Dietrich Stauffer
H. Eugene Stanley

From Newton to Mandelbrot

A Primer in Theoretical Physics

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With 50 Figures and 16 Colored Plates

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Preface

This is not a book for theoretical physicists. Rather it is addressed to professionals from other disciplines, as well as to physics students who may wish to have in one slim volume a concise survey of the four traditional branches of theoretical physics. We have added a fifth chapter, which emphasizes the possible connections between basic physics and geometry. Thus we start with classical mechanics, where Isaac Newton was the dominating force, and end with fractal concepts, pioneered by Benoit Mandelbrot. Just as reading a review article should not replace the study of original research publications, so also perusing the present short volume should not replace systematic study of more comprehensive texts for those wishing a firmer grounding in theoretical physics.

The opening paragraphs of Chapter 5 benefitted from input by B. Jorgensen. We wish to thank G. Daccord for providing us with Plates 7 and 8, F. Family for Plates 1 and 15, A.D. Fowler for Plate 3, R. Lenormand for Plate 11, P. Meakin for Plate 14 as well as the cover illustration, J. Nittmann for Plate 13, U. Oxaal for Plate 10, A. Skjeltorp for Plates 4, 9 and 16, K.R. Sreenivasan for Plate 5, R.H.R. Stanley for Plate 2, and P. Trunfio for Plates 6 and 12. We also thank A. Armstrong, A. Coniglio, J. Hajdu, F.W. Hehl, K.W. Kehr, J. Kertesz, A. Margolina, R. Selinger, P. Trunfio, and D.E. Wolf as well as many students – particularly L. Jaeger – who offered their feedback at appropriate occasions and A. Armstrong for translating Chapters 1–4 from the original German edition published by Springer.

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Contents

1. Mechanics

Theoretical physics is the first science to be expressed mathematically: the results of experiments should be predicted or interpreted by mathematical formulae. Mathematical logic, theoretical chemistry and theoretical biology arrived much later. Physics had been understood mathematically in Greece more than 2000 years earlier, for example the law of buoyancy announced by Archimedes – lacking *The New York Times* – with *Eureka*! Theoretical Physics first really came into flower, however, with Kepler's laws and their explanation by Newton's laws of gravitation and motion. We also shall start from that point.

1.1 Point Mechanics

1.1.1 Basic Concepts of Mechanics and Kinematics

A point mass is a mass whose spatial dimension is negligibly small in comparison with the distances involved in the problem under consideration. Kepler's laws, for example, describe the earth as a point mass "circling" the sun. We know, of course, that the earth is not really a point, and geographers cannot treat it in their field of work as a point. Theoretical physicists, however, find this notion very convenient for describing approximately the motion of the planets: theoretical physics is the science of successful approximations. Biologists often have difficulties in accepting similarly drastic approximations in their field.

The motion of a point mass is described by a position vector $\boldsymbol{r}$ as a function of time t, where $\boldsymbol{r}$ consists of the three components (x, y, z) of a rectangular coordinate system. (A boldface variable represents a vector. The same variable not in boldface represents the absolute magnitude of the vector, thus for example $r = |\boldsymbol{r}|$.) Its velocity $\boldsymbol{v}$ is the time derivative

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = (\dot{x}, \dot{y}, \dot{z}) \quad , \tag{1.1}$$

where a dot over a variable indicates the derivative with respect to time t. The acceleration $\boldsymbol{a}$ is

$$\boldsymbol{a}(t) = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2} = (\dot{v}_x, \dot{v}_y, \dot{v}_z) \quad , \tag{1.2}$$

the second derivative of the position vector with respect to time.

Galileo Galilei (1564–1642) discovered, reputedly by experimentally dropping objectives from the Leaning Tower of Pisa, that all objects fall to the ground equally "fast", with the constant acceleration

$$a = g \quad \text{and} \quad g = 9.81\mathrm{m/s}^2 \quad . \tag{1.3}$$

Nowadays this law can be conveniently "demonstrated" in the university lecture room by allowing a piece of chalk and a scrap of paper to drop simultaneously: both reach the floor at the same time ... don't they?

