Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann
Subseries: USSR
Adviser: L. D. Faddeev, Leningrad

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Yu. I. Manin (Ed.)

K-Theory, Arithmetic and Geometry

Seminar, Moscow 1984–1986

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Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

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ISBN 3-540-18571-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-18571-2 Springer-Verlag New York Berlin Heidelberg

Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

PREFACE

This volume contains a collection of articles that originated from lectures given at the Manin Seminar at Moscow University, in 1984-1986. One of the principal motivations of the seminar was a collective desire to understand various ramifications of motivic and K-theoretic thinking in modern geometry and arithmetics. The final product is a volume of research papers reflecting the individual tastes of contributors. We hope however that it retains some internal coherence that is difficult to verbalize.

Всего, что знал еще Евгений,
Пересказать мне недосуг ...
(А. С. Пушкин, Евгений Онегин, I.УIII)

A.A.Beilinson
Yu.I.Manin
V.V.Schechtman

February 1987

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Height pairing between algebraic cycles

A.A. Beilinson

Introduction
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2. Local indices over non-archimedean places
3. Local indices over $\mathbb{C}$ or $\mathbb{R}$
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Introduction

Let X be a smooth projective variety over Q; assume that its L-functions $L(H^j(X), s)$ satisfy the standard analytic continuation conjectures. Let $CH^i(X)$ be the group of codimension i cycles on X modulo rational equivalence, and $Ch^i(X)^o \subset CH^i(X)$ be the subgroup of cycles homologous to zero on $X(\mathbb{C})$; in particular $CH^1(X)^\circ = \mathrm{Pic}^\circ(X)(\mathbb{Q})$. The conjecture of Birch and Swinnerton-Dyer claims that at $s = 1$ the function $L(H^1(X), s)$ has zero of order $\mathrm{rk}\ CH^1(X)^\circ$ with the leading coefficient equal to the determinant of Neron-Tate canonical height pairing multiplied by the period matrix determinant up to some rational multiple (we do not need its exact value in what follows). As for the other L-functions, Swinnerton-Dyer conjectured [20] that the function $L(H^{2i-1}(X), s)$ has at $s=i$ (=the middle of the critical strip) the zero of order $\mathrm{rk}\ CH^i(X)^\circ$. The aim of this note is to define the canonical height pairing between $CH^i(X)^\circ$ and $CH^{\dim X+1-i}(X)^\circ$ that coincides with the Neron-Tate one for i=1 and whose determinant multiplied by the period matrix determinant should conjecturally be equal up to a rational multiple (of the nature I cannot imagine) to the leading coefficient of $L(H^{2i-1}(X), s)$ at $s=i$ *). This pairing should also occure in Riemann-Roch type theorems à la Arakelov-Faltings (see [15]).

---

*) In fact our height pairing is defined on a certain subgroup of $CH^\cdot(X)^\circ$; under the very plausible (:=of rank of evidence far higher then B-SwD) local conjectures this subgroups should coincide with the whole $CH^\cdot(X)^\circ$.

The paper goes as follows. To motivate the basic construction, we begin with the simpler geometric case: here our base field is a field $k(C)$ of rational functions on a smooth projective curve C. Then the height pairing $<,>$ comes from the global Poincaré duality on ℓ-adic cohomology. We may compute $<,>$ in terms of local data round the points of C : if a_1, a_2 are cycles with disjoint supports that are homologous to zero on $X \otimes \overline{k(C)}$ and $v \in C$ is a closed point then the local link index $\langle a_1, a_2 \rangle_v$ is defined, and we have

$$(*) \qquad \langle a_1, a_2 \rangle = \sum_{v \in C} \langle a_1, a_2 \rangle_v$$

In the arithmetic situation, when the base field is a number field, the global construction fails due to the lack of appropriate cohomology theory. But we may still define the local indices $<,>_v$ numbered by the places of the base field, and then use (*) as the definition of $<,>$. These indices are defined using ℓ-adic cohomology for non-archimedean v and using the absolute Hodge-Deligne cohomology (see [2], [3], [19]) for archimedean ones; in case of pairing between divisors and zero cycles they are just Neron's quasifunctions ([17], [13],[22])

We also consider the intersection pairings. In the geometric case this is just the usual intersection pairing between the cycles of complementary dimensions on the regular scheme X_C proper over C. In the arithmetic case the role of X_C plays the A-variety $X = (X_{\mathbb{Z}}, \omega)$: where $X_{\mathbb{Z}}$ is a regular scheme projective over Spec $\mathbb{Z}$ and ω is a Kahler (1,1)-form on $X_{\mathbb{Z}} \otimes \mathbb{R}$ (see [15]). We define the corresponding Chow groups $CH^{\cdot}(X)$ and the $\mathbb{R}$-valued intersection pairing between $CH^i(X)$ and $CH^{\dim X - i}(X)$. This construction was independently found by H. Gillet and Ch. Soulé [10].

The final § contains some conjectures and motivic speculations about algebraic cycles, heights, L-functions and absolute cohomology groups.

The different construction of height pairing was proposed by S. Bloch [6]; I hope that our pairings coincide.

