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Graph Theory, Combinatorics, & Applications

Vol. 1

Graph Theory, Combinatorics, and Applications

Volume 1

PROCEEDINGS OF THE SIXTH QUADRENNIAL INTERNATIONAL CONFERENCE ON THE THEORY AND APPLICATIONS OF GRAPHS

Western Michigan University

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The Sixth Conference and these Proceedings are dedicated to
Lowell W. Beineke
and
Carsten Thomassen
and the editors herewith recognize and laud their outstanding contributions to Graph Theory and its promotion around the world.

Preface

These volumes constitute the Proceedings of the Sixth Quadrennial International Conference on the Theory and Applications of Graphs, held at Western Michigan University in Kalamazoo, Michigan, 30 May–3 June 1988. Conference participants included research mathematicians from colleges, universities, and industry, as well as graduate and undergraduate students. Altogether 25 states and 18 countries were represented. The contributions to these volumes include many topics in current research in both the theory and applications in the areas of graph theory and combinatorics.

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Y.A.
G.C.
O.O.
A.S.

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Second Order Degree Regular Graphs

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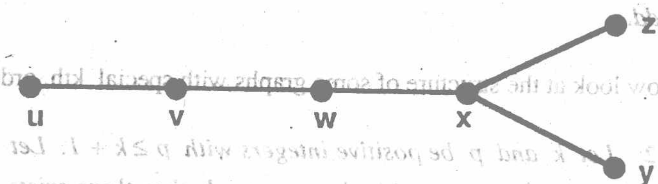
Let G be a connected graph of order p. Let $d(u, v)$ denote the distance between the vertices u and v of G. For any integer k with $1 \le k \le p - 1$, we use the notation

$$N_k(v) = \{u \mid u \in V(G) \text{ and } d(u, v) = k\}$$

and

$$N_k[v] = N_k(v) \cup \{v\}.$$

Example



$$N_2(u) = \{w\}, \; N_3(v) = \{y, z\} = N_4(u), \; N_2(z) = \{w, y\}$$

The *kth order degree of a vertex* v, denoted by $\deg_k v$, is the cardinality of the set $N_k(v)$. In the above example, $\deg_4 u = 2$, $\deg_5 u = 0$, and $\deg_3 z = 1$.

It follows that for any connected graph G that

$$\sum_{k=1}^{p-1} \deg_k v = p - 1,$$

and

$$\sum_{k=1}^{i} \deg_k v \leq p - 1,$$

where $1 < i \leq p - 1$. We can now make the following observation. If G is a connected graph of order $p \geq 2$ and k is an integer with $1 \leq k \leq p - 1$, then $\deg_k v \leq p - k$ for every $v \in V(G)$.

It is well–known that for any graph G, the sum of the degrees of the vertices of G is even. We now show an analogous result for the kth order degree of a graph. We use the following notation throughout the remainder of this paper.

$$D_k = D_k(G) = \sum_{v \in V(G)} \deg_k v.$$

Example (a) From the example above, $D_2(G) = 10$, $D_3(G) = 6$, $D_4(G) = 4$, and $D_5(G) = 0$.

(b) $D_k(K_p) = 0$ for $k > 1$.

(c) $D_2(K(1, p - 1)) = (p - 1)(p - 2)$ and $D_k(K(1, p - 1)) = 0$ for $k > 2$.

Proposition 1 *Let k and p be integers with $1 \leq k \leq p - 1$. If G is a graph of order p, then $D_k(G)$ is even.*

Proof Let k and p be integers with $1 \leq k \leq p - 1$ and let G be a graph of order p. For each pair of vertices u and v with $d(u, v) = k$, let P_{uv} be a path of length k joining u and v. Then in determining $D_k(G)$, P_{uv} is used twice, once for $\deg_k u$ and once for $\deg_k v$, and adds two the sum. Thus $D_k(G)$ is even. ☐

Corollary 1 *In any graph there is an even number of vertices whose kth order degrees are odd.*

We can now look at the structure of some graphs with special kth order degrees.

