Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1117

D. J. Aldous
I. A. Ibragimov
J. Jacod

École d'Été de Probabilités de Saint-Flour XIII – 1983

Édité par P.L. Hennequin

Springer-Verlag
Berlin Heidelberg New York Tokyo

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1117

D. J. Aldous
I. A. Ibragimov
J. Jacod

École d'Été de Probabilités de Saint-Flour XIII – 1983

Édité par P. L. Hennequin

Springer-Verlag
Berlin Heidelberg New York Tokyo

Auteurs

David J. Aldous
University of California
Department of Statistics
Berkeley, CA 94720, USA

Illdar A. Ibragimov
Math. Institute Ac.Sci.
Fontanka 27
191011 Leningrad, USSR

Jean Jacod
Laboratoire de Probabilités
Tour 56 (3ème étage)
4, Place Jussieu
75230 Paris Cedex 05, France

Editeur

P.L. Hennequin
Université de Clermont II, Complexe Scientifique des Cézeaux
Département de Mathématiques Appliquées
B.P. 45, 63170 Aubière, France

AMS Subject Classifications (1980): 60-02, 60F05, 60G05, 60G09, 60G46, 60G50, 62-02, 62A05, 60D05

ISBN 3-540-15203-2 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-15203-2 Springer-Verlag New York Heidelberg Berlin Tokyo

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

Lecture Notes in Mathematics

For information about Vols. 1–925, please contact your bookseller or Springer-Verlag.

Vol. 926: Geometric Techniques in Gauge Theories. Proceedings, 1981. Edited by R. Martini and E.M.de Jager. IX, 219 pages. 1982.

Vol. 927: Y. Z. Flicker, The Trace Formula and Base Change for GL (3). XII, 204 pages. 1982.

Vol. 928: Probability Measures on Groups. Proceedings 1981. Edited by H. Heyer. X, 477 pages. 1982.

Vol. 929: Ecole d'Eté de Probabilités de Saint-Flour X – 1980. Proceedings, 1980. Edited by P.L. Hennequin. X, 313 pages. 1982.

Vol. 930: P. Berthelot, L. Breen, et W. Messing, Théorie de Dieudonné Cristalline II. XI, 261 pages. 1982.

Vol. 931: D.M. Arnold, Finite Rank Torsion Free Abelian Groups and Rings. VII, 191 pages. 1982.

Vol. 932: Analytic Theory of Continued Fractions. Proceedings, 1981. Edited by W.B. Jones, W.J. Thron, and H. Waadeland. VI, 240 pages. 1982.

Vol. 933: Lie Algebras and Related Topics. Proceedings, 1981. Edited by D. Winter. VI, 236 pages. 1982.

Vol. 934: M. Sakai, Quadrature Domains. IV, 133 pages. 1982.

Vol. 935: R. Sot, Simple Morphisms in Algebraic Geometry. IV, 146 pages. 1982.

Vol. 936: S.M. Khaleelulla, Counterexamples in Topological Vector Spaces. XXI, 179 pages. 1982.

Vol. 937: E. Combet, Intégrales Exponentielles. VIII, 114 pages. 1982.

Vol. 938: Number Theory. Proceedings, 1981. Edited by K. Alladi. IX 177 pages. 1982.

Vol. 939: Martingale Theory in Harmonic Analysis and Banach Spaces. Proceedings, 1981. Edited by J.-A. Chao and W.A. Woyczyński. VIII, 225 pages. 1982.

Vol. 940: S. Shelah, Proper Forcing. XXIX, 496 pages. 1982.

Vol. 941: A. Legrand, Homotopie des Espaces de Sections. VII, 132 pages. 1982.

Vol. 942: Theory and Applications of Singular Perturbations. Proceedings, 1981. Edited by W. Eckhaus and E.M. de Jager. V, 363 pages. 1982.

Vol. 943: V. Ancona, G. Tomassini, Modifications Analytiques. IV, 120 pages. 1982.

Vol. 944: Representations of Algebras. Workshop Proceedings, 1980. Edited by M. Auslander and E. Lluis. V, 258 pages. 1982.

Vol. 945: Measure Theory. Oberwolfach 1981, Proceedings. Edited by D. Kölzow and D. Maharam-Stone. XV, 431 pages. 1982.

Vol. 946: N. Spaltenstein, Classes Unipotentes et Sous-groupes de Borel. IX, 259 pages. 1982.