I would like to thank S. Bloch, P. Deligne, Yu. Manin, V. Schechtman and Ch. Soulé for stimulating ideas and interest.

§1. Height pairing in geometric case
(global construction)

In this § k will be an algebraically closed field, and ℓ will be a prime different from char k.

1.0. First recall some basic facts about the intersections. Let Y be a smooth projective scheme over k of dimension N + 1. The intersection of cycles defines the ring structure on Chow group $CH^{\bullet}(Y)$. This, together with an obvious trace map $CH^{N+1}(Y) \to \mathbb{Q}$, determines the intersection pairing $(,) : CH^{i}(Y) \otimes CH^{N+1-i}(Y) \to \mathbb{Q}$. We also have étale cohomology ring and the trace $H^{2(N+1)}(Y, \mathbb{Q}_{\ell}(N+1)) \to \mathbb{Q}_{\ell}$. The class map $cl : CH^{\bullet}(Y) \to H^{2\bullet}(Y, \mathbb{Q}_{\ell}(\cdot))$ is compatible with these structures, so we may compute (,) using ℓ-adic cohomology classes. In particular, (,) factors through $\overline{CH}^{\bullet}(Y) := \mathrm{Im}(CH^{\bullet}(Y) \to H^{2\bullet}(Y, \mathbb{Q}_{\ell}(\cdot)))$. One hopes that the following standard conjectures hold

- the obvious map $\overline{CH}^{\bullet}(Y) \otimes \mathbb{Q}_{\ell} \to H^{2\bullet}(Y, \mathbb{Q}_{\ell}(\cdot))$ should be injective, so $\overline{CH}^{\bullet}(Y) \otimes \mathbb{Q}$ should be finite-dimensional $\mathbb{Q}$-vector spaces (this is obviously true in char 0)
- (,) should be non-degenerate on $\overline{CH}^{\bullet}(Y) \otimes \mathbb{Q}$
- $\overline{CH}^{\bullet} \otimes \mathbb{Q}$ and (,) should satisfy hard Lefschetz and Hodge-index theorems.
- Finally if k is an algebraic closure of a finite field, then one should have $CH^{\bullet} \otimes \mathbb{Q} = \overline{CH}^{\bullet} \otimes \mathbb{Q}$. More precisely, for an Y_o/F_q, $Y_o \otimes k = Y$, the group $CH^{\bullet}(Y_o) \otimes \mathbb{Q}_{\ell}$ should coincide with invariants of Frobenius in $H^{2\bullet}(Y, \mathbb{Q}_{\ell}(\cdot))$ (Tate's conjecture).

1.1. The height pairing arises in a slightly different situation. Fix a smooth projective irreducible curve C over k; let $\eta \in C$ be its generic point, and $\bar{\eta}/\eta$ be some geometric generic point. Now let X be a smooth projective N-dimensional η-scheme, and $X_{\bar{\eta}} := X \times_{\eta} \bar{\eta}$ be its geometric fiber. Choose some projective scheme $X_C \xrightarrow{\pi} C$ over C with the generic fiber X; for an open $U \subset C$ put $X_U = \pi^{-1}(U)$. Now take $j : U \hookrightarrow C$ s.t. U is affine and $\pi|_U$ is smooth. Put $H^{\bullet}_{!*}(X, \mathbb{Q}_{\ell}(*)) := \mathrm{Im}(H^{\bullet}_c(X_U, \mathbb{Q}_{\ell}(*)) \to H^{\bullet}(X_U, \mathbb{Q}_{\ell}(*)))$; the Poincaré duality on X_U induces the perfect pairing between $H^{\bullet}_{!*}(X, \mathbb{Q}_{\ell}(*))$ and $H^{2N+2-\bullet}_{!*}(X, \mathbb{Q}_{\ell}(N+1-*))$. We'll see in a moment that this groups and pairing depend only on X itself (and not on the choice of particular model X_C/C). To do this consider the smooth sheaf $R^{\bullet-1}\pi|_{U*}(\mathbb{Q}_\ell)$ on U and its middle extension $\mathcal{F}^\bullet := j_*R^{\bullet-1}\pi|_{U*}(\mathbb{Q}_\ell)$ on C. The Poincare duality along the fibers of π defines the perfect pairing $\langle,\rangle : \mathcal{F}^\bullet \otimes \mathcal{F}^{2N+2-\bullet} \to \mathbb{Q}_\ell(-N)$ in $D^b(C,\mathbb{Q}_\ell)$ and so the same-noted perfect duality $\langle,\rangle : H^1(C, \mathcal{F}^\bullet) \otimes H^1(C, \mathcal{F}^{2N+2-\bullet}) \to \mathbb{Q}_\ell(-N-1)$. Clearly both $\mathcal{F}^\bullet$ and $\langle,\rangle$ depend only on X (and not on X_C).

<u>Lemma 1.1.1.</u> We have $H^\bullet_{!*}(X, \mathbb{Q}_\ell(*)) = H^1(C, \mathcal{F}^\bullet(*))$ and $\langle,\rangle$ coincides with the duality on $H_{!*}$ induced by the pairing between $H_c(X_U)$ and $H(X_U)$.