Proposition 2 *Let k and p be positive integers with $p \geq k + 1$. Let G be a graph of order p. If there is a vertex v with $\deg_k v = p - k$, then there exists $k - 1$ vertices $u_1, u_2, \ldots, u_{k-1}$ of G with $\deg_k u_i = 0$ for $1 \leq i \leq k - 1$ and the subgraph S of G induced by the vertex set $V(G) - \{v, u_1, u_2, \ldots, u_{k-2}\}$ contains a star isomorphic to $K(1, p - k)$ with the center of S at u_{k-1}.*

Proof Let $N_k(v) = \{x_1, x_2, \ldots, x_{p-k}\}$ and let $v, u_1, u_2, \ldots, u_{k-1}, x_1$ be a path of length k in G. Since $d(v, x_1) = k$, there is no shorter $v - x_1$ path in G. Then $V(G) = \{v, u_1, u_2, \ldots, u_{k-1}, x_1, x_2, \ldots, x_{p-k}\}$. Since $d(v, u_i) = i$, for $1 \leq i \leq k - 1$, and $d(v, x_j) = k$, for $1 \leq j \leq p - k$, the graph induced by $\{v, u_1, u_2, \ldots, u_{k-1}\}$ is a path of length $k - 1$ and the graph induced by $\{u_{k-1}, x_1, x_2, \ldots, x_{p-k}\}$ contains a subgraph S isomorphic to the star $K(1, p-k)$ and S has center u_{k-1}. Thus $\deg_k(u_i) = 0$ for $1 \le i \le k+1$. □

Remark Under the hypothesis of Proposition 2, G is isomorphic to a subgraph of the graph H of order p where H is composed of a complete graph of order $p-k+1$ with a path of length $k-1$ attached to one of the vertices.

Corollary 2a *Let G be a connected graph of order p. Then $D_2(G) \le (p-1)(p-2)$ and this inequality is the best possible.*

Proof Let q denote the size of G. Since, for every vertex v of G, $\deg_2 v = |N_2(v)| \le (p-1) - \deg v$, $D_2(G) \le p(p-1) - 2q \le p(p-1) - 2(p-1) = (p-1)(p-2)$. This is the best possible since $D_2(K(1, p-1)) = (p-1)(p-2)$. □

Corollary 2b *Let G be a connected graph order $p \ge 3$. Then $D_2(G) = (p-1)(p-2)$ if and only if G is isomorphic to the star $K(1, p-1)$.*

Proof It is clear that $D_2(K(1, p-1)) = (p-1)(p-2)$. So we assume that G is a connected graph of order $p \ge 3$ with $D_2(G) = (p-1)(p-2)$. For every vertex v of G, $\deg v + \deg_2 v \le p-1$, and so $D_1(G) + D_2(G) \le p(p-1)$. Thus $D_1 \le p(p-1) - D_2(G) = p(p-1) - (p-1)(p-2) = 2(p-1)$, but $2q = D_1(G)$, where q denotes the size of G. We have that $q = p-1$, and so G is a tree. If $\deg_2 v \le p-3$ for every vertex v of G, then $D_2(G) \le p(p-3) = p^2 - 3p < p^2 - 3p + 2 = (p-1)(p-2)$. So at least one vertex of G, say v, has $\deg_2 v = p-2$. But then there is a vertex u with $\deg_2 u = 0$. Thus the other $p-2$ vertices also must have second order degrees $p-2$. Hence G is isomorphic to $K(1, p-1)$. □

The graph G is said to be *kth order regular of degree* d if for every vertex v of G, $\deg_k v = d$. Then, being first order regular of degree d is equivalent to being regular of degree d. Before giving some examples of kth order regular graphs of degree d, we provide some notation and a definition.

(a) Let $K_{n(m)}$ denote the complete n–partite graph with each partite set containing exactly m vertices.

