such norms appear abundantly in Harmonic Analysis where they have led to the notion of amalgams. We refer to [33] for a thorough survey and also to the work of Feichtinger, see, for example, [30, 31]. Instances of the blocking technique for functions can be found in connection with the notion of a Lebesgue point (Tandori [94]), in the context of strong Cesàro summability of functions (Borwein [21]) and in the theory of Beurling algebras and more general function spaces (Gilbert [34], Johnson [47]), among others.

## Contents

These notes are divided into four chapters and an appendix.

Chapter I develops the blocking technique as indicated above. In Section 2 we obtain transformations from block form into section form while in Section 3 we go the opposite direction. This forms the basis of all that follows.

In Chapter II we introduce two classes of sequence spaces, $c(\boldsymbol{a}, p, q)$ and $d(\boldsymbol{a}, p, q)$. These spaces contain as special cases the space $\operatorname{ces}(p)$ and many other spaces. We apply the results of Chapter I to study their basic structure (Section

6), to characterise the multipliers between these spaces and $l^p$ (Section 7) and to obtain some factorisation results (Section 8).

Chapter III presents our main applications. In Section 9 we deal with Hardy's inequality with weights or, in a different language, with factorable matrices as operators on $l^p$; these cover the weighted mean matrices as most important special case. In particular we prove the conjecture of Bennett mentioned above. Section 10 studies Copson's inequality with weights, where we also answer another problem of Bennett. In Sections 11 and 12 we treat the reverses of some classical inequalities. We end the chapter with a selection of further applications (Section 13).

In Chapter IV we indicate integral analogues of our results. In particular we complete an investigation started by Beesack and Heinig [9], thus answering a third question of Bennett (**BIV**, p. 37).

We end these notes with an Appendix in which we apply the results of Chapter I to study the equivalence of norms in section form with norms in block form à la Totik-Vincze and Leindler.

## Notation

We agree that Roman indices $n, k, \ldots$ start from 1 while Greek indices $\nu, \mu, \ldots$ start from 0, if nothing else is said.

For any sequence $\boldsymbol{x} = (x_n)$ we denote by $P_n\boldsymbol{x} = (x_1, \ldots, x_n, 0, 0, \ldots)$ its $n^{\text{th}}$ section.

The space $\omega$ is the space of all (real or complex) sequences, its subspace $\varphi$ consists of all finite sequences. If $E$ is a sequence space, any non-negative sequence $\boldsymbol{w}$ defines a weighted space $E_{\boldsymbol{w}} = \{\boldsymbol{x} : \boldsymbol{x} \cdot \boldsymbol{w} \in E\}$, where the product of two sequences is taken coordinatewise.

For $0 < p \leq \infty$ we define its *conjugate* $p^*$ by $\frac{1}{p} + \frac{1}{p^*} = 1$, with the usual convention if $p = 1$ or $p = \infty$. We remark that $p^* < 0$ if $p < 1$.

As usual the constant $K$ appearing in inequalities may vary from occurrence to occurrence.

Further notation will be introduced at the beginnings of Sections 1, 5, 7 and 8. In addition, we shall adopt the following convention: When a condition contains a sum $\sum_{k=n}^{\infty} c_k$ over non-negative numbers, then the condition is understood to imply that $\sum_k c_k < \infty$. The same applies to a supremum. Further conventions will be introduced in Remark 2.2(i).

# Chapter I

# The Blocking Technique

## 1 Norms in Section Form and Norms in Block Form

As we have seen, the norm

$$\|\boldsymbol{x}\|_{\mathrm{ces}(p)} = \left(\sum_n \left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{1/p}$$

is at the heart of Hardy's inequality. Generalising this we assume that $\boldsymbol{a} = (a_n)$ is any sequence of non-negative terms and that $0 < p, q \leq \infty$. Then, for any sequence $\boldsymbol{x} = (x_n)$, we consider

$$\|\boldsymbol{x}\| = \left(\sum_n \left[a_n \left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p}\right]^q\right)^{1/q}$$

and its companion

$$\|\boldsymbol{x}\| = \left(\sum_n \left[a_n \left(\sum_{k=n}^{\infty}|x_k|^p\right)^{1/p}\right]^q\right)^{1/q},$$

with the usual modifications if $p$ or $q$ is infinite. By abuse of language we shall refer to these extended quasi-seminorms briefly as norms, and we say that these norms are in *section form*. At times we shall allow additional weights, that is, we replace $x_k$ by $w_k x_k$.