<u>Proof</u>. I'll prove only the first statement since the second is immediate. Since U is affine, the Leray spectral sequence for π degenerates and reduces to two-step filtrations on $H^\bullet_c(X_U)$ and $H^\bullet(X_U)$ with factors $Gr_{-2}(H^\bullet_c) = H^2_c(U, R^{\bullet-2}\pi_{U*}(\mathbb{Q}_\ell))$, $Gr_{-1}(H^\bullet_c) = H^1_c(U, R^{\bullet-1}\pi_*(\mathbb{Q}_\ell))$ and $Gr_{-1}(H^\bullet) = H^1(U, R^{\bullet-1}\pi_*(\mathbb{Q}_\ell))$, $Gr_0(H^\bullet) = H^0(U, R\pi^\bullet_*(\mathbb{Q}_\ell))$. So $H^\bullet_{!*}(X, \mathbb{Q}_\ell) = \mathrm{Im}\,(H^1_c(U, R^{\bullet-1}\pi_*(\mathbb{Q}_\ell)) \longrightarrow H^1(U, R^{\bullet-1}\pi_*(\mathbb{Q}_\ell))) = H^1(C,\mathcal{F}^\bullet)$, q.e.d. ■

For a moment put $H^\bullet(X, \mathbb{Q}_\ell) := \varinjlim H^\bullet(X_U, \mathbb{Q}_\ell) = \varinjlim H^\bullet(U, R\pi_*(\mathbb{Q}_\ell))$, $H^\bullet(X, \mathbb{Q}_\ell)^0 := \mathrm{Ker}(H^\bullet(X, \mathbb{Q}_\ell) \to H^\bullet(X_{\bar\eta}, \mathbb{Q}_\ell)) = \varinjlim H^1(U,R^{\bullet-1}\pi_*(\mathbb{Q}_\ell))$. Clearly $H^\bullet_{!*}(X) \subset H^\bullet(X)^0 \subset H^\bullet(X)$ and this groups depend on X only. Now consider the cycles on X; we have the class map $cl : CH^\bullet(X) \longrightarrow H^{2\bullet}(X, \mathbb{Q}_\ell(\bullet))$, put $CH^\bullet(X)^0 := cl^{-1}(H^{2\bullet}(X, \mathbb{Q}_\ell(\bullet))^0)$ to be the subgroup of cycles whose intersection with generic geometric fiber is homologous to zero.

<u>Key-lemma 1.1.2.</u> One has $cl(CH^\bullet(X)^0) \subset H^{2\bullet}_{!*}(X, \mathbb{Q}_\ell(\bullet))$.

<u>Proof</u>. The problem is local round the points of C (and exists only at points of bad reduction). So let η_v be the generic point of a henselisation of C at some closed point, $\bar\eta_v$ be a separable closure of η_v, and $X_{\eta_v} = X \times \eta_v$, $X_{\bar\eta_v} = X \times \bar\eta_v$. We have to show that whenever $a \in CH^\bullet(X_{\eta_v})$ is a cycle s.t. $cl\ a \in H^{2\bullet}(X_{\eta_v}, \mathbb{Q}_\ell(\bullet))^0 := \mathrm{Ker}(H^{2\bullet}(X_{\eta_v}, \mathbb{Q}_\ell(\bullet)) \longrightarrow H^2(X_{\bar\eta_v}, \mathbb{Q}_\ell(\bullet)))$ then $cl\ a$ is zero. But $H^{2\bullet}(X_{\eta_v}, \mathbb{Q}_\ell(\bullet))^0 = H^{2\bullet-1}(X_{\bar\eta_v}, \mathbb{Q}_\ell(\bullet-1))_{Gal\,\bar\eta_v/\eta_v}$. According to [9] th.1.8.4. the weights on this group are $\geqslant 1$. Since the class of an algebraic cycle has weight zero, we are done. ■

Now, by lemma, we may define the height pairing $\langle,\rangle: CH^{i}(X)^{o} \times CH^{N+1-i}(X)^{o} \longrightarrow \mathbb{Q}_{\ell}$ by means of class map followed by the duality between $H^{\bullet}_{!*}$.

One may conjecture that this pairing is $\mathbb{Q}$-valued and independent of ℓ ; of course this is obviously true in char 0, and also true in any char if $i = 1$ (for the discussion of this see §2). The variant of standard conjectures claims that $\overline{CH^{\bullet}(X)^{o}} := \mathrm{Im}\,(CH^{\bullet}(X)^{o} \otimes \mathbb{Q} \longrightarrow H^{2\bullet}_{!*}(X, \mathbb{Q}_{\ell}(\cdot)))$ should be finite-dimensional $\mathbb{Q}$-vector space, that this groups satisfy hard Lefschetz, the form $\langle,\rangle$ should be non-degenerate on them and satisfy Hodge-index theorem. If k = algebraic closure of finite field, then one should have $CH^{\bullet}(X)^{o} \otimes \mathbb{Q} = \overline{CH^{\bullet}(X)^{o}}$.

Problem. Consider the case $\dim \eta/k > 1$.