Vol. 947: Algebraic Threefolds. Proceedings, 1981. Edited by A. Conte. VII, 315 pages. 1982.

Vol. 948: Functional Analysis. Proceedings, 1981. Edited by D. Butković, H. Kraljević, and S. Kurepa. X, 239 pages. 1982.

Vol. 949: Harmonic Maps. Proceedings, 1980. Edited by R.J. Knill, M. Kalka and H.C.J. Sealey. V, 158 pages. 1982.

Vol. 950: Complex Analysis. Proceedings, 1980. Edited by J. Eells. IV, 428 pages. 1982.

Vol. 951: Advances in Non-Commutative Ring Theory. Proceedings, 1981. Edited by P.J. Fleury. V, 142 pages. 1982.

Vol. 952: Combinatorial Mathematics IX. Proceedings, 1981. Edited by E. Billington, S. Oates-Williams, and A.P. Street. XI, 443 pages. 1982.

Vol. 953: Iterative Solution of Nonlinear Systems of Equations. Proceedings, 1982. Edited by R. Ansorge, Th. Meis, and W. Törnig. VII, 202 pages. 1982.

Vol. 954: S.G. Pandit, S.G. Deo, Differential Systems Involving Impulses. VII, 102 pages. 1982.

Vol. 955: G. Gierz, Bundles of Topological Vector Spaces and Their Duality. IV, 296 pages. 1982.

Vol. 956: Group Actions and Vector Fields. Proceedings, 1981. Edited by J.B. Carrell. V, 144 pages. 1982.

Vol. 957: Differential Equations. Proceedings, 1981. Edited by D.G. de Figueiredo. VIII, 301 pages. 1982.

Vol. 958: F.R. Beyl, J. Tappe, Group Extensions, Representations, and the Schur Multiplicator. IV, 278 pages. 1982.

Vol. 959: Géométrie Algébrique Réelle et Formes Quadratiques, Proceedings, 1981. Edité par J.-L. Colliot-Thélène, M. Coste, L. Mahé, et M.-F. Roy. X, 458 pages. 1982.

Vol. 960: Multigrid Methods. Proceedings, 1981. Edited by W. Hackbusch and U. Trottenberg. VII, 652 pages. 1982.

Vol. 961: Algebraic Geometry. Proceedings, 1981. Edited by J.M. Aroca, R. Buchweitz, M. Giusti, and M. Merle. X, 500 pages. 1982.

Vol. 962: Category Theory. Proceedings, 1981. Edited by K.H. Kamps D. Pumplün, and W. Tholen, XV, 322 pages. 1982.

Vol. 963: R. Nottrot, Optimal Processes on Manifolds. VI, 124 pages. 1982.

Vol. 964: Ordinary and Partial Differential Equations. Proceedings, 1982. Edited by W.N. Everitt and B.D. Sleeman. XVIII, 726 pages. 1982.

Vol. 965: Topics in Numerical Analysis. Proceedings, 1981. Edited by P.R. Turner. IX, 202 pages. 1982.

Vol. 966: Algebraic K-Theory. Proceedings, 1980, Part I. Edited by R.K. Dennis. VIII, 407 pages. 1982.

Vol. 967: Algebraic K-Theory. Proceedings, 1980. Part II. VIII, 409 pages. 1982.

Vol. 968: Numerical Integration of Differential Equations and Large Linear Systems. Proceedings, 1980. Edited by J. Hinze. VI, 412 pages. 1982.

Vol. 969: Combinatorial Theory. Proceedings, 1982. Edited by D. Jungnickel and K. Vedder. V, 326 pages. 1982.

Vol. 970: Twistor Geometry and Non-Linear Systems. Proceedings, 1980. Edited by H.-D. Doebner and T.D. Palev. V, 216 pages. 1982.

Vol. 971: Kleinian Groups and Related Topics. Proceedings, 1981. Edited by D.M. Gallo and R.M. Porter. V, 117 pages. 1983.

Vol. 972: Nonlinear Filtering and Stochastic Control. Proceedings, 1981. Edited by S.K. Mitter and A. Moro. VIII, 297 pages. 1983.

Vol. 973: Matrix Pencils. Proceedings, 1982. Edited by B. Kågström and A. Ruhe. XI, 293 pages. 1983.

Vol. 974: A. Draux, Polynômes Orthogonaux Formels – Applications. VI, 625 pages. 1983.