Our aim is to transform these norms into block form. It turns out that it does not suffice to consider dyadic blocks only. Thus let $\boldsymbol{m} = (m_\nu)_{\nu\geq 0}$ be any *index sequence*, that is, any sequence of integers with $m_0 = 1$, $m_{\nu+1} \geq m_\nu$ for all $\nu$ and $m_\nu \to \infty$ as $\nu \to \infty$. The *blocks* associated with $\boldsymbol{m}$ are defined as

$$I_\nu = [m_\nu, m_{\nu+1}) = \{n \in \mathbb{N} : m_\nu \leq n < m_{\nu+1}\}.$$

We allow the $I_\nu$ to be empty. The commonest blocks to be found in the literature are the *dyadic blocks* defined by $m_\nu = 2^\nu$. Now let $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then, for any sequence $\boldsymbol{x} = (x_n)$, we consider

$$\|\boldsymbol{x}\| = \left(\sum_\nu \left[\frac{1}{2^{\nu\alpha}}\left(\sum_{k \in I_\nu} |x_k|^p\right)^{1/p}\right]^q\right)^{1/q},$$

again with modifications if $p$ or $q$ is infinite. We call this a norm in *block form*.

We start by transforming norms from block form into section form because this turns out to be the more difficult direction and it immediately implies the opposite direction. We postpone a discussion of these results to Section 4.

## 2 Transformation from Block Form into Section Form

We first define a correlation between index sequences $\boldsymbol{m} = (m_\nu)$ and positive monotonic sequences $\boldsymbol{s} = (s_n)$ with $s_n \to 0$. We say that $\boldsymbol{m}$ and $\boldsymbol{s}$ are *correlated* if

$$\text{(2.1)} \qquad \begin{aligned} \frac{1}{2^\nu} \geq s_n &> \frac{1}{2^{\nu+1}} && \text{if } m_\nu \leq n < m_{\nu+1} \quad (\nu \geq 1), \\ s_n &> \frac{1}{2} && \text{if } n < m_1. \end{aligned}$$

Given any such sequence $\boldsymbol{s}$ we see that there is a unique index sequence $\boldsymbol{m}$ correlated to it; it is defined by $m_0 = 1$ and, for $\nu \geq 1$,

$$\text{(2.2)} \qquad m_\nu = \min\left\{n : s_n \leq \frac{1}{2^\nu}\right\}.$$

Conversely, to any index sequence $\boldsymbol{m}$ there are infinitely many sequences $\boldsymbol{s}$ correlated to it, for instance the one defined by

$$s_n = \frac{1}{2^\nu} \quad \text{for} \quad m_\nu \leq n < m_{\nu+1}, \nu \geq 0.$$

Throughout this section, let $\boldsymbol{m} = (m_\nu)$ be a fixed index sequence and $\boldsymbol{s} = (s_n)$ a fixed positive monotonic sequence converging to 0 that is correlated to $\boldsymbol{m}$.

**Theorem 2.1** *Let $0 < p, q < \infty$ and $\alpha \in \mathbb{R}$. Then, for any sequence $x = (x_n)$, the condition*

$$\sum_{\nu} \left[ \frac{1}{2^{\nu\alpha}} \left( \sum_{k \in I_\nu} |x_k|^p \right)^{1/p} \right]^q < \infty \tag{2.3}$$

*is equivalent to any of the following conditions, where $\beta \neq 0, \gamma \neq 0$ and $\delta$ are real numbers with $\gamma/q + \delta/p = \alpha$:*

$$\sum_n (s_n^\beta - s_{n+1}^\beta) s_n^{\gamma-\beta} \left( \sum_{k=1}^{n} s_k^\delta |x_k|^p \right)^{q/p} < \infty \quad \text{if } \beta > 0, \gamma > 0, \tag{2.4i}$$

$$\sum_n (s_{n+1}^\beta - s_n^\beta) s_n^\gamma s_{n+1}^{-\beta} \left( \sum_{k=1}^{n} s_k^\delta |x_k|^p \right)^{q/p} < \infty \quad \text{if } \beta < 0, \gamma > 0, \tag{2.4ii}$$

$$\sum_n (s_n^\beta - s_{n-1}^\beta) s_n^{\gamma-\beta} \left( \sum_{k=n}^{\infty} s_k^\delta |x_k|^p \right)^{q/p} < \infty \quad \text{if } \beta < 0, \gamma < 0, \tag{2.4iii}$$

$$\sum_n (s_{n-1}^\beta - s_n^\beta) s_n^\gamma s_{n-1}^{-\beta} \left( \sum_{k=n}^{\infty} s_k^\delta |x_k|^p \right)^{q/p} < \infty \quad \text{if } \beta > 0, \gamma < 0. \tag{2.4iv}$$