1.2. The both types of pairings - the intersection and the height one - are related as follows. Suppose that we have $\pi : X_C \longrightarrow C$ s.t. X_C is regular projective with generic fiber X. Define $CH^{\bullet}(X_C)^{o} := \mathrm{Ker}(CH^{\bullet}(X_C) \to H^{o}(C, R^{2\bullet}\pi_*(\mathbb{Q}_{\ell}(\cdot))))$ to be the subgroup of cycles whose intersection with any geometric fiber of π is homologous to zero. The restriction map $CH^{\bullet}(X_C) \to CH^{\bullet}(X)$, $a_C \mapsto a$ maps $CH^{\bullet}(X_C)^{o}$ into $CH^{\bullet}(X)^{o}$ and we have an obvious

Lemma 1.2.1. For any $a_{1C} \in CH^{\bullet}(X_C)^{o}$, $a_{2C} \in CH^{N+1-\bullet}(X_C)^{o}$ one has $(a_{1C}, a_{2C}) = \langle a_1, a_2\rangle$ ■

One may suppose that the image of $CH\,(X\,)^{o}$ under the restriction map coincides with $CH^{\bullet}(X)^{o}$; see §2 for details.

§2. Local indices over non-archimedean places

In this § we'll define the local pairing between cycles, and will show how to decompose the global pairings of the previous § into the sum of local ones.

In what follows C_v will be any strictly henselian trait with the generic point η_v, the special point s_v, and an algebraic generic geometric point $\bar{\eta}_v$; X_v will be a smooth projective scheme over η_v of dimension N, and $X_{\bar{v}} := X_v \times_{\eta_v} \bar{\eta}_v$ will be its geometric fiber; ℓ will be a prime $\neq$ char s.

2.0. Let me start with the local intersection pairing. Let X_{C_v} be a regular projective scheme over C_v with the generic fiber X_v and the special one X_s. If a_{C_v} is any cycle on X_{C_v} of codimension d then one has its classes $\widetilde{cl}\,(a_{C_v})$ in cohomology groups with supports:

the universal one $\widetilde{cl}_M(a_{C_v}) \in H^{2d}_{M\ \mathrm{supp}a_{C_v}}(X_{C_v}, \mathbb{Q}(d)) =$ $H^{2d}_M(X_{C_v}, X_{C_v} \setminus \mathrm{supp}a_{C_v}, \mathbb{Q}(d))$ and the ℓ-adic one $\widetilde{cl}_{\mathbb{Q}_\ell}(a_{C_v}) \in$ $H^{2d}_{\mathrm{supp}a_{C_v}}(X_{C_v}, \mathbb{Q}_\ell(d))$; clearly $cl_M \longmapsto cl_{\mathbb{Q}_\ell}$ under canonical map (see [18], [3]).

Now let $a_{1C_v} \in Z^{d_1}(X_{C_v})$, $a_{2C_v} \in Z^{d_2}(X_{C_v})$ be two cycles on X_{C_v} of supports Y_{1C_v}, Y_{2C_v} respectively such that $d_1 + d_2 = N + 1$ and $Y_{1v} \cap Y_{2v} := Y_{1C_v} \cap Y_{2C_v} \cap X_v = \emptyset$. Define the intersection index $(a_{1C_v}, a_{2C_v})_v \in \mathbb{Q}$ to be the image of $\widetilde{cl}_M(a_{1C_v}) \cup \widetilde{cl}_M(a_{2C_v}) \in$ $H^{2N+2}_{M\ Y_1 \cap Y_2}(X_{C_v}, \mathbb{Q}(N+1))$ by $H^{2N+2}_{M\ X_s}(X_{C_v}, \mathbb{Q}(N+1)) \xrightarrow{Tr_\pi} H^1_{M\ s_v}(C_v, \mathbb{Q}(1)) = \mathbb{Q}$. We may replace here H_M by ℓ-adic cohomology; since $H_M \longrightarrow H_{\mathbb{Q}_\ell}$ commutes with any canonical map, we'll get the same answer.

If we are in a global geometric situation 1.2, then for any closed point s_v of C we may consider the henselisation C_v of C at s_v and thus get our local situation. If a_{1C}, a_{2C} are cycles on X_C s.t. a_1, a_2 have disjoint supports on X, then for any s_v we get local intersection index $(a_{1C}, a_{2C})_v$ and clearly one has

<u>Lemma 2.0.1.</u> $(a_{1C}, a_{2C}) = \sum (a_{1C}, a_{2C})_v$ ∎

2.1. Now let us consider the local components for height pairing. Let a_1, a_2, $a_i \in Z^{d_i}(X_v)$, be a cycles on X_v; put $Y_i = \mathrm{supp}\, a_i$, $U_i = X_v - Y_i$. Suppose that $d_1 + d_2 = N+1$, $Y_1 \cap Y_2 = \emptyset$ and both $cl(a_i) \in H^{2d_i}(X_v, \mathbb{Q}_\ell(d_i))$ are zero. In this situation one has link index $\langle a_1, a_2\rangle_v \in \mathbb{Q}_\ell$. The intuitive picture for $\langle , \rangle_v$ is following: from the homotopy point of view η_v is a circle round s_v and X_v is $2N+1$-dimensional topological manifold fibered over this circle, a_i are $2d_i$-codimensional cycles on it that doesn't intersect and homologous to zero; the link index $\langle a_1, a_2\rangle_v$ is the intersection number of a_1 and the chain that bounds a_2. Here are the number of exact definitions of $\langle a_1, a_2\rangle_v$; the proof of their equivalence is left to the reader. In what follows $\alpha_i \in H^{2d_i-1}(U_i, \mathbb{Q}_\ell(d_i))$ are classes that bound a_i: this means that $\alpha_i \longmapsto \widetilde{cl}(a_i)$ under the boundary map $H^{2d_i-1}(U_i) \longrightarrow H^{2d_i}_{Y_i}(X_v)$.