It will be observed that theoretical physics is often concerned with asymptotic limiting cases: equation (1.3) is valid only in the limiting case of vanishing friction, never fully achieved experimentally, just as good chemistry can be carried out only with "chemically pure" materials. Nature is so complex that natural scientists prefer to observe unnatural limiting cases, which are easier to understand. A realistic description of Nature must strive to combine the laws so obtained, in such a way that they describe the reality, and not the limiting cases.

The differential equation (1.3), $d^2\boldsymbol{r}/dt^2 = (0, 0, -g)$ has for its solution the well known parabolic trajectory

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + (0, 0, -g)t^2/2 \quad ,$$

where the z axis is taken as usual to be the upward vertical. Here $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ are the position and the velocity initially (at $t = 0$). It is more complicated to explain the motion of the planets around the sun; in 1609 and 1619 Johann Kepler accounted for the observations known at that time with the three Kepler laws:

(1) Each planet moves on an ellipse with the sun at a focal point.
(2) The radius vector $\boldsymbol{r}$ (from the sun to the planet) sweeps out equal areas in equal times.
(3) The ratio (orbital period)^2/(major semi-axis)^3 has the same value for all planets in our solar system.

Ellipses are finite conic sections and hence differ from hyperbolae; the limiting case between ellipses and hyperbolae is the parabola. In polar coordinates (distance r, angle ϕ) we have

$$r = \frac{p}{1 + \varepsilon \cos \phi} \quad ,$$

where $\varepsilon < 1$ is the eccentricity of the ellipse and the planetary orbit. (Circle $\varepsilon = 0$; parabola $\varepsilon = 1$; hyperbola $\varepsilon > 1$; see Fig. 1.1.) Hyperbolic orbits are exhibited by comets; however, Halley's Comet is not a comet *in this sense*, but a very eccentric planet.

It is remarkable, especially for modern science politicians, that from these laws of Kepler for the motion of remote planets, theoretical physics and Newton's law of motion resulted. Modern mechanics was derived, not from practical,

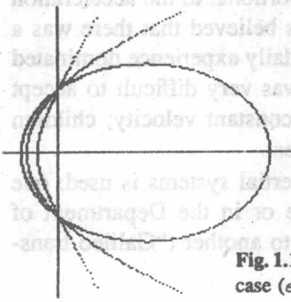

Fig. 1.1. Examples of an ellipse, an hyperbola, and a parabola as limiting case ($\epsilon = 1/2$, 2 and 1, respectively)

"down to earth" research, but from a desire to understand the motion of the planets in order to produce better horoscopes. Kepler also occupied himself with snowflakes (see Chap. 5), a still controversial theme of research in computer physics in 1987. That many of his contemporaries ignored Kepler's work, and that he did not always get his salary, places many of us today on a par with him, at least in this respect.

1.1.2 Newton's Law of Motion

Regardless of fundamental debates on how one defines "force" and "mass", we designate a reference system as an inertial system if a force-free body moves in a straight line with a steady velocity. We write the law of motion discovered by Isaac Newton (1642–1727) thus:

$$\boldsymbol{f} = m\boldsymbol{a}$$

$$\text{force} = \text{mass} \times \text{acceleration} \qquad (1.4)$$

For free fall we state Galileo's law (1.3) as

$$\text{weight} = mg \quad . \qquad (1.5)$$

Forces are added as vectors ("parallelogram of forces"), for two bodies we have action = – reaction, and masses are added arithmetically. So long as we do not need to take account of Einstein's theory of relativity, masses are independent of velocity.

The *momentum* $\boldsymbol{p}$ is defined by $\boldsymbol{p} = m\boldsymbol{v}$, so that (1.4) may also be written as:

$$\boldsymbol{f} = \frac{d\boldsymbol{p}}{dt} \quad , \qquad (1.6)$$

which remains valid even with relativity. The law action = – reaction then states that:

The sum of the momenta of two mutually interacting point masses remains constant. (1.7)

It is crucial to these formulae that the force is proportional to the acceleration and not to the velocity. For thousands of years it was believed that there was a connection with the velocity, as is suggested by one's daily experience dominated by friction. For seventeenth century philosophers it was very difficult to accept that force-free bodies would continue to move with constant velocity; children of the space age have long been familiar with this idea.