(b) Let G_1 and G_2 be two non–empty graphs. The *cartesian product* $G = G_1 \times G_2$ of G_1 and G_2 has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either

$$u_1 = v_1 \text{ and } u_2v_2 \in E(G_2)$$

or

$$u_2 = v_2 \text{ and } u_1v_1 \in E(G_1).$$

For $n \ge 3$, the graphs $C_n \times K_2$ are called *double cycles*.

(c) Let G_1 and G_2 be two non-empty graphs. The *lexicographic product* $G = G_1[G_2]$ of G_1 and G_2 has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1v_1 \in E(G_1)$ or $u_1 = v_1$ and $u_2v_2 \in E(G_2)$. The *generalized cycle* $C(m, n)$, for integers $m \geq 3$ and $n \geq 1$, is the lexicographic product $C_m[\overline{K}_n]$.

Example

(a) The complete graphs K_n are second order regular of degree 0. In fact these are the only connected graphs that are second order regular of degree 0.

(b) The path P_4 of length 3 and the four cycle C_4 are second order regular of degree 1. The graphs $K_{n(2)}$ are second order regular of degree 1. More generally, the path P_{2k} of length $2k - 1$ and the $2k$ cycle C_{2k} are kth order regular of degree 1.

(c) The cycles C_n, $n \geq 5$, and the double star $S(2, 2)$ are second order regular of degree 2. The double cycle $C_3 \times K_2$ is second order regular of degree 2.

(d) The double cycle $C_4 \times K_2$ is second order regular of degree 3.

(e) The double cycles $C_n \times K_2$, $n \geq 5$, are second order regular of degree 4.

(f) Let m be any positive integer. Then $K_{n(m+1)}$ is second order regular of degree m and the double star $S(m, m)$ is second order regular of degree m.

(g) For $n \geq 2$, the complete bipartite graph $K(n + 1, n + 1)$ with a one factor removed is third order regular of degree one and regular of degree n.

(h) For the n-cube Q_n we have for any k, $1 \leq k \leq n$, that Q_n is kth order regular of degree $\binom{n}{k}$, the combinatorial coefficient.

Proposition 2 provides the following.

Corollary 2c *Let k and p be integers with $p \geq k + 1 \geq 3$. There exist no connected graphs G of order p that are kth order regular of degree $p - k$.*

Proposition 3 *Let k and d be positive integers. There exists graph $G(k, d)$ which is kth order regular of degree d.*

Proof 1. If $k = 1$, then $G(1, d)$ can be taken to be any regular graph of degree d.

2. If $k = 2$ and n is an integer at least 2, then $K_{n(d+1)}$ is second order regular of degree d.

3. Assume $k \geq 3$. The generalized cycle $C(2k, d)$ is a kth order regular graph of degree d. ❑

We now look at the special case of second order regular graphs.

Proposition 4 *Let G be a connected graph of order $p \geq 4$. Then G is second order regular of degree $p-3$ if and only if G is isomorphic to P_4, C_4, or C_5.*

Proof If G is isomorphic to P_4, C_4, or C_5, then G is second order regular of degree $p-3$. so we assume that G is second order regular of degree $p-3$. Then $\deg v + \deg_2 v \leq p-1$ and so $\deg v \leq (p-1) - \deg_2 v = (p-1) - (p-3) = 2$. Thus, since G is connected, $1 \leq \deg v \leq 2$. Hence G is a path or G is a cycle. If G is a path, then G is isomorphic to P_4 and if G is a cycle, then G is isomorphic to C_4 or C_5. ❑

Theorem 1 *The connected graph G is second order regular of degree 1 if and only if G is either a path of length 3 or G is $K_{n(2)}$.*