<u>Lemma-definition 2.1.1.</u> The following definitions of $\langle a_1, a_2)_v$ are equivalent.

a) $\langle a_1, a_2\rangle_v$ is the image of $\alpha_1 \cup \widetilde{cl}(a_2)$ by $H^{2N+1}_{Y_2}(U_1, \mathbb{Q}\,(N+1)) \longrightarrow H^{2N+1}(X_v, \mathbb{Q}_\ell(N+1)) \xrightarrow{Tr} H^1(\eta_v, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$

b) $\langle a_1, a_2\rangle_v$ is the image of $\alpha_1 \cup \alpha_2$ by $H^{2N}(U_1 \cap U_2, \mathbb{Q}_\ell(N+1)) \longrightarrow H^{2N+1}(X_v, \mathbb{Q}_\ell(N+1)) \xrightarrow{Tr} \mathbb{Q}_\ell$; here the first arrow comes from the Myer-Vietoris exact sequence for the covering $U_1 \cup U_2 = X_v$.

c) Choose a projective $\pi : X_{C_v} \longrightarrow C_v$ with the generic fiber X_v, $\beta_1 \in H^{2d_1}_{\pi^{-1}(s_v)\cup Y_1}(X_{C_v}, \mathbb{Q}_\ell(d_1))$ and $\beta_2 \in H^{2d_2-2N}_{\pi^{-1}(s_v)\cup Y_2}(X_{C_v}, R\pi^!\mathbb{Q}_\ell(d_2-N))$ s.t. the restriction of β_i on X_v coincide with $\widetilde{cl}(a_i)$ and the image of β_1 in $H^{2d_1}(X_{C_v}, \mathbb{Q}_\ell(d_1))$ is zero (e.g. you may take β_1 to be the boundary of α_1 in X_{C_v}). Then $\langle a_1, a_2\rangle_v$ is the image of $\beta_1 \cup \beta_2$ by $H^2_{\pi^{-1}(s_v)}(X_{C_v}, R\pi^!\mathbb{Q}_\ell(1)) \longrightarrow H^2_{s_v}(C_v, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$ ■

This way we get link pairing between the cycles with disjoint supports; clearly 2.1.1. b) shows that this pairing is bilinear and symmetric. Now we are going to show that it behaves well under the action of correspondences. To do this first let us see that the above definition may be easily generalised to the case of arbitrary many cycles. Namely let $a_1, \ldots, a_n$ be cycles of codimensions d_i on X_v s.t. $\sum d_i = N+1$ and $\bigcap Y_i = \emptyset$ (here $Y_i = \operatorname{supp} a_i$, $U_i = X_v \setminus Y_i$) assume that at least one of them is homologous to zero in X. Choose a non-empty subset $S \subset \{1,\ldots,n\}$ s.t. for any $j \in S$ the cycle a_j is homologous to zero, and for $a_j \in S$ some $\alpha_j \in H^{2d_j-1}(U_j, \mathbb{Q}_\ell(d_j))$ that bounds a_j. Define $\langle a_1,\ldots,a_n\rangle_v$ to be $Tr\,[\partial(\bigcup_{j\in S}\alpha_j) \cup (\bigcup_{j\notin S} a_j)]$; here $\partial : H^\cdot(\bigcap_{j\in S} U_j) \longrightarrow H^{\cdot+\#S-1}(\bigcup_{j\in S} U_j)$ is the differential in the spectral sequence of the covering $\{U_j\}_{j\in S}$ of $\bigcup_{j\in S} U_j$, and

We have the following easy generalisation of 2.1.1:

<u>Lemma 2.1.2.</u> If at least two of a_i are homologous to zero, then $\langle a_1,\ldots,a_n\rangle_v$ depends on $a_1,\ldots,a_n$ only (and not on the choice of S and α_j) ■

Clearly in this case the pairing is also bilinear and symmetric.