It is not stipulated which of the many possible inertial systems is used: one can specify the origin of coordinates in one's office or in the Department of Education. Transformations from one inertial system to another ("Galileo transformations") are written mathematically as:

$$\boldsymbol{r}' = \mathcal{R}\boldsymbol{r} + \boldsymbol{v}_0 t + \boldsymbol{r}_0 \quad ; \quad t' = t + t_0 \tag{1.8}$$

with arbitrary parameters $\boldsymbol{v}_0$, $\boldsymbol{r}_0$, t_0 (Fig. 1.2). Here $\mathcal{R}$ is a rotational matrix with three "degrees of freedom" (three angles of rotation); there are three degrees of freedom also in each of $\boldsymbol{v}_0$ and $\boldsymbol{r}_0$, and the tenth degree of freedom is t_0. Corresponding to these ten continuous variables in the general Galileo transformation we shall later find ten laws of conservation.

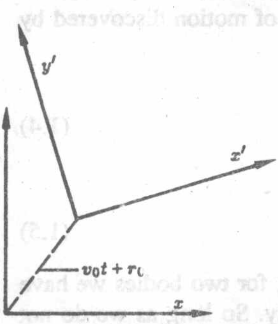

Fig. 1.2. Example of a transformation (1.8) in two-dimensional space

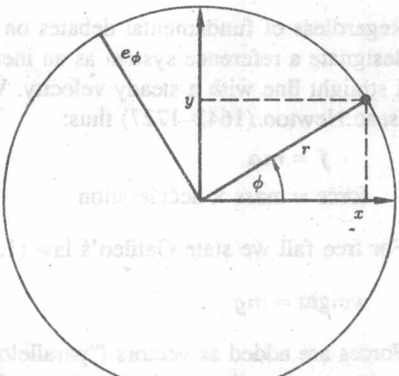

Fig. 1.3. Polar coordinates (r, ϕ) on a flat disk rotating with angular velocity ω, viewed from above

There are interesting effects if the system of reference is not an inertial system. For example we can consider a flat disk rotating (relative to the fixed stars) with an angular velocity $\omega = \omega(t)$ (Fig. 1.3). The radial forces then occurring are well known from rides on a carousel. Let the unit vector in the $\boldsymbol{r}$ direction be $\boldsymbol{e}_r = \boldsymbol{r}/|\boldsymbol{r}|$, and the unit vector perpendicular to it in the direction of rotation be $\boldsymbol{e}_\phi$, where ϕ is the angle with the x-axis: $x = r\cos\phi, y = r\sin\phi$. The time derivative of $\boldsymbol{e}_r$ is $\omega\boldsymbol{e}_\phi$, that of $\boldsymbol{e}_\phi$ is $-\omega\boldsymbol{e}_r$, with the *angular velocity* $\omega = d\phi/dt$. The velocity is

$$v = \frac{d(re_r)}{dt} = e_r \frac{dr}{dt} + r\omega e_\phi$$

according to the rule for the differentiation of a product. Similarly for the acceleration a and the force f we have

$$\frac{f}{m} = a = \dot{v} = \left(\frac{d^2r}{dt^2} - \omega^2 r \right) e_r + (2\dot{r}\omega + r\dot{\omega}) e_\phi \quad . \tag{1.9}$$

Of the four terms on the right hand side the third is especially interesting. The first is "normal", the second is "centrifugal", the last occurs only if the angular velocity varies. In the case when, as at the north pole on the rotating earth, the angular velocity is constant, the last term disappears. The penultimate term in (1.9) refers to the Coriolis force and implies that in the northern hemisphere of the earth swiftly moving objects are deflected to the right, as observed with various phenomena on the rotating earth: Foucault's pendulum (1851), the precipitous right bank of the Volga, the direction of spin of European depressions, Caribbean hurricanes and Pacific typhoons. For example, in an area of low pressure in the North Atlantic the air flows inwards; if the origin of our polar coordinates is taken at the centre of the depression (and for the sake of simplicity this is taken at the north pole), dr/dt is then negative, ω is constant, and the "deflection" of the wind observed from the rotating earth is always towards the right; at the south pole it is reversed. (If the observer is not at the north pole, ω has to be multiplied by $\sin\psi$, where ψ is the latitude: at the equator there is no Coriolis force.)