Proof Suppose that G is second order regular of degree 1 and that G is not a path of length 3. For each vertex v of G there is exactly one vertex v' of G whose distance from v is 2. Since the vertices of G can be paired this way, the order of G must be even, say $|V(G)| = 2n$. Suppose there is a vertex w of G with $d(v, w) = 3$. The path joining v and w must be of the form $vw'v'w$, where w' is the unique vertex of G whose distance from w is 2. If $V(G) = \{v, v', w, w'\}$, then, necessarily, G is a path of length 3, contradicting our choice of G. Hence, since the order of G is even, G has order at least six. Since G is connected, there is a vertex u in $G - \{v, v', w, w'\}$ which is adjacent to one of the vertices v, v', w, w'. If u is adjacent to v (or w), then u must also be adjacent to w', for otherwise, both u and w have distance 2 from w'. Now u must be adjacent to v', for otherwise, both u and v have distance 2 from v'. Also, u must be adjacent to w, for otherwise, both u and w' have distance 2 from w. Thus u is adjacent to each of the vertices v, v', w, w', and $d(v, w) \leq 2$. This contradicts the assumption that $d(v, w) = 3$. On the other hand, if u is adjacent to w' (or v'), then u must be adjacent to v, for otherwise, both u and v' has distance 2 from v. Thus u and v are adjacent and we have the previous case.

The above consideration shows that the diameter of G is 2. Let v be any vertex of G and let z be any vertex of G other than v and v'. Since the distance between v and z is at most 2 and cannot be 2, since $d(v, v') = 2$, v and z must be adjacent. That is, v is adjacent to every vertex of G except v'. Since v was an arbitrary vertex of G, G must be the complete graph K_{2n} with a one factor removed and so G is $K_{n(2)}$. ❑

This result indicates that if G is second order regular of degree one and G is not the path of length three, then G is $K_{n(2)}$. In all of the examples above, except for the paths, the graphs which are kth order regular are also regular. One might be led to the conclusion that if a graph is kth order regular and not a path, then it is regular. Figure 2 provides an example of a graph which is third order regular of degree one which is not regular. The graph in Figure 2 is an example of a third order regular graph of degree one which is not K(n, n) with a one factor removed.

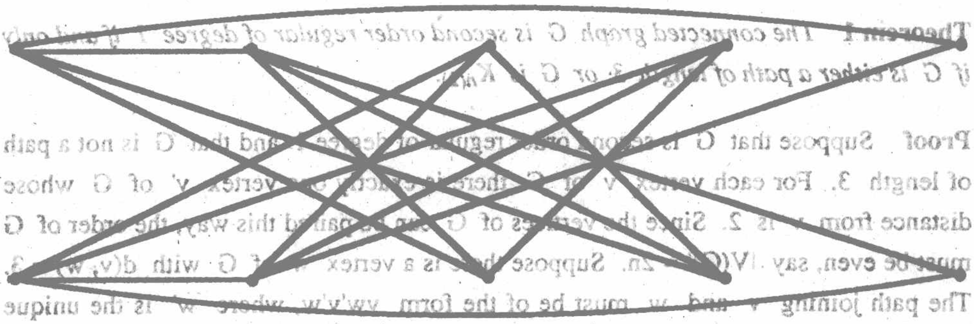


Figure 2

Theorem 2 *If G is a connected third order regular graph of degree 1, then G is either a path of length 5 or G has diameter 3.*

Proof. Suppose that G is a connected third order regular graph of degree 1, and that G is not a path of length 5. Necessarily, the diameter of G is at least 3. First we show that the diameter is at most 4. If this is not the case, then there are vertices v and w of G with $d(v, w) = 5$; let $P: v, x, w', v', x', w$ be a shortest v–w path in G. Since G is connected and third order regular of degree 1, it follows that every vertex in $V(G) - V(P)$ is at distance at most 2 from some vertex in $V(P)$, i.e., $d(u, V(P)) \leq 2$, for all vertices $u \in V(G) - V(P)$.

If there is a vertex u belonging to $V(G) - V(P)$ with $d(u, V(P)) = 2$, then let u, u_1, u_2 be a shortest path from u to a vertex in $V(P)$. Thus u is not adjacent to any vertex of $V(P)$, since otherwise there exists neighboring vertices y_1 and y_2 in $V(P)$ such that $d(u, y_1) = 2$, $d(u, y_2) > 2$, and hence $\deg_3 y_2 > 1$, contradicting the hypothesis. This implies, however, that $d(v, w) \leq d(v, u) + d(u, w) \leq 4$, contrary to our assumption. Hence every vertex in $V(G) - V(P)$ is adjacent to some vertex in $V(P)$.