Now we may look at correspondences. Consider two schemes X_{1v}, X_{2v} of dimensions N_1, N_2, a cycles a_i of codimensions d_i on

X_{iv} and a cycle b of codimension d_3 on $X_{1v} \times X_{2v}$. Assume that both a_i are homologous to zero, $d_1 + d_2 + d_3 = N_1 + N_2 + 1$ and $p_1^{-1}(\mathrm{supp}\ a_1) \cap \mathrm{supp}\ b \cap p_2^{-1}(\mathrm{supp}\ a_2) = \emptyset$ (here p_i are projections $X_{1v} \times X_{2v} \longrightarrow X_{iv}$). Then 2.1.1 implies

Lemma 2.1.3. $\langle p_1^*(a_1), b, p_2^*(a_2)\rangle_v = \langle b(a_1), a_2\rangle_v = \langle a_1, b(a_2)\rangle_v$ ∎

Here $b(a_1)$ is the image of a_1 under the action of correspondence b. (Note that if $p_1^*(a_1)$ and b doesn't intersect properly, then the cycle $b(a_1)$ is not defined in a unique way, but it has correctly defined class in the cohomology group with support in $p_2(\mathrm{supp}\ b \cap p_1^{-1}(\mathrm{supp}\ a_1))$ that suffice for our purposes.)

The lemma shows for example that the computation of $\langle a_1, a_2\rangle_v$ in case when one of a_i is algebraically equivalent to zero, may be reduced to the computation of link index between a zero cycles on a curve.

Our pairing behaves in a usual way under the change of the base field. Namely let $\eta_{v'}/\eta_v$ be a degree n extension, X_v is a scheme over η_v and $Y_{v'}$ is one over $\eta_{v'}$. Put $X_{v'} = X_v \times_{\eta_v} \eta_{v'}$ and Y_v be $Y_{v'}$ considered as a scheme over η_v. We have the obvious arrows $\mathcal{Z}^{\cdot}(X_v) \hookrightarrow \mathcal{Z}^{\cdot}(X_{v'})$, $\mathcal{Z}^{\cdot}(Y_{v'}) = \mathcal{Z}^{\cdot}(Y_v)$ and the following holds (recall that k is separably closed):

Lemma 2.1.4. Let a_i be cycles on X_v and b_i the ones on $Y_{v'}$. Then $\langle a_1, a_2\rangle_v = 1/n\ \langle a_1, a_2\rangle_{v'}$, $\langle b_1, b_2\rangle_{v'} = \langle b_1, b_2\rangle_v$ ∎

In particular by means of first formula we may define the height pairing between cycles on $X_{\bar{\eta}_v}$; this pairing is clearly Galois-invariant.

Let us relate local link pairings with the global height pairing of §1. Assume that we are in global geometric situation of 1.1. As in 2.0 to each closed point on C corresponds the local picture over corresponding local field η_v. Let a_1, a_2 be cycles on X of codimensions d_i s.t. $d_1 + d_2 = N+1$ and $\mathrm{supp}a_1 \cap \mathrm{supp}a_2 = \emptyset$. If a_i belongs to $CH^{d_i}(X)^{\circ}$ then 1.12.shows that a_i is homologous to zero on any X_v. So the link index $\langle a_1, a_2\rangle_v$ is defined, and it is easy to see that the definition 2.1.1.c implies the formula (*) from the introduction:

Lemma 2.1.5. We have $\langle a_1, a_2\rangle = \sum \langle a_1, a_2\rangle_v$ (sum over all closed points of C). ∎

Finally let me compare the local pairings from 2.0 and 2.1. Suppose that we are in a situation 2.0. Put $CH^{\cdot}(X_{C_v})^{o} := \mathrm{Ker}(CH^{\cdot}(X_{C_v}) \longrightarrow H^{2\cdot}(X_{C_v}, \mathbb{Q}_\ell(\cdot)))$. Let a_{1C_v}, a_{2C_v} be cycles on X_{C_v} as in 2.0; assume that both a_i are homologous to zero on Xv (here $a_i :=$

$a_{iC_v} \cap X_v$). Then 2.1.1c implies.

Lemma 2.1.6. If one of a_{iC_v} also belongs to $CH^\bullet(X_{C_v})^o$, then $(a_{1C_v}, a_{2C_v})_v = \langle a_1, a_2\rangle_v$. ■

This is local analog of 1.2.1. In particular in this situation $\langle a_1, a_2\rangle_v \in \mathbb{Q}$ and doesn't depends on the choice of $\ell \neq$ char k.

2.2. Here are some conjectures about local pairings. Put

$CH^\bullet(X_v)^o := \mathrm{Ker}\,(CH^\bullet(X_v) \longrightarrow H^{2\bullet}(X_{\bar\eta_v}, \mathbb{Q}_\ell(\cdot)))$

Conjecture 2.2.1. One has $CH^i(X_v)^o = \mathrm{Ker}\,(CH^i(X_v) \to H^{2i}(X_v, \mathbb{Q}_\ell(i)))$ i.e. any cycle whose intersection with a generic geometric fiber is homologous to zero is homologous to zero on X_v ■

Lemma 2.2.2. This conjecture is true in following cases:

a. Good reduction case.

b. For cycles algebraically equivalent to zero (in particular the case $i = 1$ and $i = N$).

c. Geometric case.

Proof. a is obvious; c was proved in 1.1.2; b may be reduced using correspondences to the case of zero cycles on curve, where it follows, say, from 2.2.6b. ■

Remark. The conjecture would follow if one knows the information on weights on $H^\bullet(X_{\bar\eta_v}, \mathbb{Q}_\ell)$ similar to those one has in geometric case. If η_v is a p-adic local field, then the thing we need is the usual conjecture on poles of local L-multiples.