Vol. 975: Radical Banach Algebras and Automatic Continuity. Proceedings, 1981. Edited by J.M. Bachar, W.G. Bade, P.C. Curtis Jr., H.G. Dales and M.P. Thomas. VIII, 470 pages. 1983.

Vol. 976: X. Fernique, P.W. Millar, D.W. Stroock, M. Weber, Ecole d'Eté de Probabilités de Saint-Flour XI – 1981. Edited by P.L. Hennequin. XI, 465 pages. 1983.

Vol. 977: T. Parthasarathy, On Global Univalence Theorems. VIII, 106 pages. 1983.

Vol. 978: J. Ławrynowicz, J. Krzyż, Quasiconformal Mappings in the Plane. VI, 177 pages. 1983.

Vol. 979: Mathematical Theories of Optimization. Proceedings, 1981. Edited by J.P. Cecconi and T. Zolezzi. V, 268 pages. 1983.

continued on page 413

INTRODUCTION

La Treizième Ecole d'Eté de Calcul des Probabilités de Saint-Flour s'est tenue du 3 au 20 Juillet 1983 et a rassemblé, outre les conférenciers, une quarantaine de participants dans les locaux acceuillants du Foyer des Planchettes.

Les trois conférenciers, Messieurs ALDOUS, IBRAGIMOV et JACOD ont entièrement repris la rédaction de leurs cours qui constitue maintenant un texte de référence et ceci justifie le nombre d'années mis pour les publier.

En outre plusieurs exposés ont été faits par les participants durant leur séjour à Saint-Flour :

A. BADRIKIAN	"Approximation de la mécanique quantique"
J. DESHAYES	"Ruptures de modèles pour des processus de Poisson"
A. EHRHARD	"L'inégalité isopérimétrique de Borell et l'opérateur $-\Delta + x\nabla$ "
L. GALLARDO	"Une formule de Lévy-Khintchine sur les hypergroupes (au sens de Jewett ou de Spector) commutatifs dénombrables"
M. LEDOUX	"Théorèmes limite central dans les espaces $\ell_p(B)$ $(1\leq p<\infty)$"
D. LOTI VIAUD	"Modélisation dans l'asymptotique des grandes déviations de processus de branchements spatiaux multitypes non homogènes saturés par le vecteur des proportions des types"
A. MILLET	"Processus à deux indices : problèmes de convergence et d'arrêt optimal"
P. MORALES	"La propriété de Bochner dans les espaces vectoriels topologiques localement convexes"
J.L. PALACIOS	"The exchangeable sigma-field of a markov chain"
J. PICARD	"Un problème de filtrage avec un petit terme non linéaire"
P. SPREIJ	"Filtering and parameter estimation for software retrability models"

A. TOUATI "Grandes déviations pour les mesures aléatoires et applications à la statistique des processus"

M. WEBER "Mesures de Hausdorff et points multiples du mouvement brownien fractionnaire dans $\mathbb{R}^n$ "

M. WSCHEBOR "Ensembles de niveau des surfaces aléatoires"

La frappe du manuscrit a été assurée par les départements de l'Université de Californie, de Clermont-Ferrand et par l'auteur et nous remercions pour leur soin et leur efficacité les secrétaires qui se sont chargées de ce travail délicat.

Nous exprimons enfin notre gratitude à la Société Springer-Verlag qui permet d'accroître l'audience internationale de notre Ecole en accueillant une nouvelle fois ces textes dans la collection Lecture Notes in Mathematics.

P.L. HENNEQUIN
Professeur à l'Université de Clermont II
B.P. n° 45
F-63170 AUBIERE