1.1.3 Simple Applications of Newton's Law

a) Energy Law. Since $f = ma$ we have:

$$f\frac{dr}{dt} = m\frac{dr}{dt}\frac{d^2r}{dt^2} = \frac{d(mv^2/2)}{dt} = \frac{dT}{dt}$$

where $T = mv^2/2$ is the *kinetic energy*. Accordingly the difference between the kinetic energy at position 1 (or time 1) and that at position 2 is given by:

$$T(t_2) - T(t_1) = \int_1^2 fv\,dt = \int_1^2 f\,dr \quad ,$$

which corresponds to the mechanical work done on the point mass ("work = force times displacement"). (The product of two vectors such as f and v is here the scalar product, viz. $f_x v_x + f_y v_y + f_z v_z$. The multiplication point is omitted. The cross product of two vectors such as $f \times v$ comes later.) The power dT/dt ("power = work/time") is therefore equal to the product of force f and velocity v, as one appreciates above all on the motorway, but also in the study.

A three-dimensional force field $f(r)$ is called *conservative* if the above integral over $f\,dr$ between two fixed endpoints 1 and 2 is independent of the path followed from 1 to 2. The gravity force $f = mg$, for example, is conservative:

$$\int f\,dr = -mgh \quad ,$$

where the height h is independent of the path followed. Defining the *potential energy*

$$U(r) = -\int f\,dr$$

we then have:

The force f is conservative if and only if a potential U exists such that $$f = -\mathrm{grad}U = -\nabla U \quad . \qquad (1.10)$$

Here we usually have conservative forces to deal with and often neglect frictional forces, which are not conservative. If a point mass now moves from 1 to 2 in a conservative field of force, we have:

$$T_2 - T_1 = \int_1^2 f\,dr = -(U_2 - U_1) \quad ,$$

so that $T_1 + U_1 = T_2 + U_2$, i.e. $T + U = \mathrm{const}$:

The *energy* $T + U$ is constant in a conservative field of force. (1.11)

Whoever can find an exception to this law of energy so central to our daily life can produce perpetual motion. We shall later introduce other forms of energy besides T and U, so that frictional losses ("heat") etc. can also be introduced into the energy law, allowing non-conservative forces also to be considered. Equation (1.11) shows mathematically that one can already predict important properties of the motion without having to calculate explicitly the entire course of the motion ("motion integrals").

b) One-dimensional Motion and the Pendulum. In one dimension all forces (depending on x only and thus ignoring friction) are automatically conservative, since there is only a unique path from one point to another point in a straight line. Accordingly $E = U(x) + mv^2/2$ is always constant, with $dU/dx = -f$ and arbitrary force $f(x)$. (Mathematicians should know that physicists pretend that all reasonable functions are always differentiable and integrable, and only now consider that known mathematical monsters such as "fractals" (see Chap. 5) also have physical meaning.) One can also see this directly:

$$\frac{dE}{dt} = \frac{dU}{dx}\frac{dx}{dt} + mv\frac{dv}{dt} = -fv + mva = 0 \quad .$$

Moreover we have $dt/dx = 1/v = [(E-U)2/m]^{-1/2}$, and hence

$$t = t(x) = \int \frac{dx}{\sqrt{(E-U(x))2/m}} \quad . \tag{1.12}$$

Accordingly, to within an integration constant, the time is determined as a function of position x by a relatively simple integral. Many pocket calculators can already carry out integrations automatically at the push of a button. For harmonic oscillators, such as the small amplitude pendulum, or the weight oscillating up and down on a spring, $U(x)$ is proportional to x^2, and this leads to sine and cosine oscillations for $x(t)$, provided that the reader knows the integral of $(1-x^2)^{-1/2}$. In general, if the energy E results in a motion in a potential trough of the curve $U(x)$, there is a periodic motion (Fig. 1.4), which however need not always be sin (ωt). In the anharmonic pendulum, for example, the restoring force is proportional to sin (x) (here x is the angle), and the integral (1.12) leads to elliptic functions, which I do not propose to pursue any further.