Conjecture 2.2.3. The local link pairing is Q-valued and independent of $\ell \neq$ char k.

Lemma 2.2.4. This conjecture is true:

a. In good reduction case.

b. In case char $s = 0$.

c. When one of the cycles is algebraically equivalent to zero (in particular for the pairing between divisors and zero cycles).

Proof. b is obvious; a and c follow from 2.2.6, for c use correspondences to reduce to 2.2.6b. ■

Suppose that we are in a situation 2.0. Clearly the restriction arrow $CH^\bullet(X_{C_v}) \to CH^\bullet(X_v)$ maps $CH^\bullet(X_{C_v})^o$ into $CH^\bullet(X_v)^o$.

Conjecture 2.2.5. $CH^\cdot(X_C)^o \twoheadrightarrow CH^\cdot(X_v)^o$. ■

Clearly 2.2.5 implies both 2.2.1 and 2.2.3.

Lemma 2.2.6. This conjecture is true

a. In good reduction case.

b. If X_v is a curve.

Proof. a is obvious; b follows from the well-known fact that the intersection matrix between components of the special fiber is almost negative-definite. ■

The property, 2.2.6b together with 2.1.6, shows that in case of zero cycles on a curve our link index coincides with Néron's local symbol; using the correspondences one may see that the same is true for the pairing between divisors and zero cycles in arbitrary dimensions (or you may directly verify that $\langle , \rangle_v$ satisfies the conditions of Néron's theorem [12].) For the review of heights on curves together with the study of important examples see [7].

We wish to use the formula 2.1.5. for varieties over number field as the definition of left-hand side, just as Neron did in the case of divisors and zero cycles. The only thing that remains to be defined is the link index for archimedean local fields.

§3. Local indices over $\mathbb{C}$ and $\mathbb{R}$

In this § our base field will be $\mathbb{C}$ or $\mathbb{R}$

3.0. We have the following dictionary (see [2] for notations and details)

Non-archimedean case (η_v is the spectrum of a p-adic local field)	Archimedean case (η_v = Spec $\mathbb{C}$ or Spec $\mathbb{R}$)
Gal $\bar{\eta}_v/\eta_v$-modules	R-mixed Hodge structures
Étale cohomology groups $H^{\bullet}(X_{\bar{v}}, \mathbb{Q}_\ell(*))$ of the geometric fiber with the Galois action	Ordinary cohomology groups $H^{\bullet}(X_v(\mathbb{C}), \mathbb{R}(*))$ with Deligne's mixed Hodge structure
Étale cohomology groups $H^{\bullet}(X_v, \mathbb{Q}_\ell(*))$	Absolute Hodge-Deligne cohomology groups $H^{\bullet}_{\mathcal{H}}(X_v, \mathbb{R}(*))$
Canonical arrow $H^{\bullet}(X_v, \mathbb{Q}_\ell(*)) \to H^{\bullet}(X_{\bar{v}}, \mathbb{Q}_\ell(*))$	Canonical arrow $H^{\bullet}_{\mathcal{H}}(X_v, \mathbb{R}(*)) \to H^{\bullet}_{\mathcal{B}}(X_v, \mathbb{R}(*)) \subset H^{\bullet}(X_v(\mathbb{C}), \mathbb{R}(*))$

Note that the canonical arrow $H^{2i}(X_v, \mathbb{R}(i)) \to H^{2i}(X_v, \mathbb{R}(i))$ is injective; so whenever we have an algebraic cycle homologous to zero as topological cycle it has zero class in $H_{\mathcal{H}}$. This shows that the analog of conjecture 2.2.1 is obviously true in archimedean case (this is one of the sides of the fact that smooth proper varieties have "good reduction" at archimedean places from the Hodge-theoretic point of view). We may translate the definition 2.1.1 a or b using the above dictionary to define the link index in our situation. Namely, this way for any two cycles a_1, a_2 on X_v of codimensions d_1, d_2 s.t. $d_1 + d_2 = N+1$, the supports of a_i doesn't intersect and both a_i are homologous to zero, we get the link index $\langle a_1, a_2\rangle_v \in \mathbb{R}$. The lemmas 2.1.1 (a $\Longleftrightarrow$ b), 2.1.2. - 2.1.4. remain true, together with their proofs, in our situation.