LISTE DES AUDITEURS

Mr. AZEMA J.	Université de Paris V
Mr. BADRIKIAN A.	Université de Clermont-Ferrand II
Mr. BRETAGNOLLE J.	Université de Paris XIII
Mle CHEVET S.	Université de Clermont-Ferrand II
Mr. COMETS F.	Université de Paris XI
Mr. DESHAYES J.	Ecole Nationale Supérieure des Télécommunications à Paris
Mr. DOUKHAN P.	Université de Paris XI
Mr. EHRHARD A.	Université de Clermont-Ferrand II
Mr. FERNIQUE X.	Université de Strasbourg I
Mr. FOURT G.	Université de Clermont-Ferrand II
Mr. GALLARDO L.	Université de Nancy I
Mr. HENNEQUIN P.L.	Université de Clermont-Ferrand II
Mr. HEBUTERNE G.	C.N.E.T. à Lannion
Mme HUBER C.	Université de Paris XIII
Mle KADI Y.	Université de Paris XI
Mr. KREE P.	Université de Paris VI
Mr. LEDOUX	Université de Strasbourg I
Mr. LEONARD C.	Université de Paris XI
Mr. LETAC G.	Université de Toulouse III
Mle MILLET A.	Université d'Angers
Mr. MORALES P.	Université de Sherbrooke (Canada)
Mr. NGUYEN Van HO	Université de Paris XI
Mr. PALACIOS J.	Université Simon Bolivar à Caracas (Vénézuela)
Mr. PARDOUX E.	Université d'Aix-Marseille I
Mle PICARD D.	Université de Paris XI
Mr. PICARD J.	I.N.R.I.A. à Valbonne
Mr. PORTAL F.	Université de Paris XI
Mr. ROUX D.	Université de Clermont-Ferrand II
Mr. SANTIBANEZ J.	Université de Mexico (Mexique)
Mr. SCHOTT R.	Université de Nancy I
Mr. SPREIJ P.	Mathematisch Centrum à Amsterdam (Pays-Bas)
Mr. SUBIA CEPEDA N.	Escuela Politècnica Nacional à Quito (Equateur)

Mr. TALAY D.	I.N.R.I.A. à Valbonne
Mr. TOUATI A.	Ecole Normale Supérieure de Bizerte (Tunisie)
Mr. TU LY HOANG	Université de Paris XI
Mr. WEBER M.	Université de Strasbourg I
Mr. WONG H.	Université d'Ottawa (Canada)
Mr. WSCHEBOR M.	Université de Paris XI

TABLE DES MATIERES

D.J. ALDOUS : "EXCHANGEABILITY AND RELATED TOPICS"

I.A. IBRAGIMOV : "THEOREMES LIMITES POUR LES MARCHES ALEATOIRES"

J. JACOD : "THEOREMES LIMITE POUR LES PROCESSUS"

EXCHANGEABILITY AND RELATED TOPICS

PAR David J. ALDOUS

0. Introduction

If you had asked a probabilist in 1970 what was known about exchangeability, you would likely have received the answer "There's de Finetti's theorem: what else is there to say?" The purpose of these notes is to dispel this (still prevalent) attitude by presenting, in Parts II-IV, a variety of mostly post-1970 results relating to exchangeability. The selection of topics is biased toward my own interests, and away from those areas for which survey articles already exist. Any student who has taken a standard first year graduate course in measure-theoretic probability theory (e.g. Breiman (1968)) should be able to follow most of this article; some sections require knowledge of weak convergence.

In Bayesian language, de Finetti's theorem says that the general infinite exchangeable sequence (Z_i) is obtained by first picking a distribution θ at random from some prior, and then taking (Z_i) to be i.i.d. with distribution θ. Rephrasing in the language of probability theory, the theorem says that with (Z_i) we can associate a random distribution $\alpha(\omega,\cdot)$ such that, conditional on $\alpha = \theta$, the variables (Z_i) are i.i.d. with distribution θ. This formulation is the central fact in the circle of ideas surrounding de Finetti's theorem, which occupies most of Part I. No previous knowledge of exchangeability is assumed, though the reader who finds my proofs overly concise should take time out to read the more carefully detailed account in Chow and Teicher (1978), Section 7.3.

Part II contains results complementary to de Finetti's theorem. Dacunha-Castelle's "spreading-invariance" property and Kallenberg's stopping time property give conditions on an infinite sequence which turn out to be equivalent to exchangeability. Kingman's "paintbox" description of exchangeable random partitions leads to Cauchy's formula for the distribution of cycle lengths in a uniform random permutation, and to results about components of random functions. Continuous-time processes with interchangeable increments are discussed; a notable result is that any continuous-path process on $[0,\infty)$ (resp. $[0,1]$) with interchangeable increments is a mixture of processes which are linear transformations of Brownian motion (resp. Brownian bridge). The subsequence principle reveals exchangeable-like sequences lurking unsuspectedly within arbitrary sequences of random variables. And we discuss exchangeability in abstract spaces, and weak convergence issues.