*In addition, for real numbers $\gamma \neq 0$ and $\delta$ with $\gamma/q+\delta/p = \alpha$,* (2.3) *is equivalent to*

$$\sum_n s_n^{\gamma+\delta} |x_n|^p \left( \sum_{k=1}^{n} s_k^\delta |x_k|^p \right)^{q/p-1} < \infty \quad \text{if } \gamma > 0, \tag{2.4v}$$

$$\sum_n s_n^{\gamma+\delta} |x_n|^p \left( \sum_{k=n}^{\infty} s_k^\delta |x_k|^p \right)^{q/p-1} < \infty \quad \text{if } \gamma < 0. \tag{2.4vi}$$

**Remark 2.2** (i) In conditions (2.4iii) and (2.4iv) the coefficient for $n = 1$ is undefined. Obviously, its value has no influence on the validity of the theorem. However, it turns out that the most natural choice is to take its value as $s_1^\gamma$, that is, to choose $s_0 = \infty$, if one likes. In later sections, $s_n$ will be substituted by certain expressions in $a_k$. As a consequence, we shall there interpret $\sum_{k=0}^{\infty} a_k^q$ as $\infty$ and, as usual, $\sum_{k=1}^{0} a_k^q$ as 0, similarly for suprema. In conditions (2.4v) and (2.4vi), $0 \cdot 0^r$ has to be interpreted as 0 even if $r < 0$. *These interpretations are in effect throughout these notes.*

(ii) If one is given an abstract sequence $(m_\nu)$ (or $(s_n)$) and if $\alpha \neq 0$, then the simplest(-looking) equivalent condition for (2.3) is obtained by taking $\beta = \gamma = \alpha q$ and $\delta = 0$:

$$\text{(2.5i)} \qquad \sum_n (s_n^{\alpha q} - s_{n+1}^{\alpha q}) \left( \sum_{k=1}^{n} |x_k|^p \right)^{q/p} < \infty \qquad \text{if} \quad \alpha > 0,$$

$$\text{(2.5ii)} \qquad \sum_n (s_n^{\alpha q} - s_{n-1}^{\alpha q}) \left( \sum_{k=n}^{\infty} |x_k|^p \right)^{q/p} < \infty \qquad \text{if} \quad \alpha < 0.$$

However, as we shall see in Chapter II, in many applications one is confronted with a specific sequence $(m_\nu)$ (or $(s_n)$), in which case it will be useful to have available the additional parameters $\beta, \gamma$ and $\delta$.

*Proof of Theorem 2.1.* Replacing $|x_k|^p$ by $|x_k|$, $q/p$ by $q$ and $\alpha p$ by $\alpha$ shows that we need only consider the case $p = 1$.

(2.3)⇒(2.4i). First let $q > 1$. Then, by (2.1), we have $1/2^{\nu+1} < s_n \leq 1/2^\nu$ for $n \in I_\nu$ and $\nu \geq 1$, hence

$$\sum_{n \in I_\nu} (s_n^\beta - s_{n+1}^\beta) s_n^{\gamma - \beta} \left( \sum_{k=1}^{n} s_k^\delta |x_k| \right)^q$$

$$\leq K \sum_{n \in I_\nu} (s_n^\beta - s_{n+1}^\beta) \frac{1}{2^{\nu(\gamma-\beta)}} \left( \sum_{k=1}^{m_{\nu+1}-1} s_k^\delta |x_k| \right)^q$$

$$\leq K (s_{m_\nu}^\beta - s_{m_{\nu+1}}^\beta) \frac{1}{2^{\nu(\gamma-\beta)}} \left( \sum_{\mu=0}^{\nu} \sum_{k \in I_\mu} \frac{1}{2^{\mu\delta}} |x_k| \right)^q,$$

now using $\delta = \alpha - \gamma/q$,

$$\leq K \frac{1}{2^{\nu\gamma}} \left( \sum_{\mu=0}^{\nu} \frac{2^{\mu\gamma/q}}{2^{\nu\gamma/q}} \frac{2^{\nu\gamma/q}}{2^{\mu\alpha}} \sum_{k \in I_\mu} |x_k| \right)^q;$$

since $\gamma > 0$, we have $\sum_{\mu=0}^{\nu} 2^{\mu\gamma/q} \sim 2^{\nu\gamma/q}$, so that we can continue by applying Jensen's inequality:

$$\leq K \frac{1}{2^{\nu\gamma}} \sum_{\mu=0}^{\nu} \frac{2^{\mu\gamma/q}}{2^{\nu\gamma/q}} \left( \frac{2^{\nu\gamma/q}}{2^{\mu\alpha}} \sum_{k \in I_\mu} |x_k| \right)^q.$$