3.1. Now let me sketch the definition of the analog of intersection index à la Arakelov; since this will not be needed in the main body of the paper I'll omit the details. The similar construction was found by H. Gillet and Ch. Soulé [10]. The role of the model X_{C_v} of X_v over the ring of integers of p-adic field plays now (in our archimedean situation) the Kähler metrics ω on X_v (see [15]). Let us define the $\mathbb{R}$-vector space $Z^i(X_v, \omega)$ - an analog of the group $Z^i(X_{C_v})$. In what follows for $c \in H^{\cdot}(X_v, \mathbb{C})$ we denote by $\tilde{c}$ its ω-harmonic representative, and for a cycle $z \in Z^{\cdot}(X_v) \otimes \mathbb{R}$ let δ_z be its δ-current. Say that a current α is i-Green current if it is $R(i-1)$-valued of type $(i-1, i-1)$ and for certain (unique) $\alpha_f \in Z^i(X_v) \otimes \mathbb{R}$ one has $\partial\bar{\partial}\alpha = \delta_{\alpha_f} - \widetilde{cl\,\alpha_f}$; say that α is regular if it is of C^∞-class off the support of α_f. A (regular) Green current α is trivial if $\alpha = \partial\nu \pm \bar{\partial}\nu$ for certain (C^∞-class) current ν. Put $Z^i(X_v, \omega)$ to be the factorspace of i-Green currents modulo trivial ones (you may take all Green currents or regular ones only - this doesn't changes $Z^i(X_v, \omega)$). For $\alpha \in Z^i(X_v, \omega)$ let $\alpha_\infty \in H^{i-1,i-1}(X_v, R(i-1)) (:= H^{i-1,i-1}(X_v) \cap H^{2i-2}_B(X_v, R(i-1))$ be the (only) class such that for any ω-harmonic $2(N-i+1)$-form ν one has $\int_{X_v} \alpha \wedge \nu = \int_{X_v} \alpha_\infty \nu$. One may see that the arrow $Z^i(X_v, \omega) \longrightarrow H^{i-1,i-1}(X_v, R(i-1)) \oplus Z^i(X_v) \otimes \mathbb{R}$, $\alpha \longmapsto (\alpha_\infty, \alpha_f)$, is isomorphism. This direct sum decomposition of $Z^i(X_v, \omega)$ is analogous to the decomposition of $Z^i(X_C)$ into the sum of the group of cycles supported in the special fiber and the group of cycles that intersect the special fiber properly.

Here is an important example of Green current; we'll need it in n° 4.1. For a scheme S put $C^i(S) = \bigoplus_{\substack{\eta \in S \\ \mathrm{Codim}\,\eta = i}} \mathcal{O}^*(\eta)$: an element $\varphi \in C^i(S)$

is a finite number of invertible meromorphic functions φ_η defined at the generic points of codimension i subschemes η (this is $E^{-i-1,i}$-term of Quillen's spectral sequence for $K'_\cdot(S)$); we have the exact sequence $C^{i-1}(S) \xrightarrow{div} Z^i(S) \longrightarrow CH^i(S) \longrightarrow 0$. Now in our situation for $\varphi \in C^{i-1}(X_v)$ we have Green current $\overline{div}(\varphi)$ defined by $\int \overline{div}(\varphi) \wedge \nu = \sum_\eta \int_\eta \log|\varphi_\eta| \cdot \nu|_\eta$; clearly $(\overline{div}\ \varphi)_f = div\ \varphi$. This way we get an arrow $\overline{div} : C^{i-1}(X_v) \longrightarrow Z^i(X_v, \omega)$.

Now let me define the intersection index. Let $a_1 \in Z^{d_1}(X_v, \omega)$, $a_2 \in Z^{d_2}(X_v, \omega)$ be elements such that $d_1 + d_2 = N+1$ and the supports of a_{if} are disjoint. Put $(a_1, a_2)_v = \int_{X_v} (\alpha_1 \wedge \delta_{a_{2f}} + \delta_{a_{1f}} \wedge \widetilde{a_{2\infty}}) \in \mathbb{R}$ (here α_1 is a regular Green current that represents a_1). One may see that $(a_1, a_2)_v$ so defined is independent of the choice of α_1, that $(,)_v$ is symmetric, and in case when both a_{if} are homologous to zero one has $(a_1, a_2)_v = \langle a_{1f}, a_{2f} \rangle_v$. In $N = 1$ case this construction coincides with the original Arakelov one [1] (see also [13]).

§4. Height pairing over number fields

Let us return to the global situation. In this § our base field K will be a finite extension of $\mathbb{Q}$; let $\bar{K}$ be an algebraic closure of K, $\eta = \operatorname{Spec} K$, $\bar{\eta} = \operatorname{Spec} \bar{K}$. For any place v of K denote by K_v the corresponding local field, and by η_v the spectrum of K_v^{nr} = maximal non-ramified extension of K_v. Define the real number $r(v)$ to be log (number of elements in the residese field of K_v) for non-archimedean v, and $r(v) = 1$ for $K_v = \mathbb{R}$, $r(v) = 2$ for $K_v = \mathbb{C}$.

4.0. Let $X = X_K$ be a N-dimensional smooth projective variety over K; put $CH^i(X)^\circ := \operatorname{Ker}(CH^i(X) \longrightarrow H^{2i}(X_{\bar{\eta}}, \mathbb{Q}_\ell(i)))$, $\overline{CH}^i(X) := CH^i(X)/CH^i(X)^\circ \subset H^{2i}(X_{\bar{\eta}}, \mathbb{Q}_\ell(i))$. We'll assume that for any non-archimedean v the η_v-scheme $X_v = X \times \eta_v$ satisfies the conjectures 2.2.1 and 2.2.3.

Remark 4.0.1. a. If you do not wish to assume this, then change the notations and put $CH^i(X)^\circ := \operatorname{Im}(CH^i(X_{\mathbb{Z}})^\circ \longrightarrow CH^i(X))$; here $X_{\mathbb{Z}}$ is certain regular model of X over the ring of integers of K (we