The class of exchangeable sequences is the class of processes whose distributions are invariant under a certain group of transformations; in Part III related invariance concepts are described. After giving the abstract result on ergodic decompositions of measures invariant under a group of transformations, we specialize to the setting of partial exchangeability, where we study the class of processes $(X_i : i \in I)$ invariant under the action of some group of transformations of the index set I. Whether anything can be proved about partially exchangeable classes in general is a challenging open problem; we can only discuss three particular instances. The most-studied instance, investigated by Hoover and by myself, is partial exchangeability for arrays of random variables, where the picture is fairly complete. We also discuss partial exchangeability on trees of infinite degree, where the basic examples are reversible Markov chains; and on infinite-dimensional cubes, where it appears that the basic examples are random walks, though here the picture remains fragmentary.

Part IV outlines other topics of current research. A now-classical result on convergence of partial sum processes from sampling without replacement to Brownian bridge leads to general questions of convergence for triangular arrays of finite exchangeable sequences, where the present picture is unsatisfactory for applications. Kingman's uses of exchangeability in mathematical genetics will be sketched. The theory of sufficient statistics and mixtures of processes of a specified form will also be sketched--actually, this topic is perhaps the most widely studied relative of exchangeability, but in view of the existing accounts in Lauritzen (1982) and Diaconis and Freedman (1982), I have not emphasized it in these notes. Kallenberg's stopping time approach to continuous-time exchangeability is illustrated by the study of exchangeable subsets of $[0,\infty)$. A final section provides references to work related to exchangeability not elsewhere discussed: I apologize in advance to those colleagues whose favorite theorems I have overlooked.

General references. Chow and Teicher (1978) is the only textbook (known to me) to give more than a cursory mention to exchangeability. A short but elegant survey of exchangeability, whose influence can be seen in these notes, has been given by Kingman (1978a). In 1981 a conference on "Exchangeability in Probability and Statistics" was held in Rome to honor Professor Bruno de Finetti: the conference proceedings (EPS in the References) form a sample of the current interests of workers in exchangeability. Dynkin (1978) gives a concise abstract treatment of the "sufficient statistics" approach in several areas of probability including exchangeability.

The material in Sections 13 and 16 is new, and perhaps a couple of proofs elsewhere may be new; otherwise no novelty is claimed.

Notation and terminology. The mathematical notation is intended to be standard, so the reader should seldom find it necessary to consult the list of notation at the end. As for terminology, "exchangeable" is more popular and shorter than the synonyms "symmetrically dependent" and "interchangeable". I have introduced "directing random measure" in place of Kallenberg's "canonical random measure", partly as a more vivid metaphor and partly for more grammatical flexibility, so one can say "directed by ...". I use "partial exchangeability" in the narrow sense of Section 12 (processes with certain types of invariance) rather than in the wider context of Section 18 (processes with specified sufficient statistics). "Problem" means "unsolved problem" rather than "exercise": if you can solve one, please let me know.

Acknowledgements. My thanks to Persi Diaconis for innumerable invaluable discussions over the last several years; and to the members of the audiences at St. Flour and the Berkeley preview who detected errors and contributed to the presentation. Research supported by National Science Foundation Grant MCS80-02698.

PART I

The purpose of Part I is to give an account of de Finetti's theorem and some straightforward consequences, using the language and techniques of modern probability theory. I have not attempted to assign attributions to these results: historical accounts of de Finetti's work on exchangeability and the subsequent development of the subject can be found in EPS (Foreword, and Furst's article) and in Hewitt and Savage (1955).

1. Definitions and immediate consequences

A finite sequence $(Z_1,\dots,Z_N)$ of random variables is called exchangeable (or N-exchangeable, to indicate the number of random variables) if

(1.1) $$(Z_1,\dots,Z_N) \stackrel{\mathcal{D}}{=} (Z_{\pi(1)},\dots,Z_{\pi(N)}) \ ;$$

each permutation π of $\{1,\dots,N\}$. An infinite sequence $(Z_1,Z_2,\dots)$ is called exchangeable if

(1.2) $$(Z_1,Z_2,\dots) \stackrel{\mathcal{D}}{=} (Z_{\pi(1)},Z_{\pi(2)},\dots)$$

for each finite permutation π of $\{1,2,\dots\}$, that is each permutation for which $\#\{i: \pi(i)\neq i\} < \infty$. Throughout Part I we shall regard random variables Z_i as real-valued; but we shall see in Section 7 that most results remain true whenever the Z_i have any "non-pathological" range space.

There are several obvious reformulations of these definitions. Any finite permutation can be obtained by composing permutations which transpose 1 and $n > 1$; so (1.2) is equivalent to the at first sight weaker condition