Assume that $u \in V(G) - V(P)$ and $uw \in E(G)$. Since $\deg_3 v = 1$, it follows that $d(u, v) \leq 2$. Hence $d(u, w) \geq 3$. However, $\deg_3 w = 1$ and so $d(u, w) \geq 4$. Hence

$d(u, w) = 4$ (w, x', v', w', u is a u–w path of length 4) and there is no u–x' path in G of length at most 2. This implies that both x and u have distance 3 from x', contradicting the hypothesis that $\deg_3 x' = 1$. Hence there can be no vertex of $V(G) - V(P)$ adjacent to w' and so $\deg w' = 2$ in G. In a similar manner, it follows that $\deg v' = 2$ in G.

Suppose that $u \in V(G) - V(P)$ and $ux \in E(G)$. Since $\deg_3 v' = 1$, we have that $d(u, v') \le 2$. Since $\deg w' = 2 = \deg v'$, we must have that $ux' \in E(G)$. (The only u-v' path of length 2 is u, x', v'.) Then w, x', u, x, v is a v–w path of length 4 in G, which produces a contradiction. Hence $\deg x = 2$ and in a similar manner it follows that $\deg x' = 2$.

If $u \in v(G) - V(P)$ and $uv \in E(G)$, since $\deg_3 w' = 1$, $d(u, w') \le 2$. This produces a contradiction since $\deg x = \deg w' = \deg v' = 2$. Hence $\deg v = 1$, and similarly, $\deg w = 1$. Thus G is a path of length 5 which is contrary to the hypothesis. We conclude that G has diameter at most 4.

We next show that the diameter of G is at most 3. If the diameter of G is 4, then there are vertices v and w of G such that $d(v, w) = 4$; let $P : v, w', x, v', w$ be a shortest v–w path in G. Let x' be the unique vertex of G whose distance from x is 3 and let $Q : x, x_1, x_2, x'$ be a shortest x–x' path in G. If $x_2 = v$, then, since the diameter of G is 4, it follows that x', v, w', x, v', w is not a shortest x'–w path of G. Let $Q' : x', z_1, z_2, \ldots, z_r = w$ be a shortest x'–w path in G. Since $d(v, w) = 4$, $r \ge 3$. However, since $\deg_3 x' = 1$, $z_3 = x$ and $r = 5$ (since $d(x, w) = 2$), contrary to the fact that r is at most the diamter of G. Hence $x_2 \ne v$, and in a similar manner, $x_2 \ne w$.

If $x_1 = w'$, since the diameter of G is 4, it follows that x', x_2, w', x, v', w is not a shortest x'–w path in G. Let $Q' : x', z_1, z_2, \ldots, z_r = w$ be a shortest x'–w path in G. If $r \ge 3$, then, since $\deg_3 x' = 1$, $z_3 = x$ and so $r = 5$, producing a contradiction. Thus $d(x', w) \le 2$. However, since $\deg_3 x' = 1$, and x', x_2, w', v is an x'–v path of length 3, it follows that $d(x', v) = 2$. Let x', u, v be a shortest x'–v path. Since $d(v, w) = 4$, any u–w path must have length at least 3. In view of the fact that w' is the unique vertex of distance 3 from w, $d(u, w) \ge 4$. However, $d(u, w) \le d(u, x') + d(x', w) \le 3$, producing a contradiction. Hence, $x_1 \ne w'$; and by a similar argument, $x_1 \ne v'$.

We deduce that $V(P) \cap V(Q) = \{x\}$. This implies that some vertex of $V(Q) - \{x\}$ is at a distance 3 from v or w', producing a contradiction. Hence, if G is not a path of length 5, its diameter is 3. ❑

The above results lead us to the following conjecture.