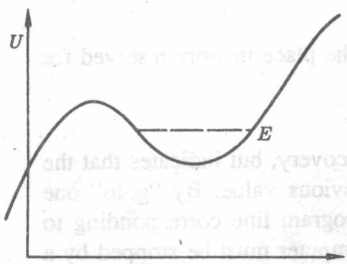

Fig. 1.4. Periodic motion between the points a and b, when the energy E lies in the trough of the potential $U(x)$

Notwithstanding the exact solution by (1.12), it is also useful to consider a computer program, with which one can solve $f = ma$ directly. Quite basically (I leave better methods to the numerical mathematicians) one divides up the time into individual time steps Δt. If I know the position x at that time I can calculate the force f and hence the acceleration $a = f/m$. The velocity v varies in the interval Δt by $a\Delta t$, the position x by $v\Delta t$. I thus construct the command sequence of the program PENDULUM, which is constantly to be repeated.

calculate $f(x)$ replace v by $v + (f/m)\Delta t$ replace x by $x + v\Delta t$ return to calculation of f

At the start we need an initial velocity v_0 and an initial position x_0. By suitable choice of the unit of time the mass can be set equal to unity. Programmable pocket calculators can be eminently suitable for executing this program. It is presented here in the computer language BASIC for $f = -\sin x$. It is clear that programming can be very easy; one should not be frightened by textbooks, where a page of programming may be devoted merely to the input of the initial data.

PROGRAM PENDULUM

```
10 x =0.0
20 v =1.0
30 dt=0.1
40 f=-sin(x)
50 v =v+f*dt
60 x =x+v*dt
70 print x,v
80 goto 40
90 end
```

In BASIC and FORTRAN

`a = b + c`   (`a := b + c;` in PASCAL)

signifies that the sum of `b` and `c` is to be stored at the place in store reserved for the variable `a`. The command

`n = n + 1`

is therefore not a sensational new mathematical discovery, but indicates that the variable n is to be increased by one from its previous value. By "goto" one commands the computer control to jump to the program line corresponding to the number indicated. In the above program the computer must be stopped by a command. In line 40 the appropriate force law is declared. It is of course still shorter if one simply replaces lines 40 and 50 by

`40v = v - sin(x)*dt`

c) Angular Momentum and Torque. The cross product $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ of position and momentum is the *angular momentum*, and $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{f}$ is the *torque*. Pedantic scientists might maintain that the cross product is not really a vector but an antisymmetric 3×3 matrix. We three-dimensional physicists can quite happily live with the pretence of handling $\boldsymbol{L}$ and $\boldsymbol{M}$ as vectors.

As the analogue of $\boldsymbol{f} = d\boldsymbol{p}/dt$ we have

$$\boldsymbol{M} = \frac{d\boldsymbol{L}}{dt} \quad , \tag{1.13}$$

which can also be written as

$$\boldsymbol{M} = \boldsymbol{r} \times \dot{\boldsymbol{p}} = \frac{d(\boldsymbol{r} \times \boldsymbol{p})}{dt} - \dot{\boldsymbol{r}} \times \boldsymbol{p} = \dot{\boldsymbol{L}} \quad ,$$

and since the vector $d\boldsymbol{r}/dt$ is parallel to the vector $\boldsymbol{p}$, the cross product of the two vectors vanishes. Geometrically $\boldsymbol{L}/m = \boldsymbol{r} \times \boldsymbol{v}$ is twice the rate at which

Fig. 1.5. The triangular area swept out by the radius vector $\boldsymbol{r}$ per unit time is a half of the cross-product $\boldsymbol{r} \times \boldsymbol{v}$. The upper picture is as seen, looking along the axis. The lower picture shows in three dimensions the angle ϕ and the vectors $\boldsymbol{L}$ and ω

area is swept out by the radius vector $\boldsymbol{r}$ (Fig. 1.5); the second law of Kepler therefore states that the sun exerts no torque on the earth and therefore the angular momentum and the rate at which area is swept out remain constant.

d) Central Forces. *Central forces* are those forces $\boldsymbol{F}$ which act in the direction of the radius vector $\boldsymbol{r}$, thus $\boldsymbol{F}(\boldsymbol{r}) = f(\boldsymbol{r})\boldsymbol{e}_r$ with an arbitrary scalar function f of the vector $\boldsymbol{r}$. Then the torque $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = (\boldsymbol{r} \times \boldsymbol{r})f(\boldsymbol{r})/|\boldsymbol{r}| = 0$:

Central forces exert no torque and leave the angular momentum unchanged. (1.14)

For all central forces the motion of the point mass lies in a plane normal to the constant angular momentum $\boldsymbol{L}$:

$$\boldsymbol{r}\boldsymbol{L} = \boldsymbol{r}(\boldsymbol{r} \times \boldsymbol{p}) = \boldsymbol{p}(\boldsymbol{r} \times \boldsymbol{r}) = 0$$

using the triple product rule

$$\boldsymbol{a}(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{c}(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b}(\boldsymbol{c} \times \boldsymbol{a}) \quad .$$

The calculation of the angular momentum in polar coordinates shows that for this motion ωr^2 remains constant: the nearer the point mass is to the centre of force, the faster it orbits round it. *Question:* Does this mean that winter is always longer than summer?

e) Isotropic Central Forces. Most central forces with which theoretical physicists have to deal are isotropic central forces. These are central forces in which the function $f(\boldsymbol{r})$ depends only on the magnitude $|\boldsymbol{r}| = r$ and not on the direction: $\boldsymbol{F} = f(r)\boldsymbol{e}_r$. With

$$U(r) = -\int f(r)dr$$

we then have $\boldsymbol{F} = -\mathrm{grad}\ U$ and $f = -dU/dr$: the potential energy U also depends only on the distance r. Important examples are:

$U \sim 1/r$, so $f \sim 1/r^2$:	gravitation, Coulomb's law;
$U \sim \exp(-r/\xi)/r$:	Yukawa potential; screened Coulomb potential;
$U = \infty$ for $r < a$, $U = 0$ for $r > a$:	hard spheres (billiard balls);
$U = \infty, -U_0$ and 0 for $r < a$, $a < r < b$ and $r > b$:	spheres with potential well;
$U \sim (a/r)^{12} - (a/r)^6$:	Lennard-Jones or "6–12" potential;
$U \sim r^2$:	harmonic oscillator.

(Here $\sim$ is the symbol for proportionality.)

For the computer simulation of real gases such as argon the Lennard-Jones potential is the most important: one places 10^6 such point masses in a large computer and moves each according to force = mass $\times$ acceleration, where the force is the sum of the Lennard-Jones forces from the neighbouring particles. This method is called "molecular dynamics" and uses a lot of computer time.

Since there is always a potential energy U, isotropic central forces are always conservative. If one constructs any apparatus in which only gravity and electrical forces occur, then the energy $E = U + T$ is necessarily constant. In a manner similar to the one-dimensional case the equation of motion can here be solved exactly, by resolving the velocity $\boldsymbol{v}$ into a component dr/dt in the $\boldsymbol{r}$-direction and a component $r d\phi/dt = r\omega$ perpendicular thereto and applying $L = m\omega r^2$:

$$E = U + T = U + \frac{mv^2}{2}$$
$$= U + \frac{m(dr/dt)^2 + r^2\omega^2}{2} = U + \frac{m[(dr/dt)^2 + L^2/m^2r^2]}{2} .$$

[In order to economise on parentheses, physicists often write a/bc for the fraction $a/(bc)$.] Accordingly, with $U_{\mathrm{eff}} = U + L^2/2mr^2$, we have:

$$\frac{dr}{dt} = \sqrt{2(E - U_{\mathrm{eff}})/m} \quad , \quad t = \int \frac{dr}{\sqrt{2(E - U_{\mathrm{eff}})/m}} . \tag{1.15}$$

By defining the effective potential U_{eff} we can thus reduce the problem to the same form as in one dimension (1.12). However, we now want to calculate also the angle $\phi(t)$, using

$$L = mr^2\omega = mr^2 \frac{d\phi}{dr}\frac{dr}{dt} : \quad \frac{d\phi}{dr} = \frac{L}{mr^2}\sqrt{2(E - U_{\mathrm{eff}})/m} . \tag{1.16}$$

Integration of this yields $\phi(r)$ and everything is solved.

f) Motion in a Gravitational Field. Two masses M and m separated by a distance r attract each other according to Newton's law of gravity