国外数学名著系列
（影印版）21

Dennis P. Sullivan

Geometric Topology: Localization, Periodicity and Galois Symmetry

几何拓扑：局部性、周期性和伽罗瓦对称性

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格（Springer）出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

Editor's Preface

The seminal 'MIT notes' of Dennis Sullivan were issued in June 1970 and were widely circulated at the time. The notes had a major influence on the development of both algebraic and geometric topology, pioneering

- the localization and completion of spaces in homotopy theory, including p-local, profinite and rational homotopy theory, leading to the solution of the Adams conjecture on the relationship between vector bundles and spherical fibrations,
- the formulation of the 'Sullivan conjecture' on the contractibility of the space of maps from the classifying space of a finite group to a finite dimensional CW complex,
- the action of the Galois group over $\mathbb{Q}$ of the algebraic closure $\widetilde{\mathbb{Q}}$ of $\mathbb{Q}$ on smooth manifold structures in profinite homotopy theory,
- the K-theory orientation of PL manifolds and bundles.

Some of this material has been already published by Sullivan himself: in an article[1] in the Proceedings of the 1970 Nice ICM, and in the 1974 Annals of Mathematics papers *Genetics of homotopy theory and the Adams conjecture* and *The transversality characteristic class and linking cycles in surgery theory*[2]. Many of the ideas originating in the notes have been the starting point of subsequent

[1] reprinted at the end of this volume
[2] joint with John Morgan

developments[3]. However, the text itself retains a unique flavour of its time, and of the range of Sullivan's ideas. As Wall wrote in section 17F *Sullivan's results* of his book *Surgery on compact manifolds* (1971) : *Also, it is difficult to summarise Sullivan's work so briefly: the full philosophical exposition in (the notes) should be read.* The notes were supposed to be Part I of a larger work; unfortunately, Part II was never written. The volume concludes with a Postscript written by Sullivan in 2004, which sets the notes in the context of his entire mathematical oeuvre as well as some of his family life, bringing the story up to date.

The notes have had a somewhat underground existence, as a kind of Western samizdat. Paradoxically, a Russian translation was published in the Soviet Union in 1975[4], but this has long been out of print. As noted in Mathematical Reviews, the translation *does not include the jokes and other irrelevant material that enlivened the English edition.* The current edition is a faithful reproduction of the original, except that some minor errors have been corrected.

The notes were TeX'ed by Iain Rendall, who also redrew all the diagrams using METAPOST. The 1970 Nice ICM article was Tex'ed by Karen Duhart. Pete Bousfield and Guido Mislin helped prepare the bibliography, which lists the most important books and papers in the last 35 years bearing witness to the enduring influence of the notes. Martin Crossley did some preliminary proofreading, which was completed by Greg Brumfiel ("ein Mann der ersten Stunde"[5]). Dennis Sullivan himself has supported the preparation of this edition via his Albert Einstein Chair in Science at CUNY. I am very grateful to all the above for their help.

Andrew Ranicki

Edinburgh, October, 2004

[3]For example, my own work on the algebraic L-theory orientations of topological manifolds and bundles.

[4]The picture of an infinite mapping telescope on page 34 is a rendering of the picture in the Russian edition.

[5]A man of the first hour.

Preface

This compulsion to localize began with the author's work on invariants of combinatorial manifolds in 1965-67. It was clear from the beginning that the prime 2 and the odd primes had to be treated differently.

This point arises algebraically when one looks at the invariants of a quadratic form[1]. (Actually for manifolds only characteristic 2 and characteristic zero invariants are considered.)

The point arises geometrically when one tries to see the extent of these invariants. In this regard the question of representing cycles by submanifolds comes up. At 2 every class is representable. At odd primes there are many obstructions. (Thom).

The invariants at odd primes required more investigation because of the simple non-representing fact about cycles. The natural invariant is the signature invariant of M – the function which assigns the "signature of the intersection with M" to every closed submanifold of a tubular neighborhood of M in Euclidean space.

A natural algebraic formulation of this invariant is that of a canonical K-theory orientation

$$\triangle_M \in K\text{-homology of } M\,.$$

[1]Which according to Winkelnkemper "... is the basic discretization of a compact manifold."

In Chapter 6 we discuss this situation in the dual context of bundles. This (Alexander) duality between manifold theory and bundle theory depends on transversality and the geometric technique of surgery. The duality is sharp in the simply connected context.

Thus in this work we treat only the dual bundle theory – however motivated by questions about manifolds.

The bundle theory is homotopy theoretical and amenable to the arithmetic discussions in the first Chapters. This discussion concerns the problem of "tensoring homotopy theory" with various rings. Most notable are the cases when $\mathbb{Z}$ is replaced by the rationals $\mathbb{Q}$ or the p-adic integers $\hat{\mathbb{Z}}_p$.

These localization processes are motivated in part by the 'invariants discussion' above. The geometric questions do not however motivate going as far as the p-adic integers.[2]

One is led here by Adams' work on fibre homotopy equivalences between vector bundles – which is certainly germane to the manifold questions above. Adams finds that a certain basic homotopy relation should hold between vector bundles related by his famous operations ψ^k.

Adams proves that this relation is *universal* (if it holds at all) – a very provocative state of affairs.

Actually Adams states infinitely many relations – one for each prime p. Each relation has information at every prime not equal to p.

At this point Quillen noticed that the Adams conjecture has an analogue in characteristic p which is immediately provable. He suggested that the etale homotopy of mod p algebraic varieties be used to decide the topological Adams conjecture.

Meanwhile, the Adams conjecture for vector bundles was seen to influence the structure of piecewise linear and topological theories.

The author tried to find some topological or geometric understanding of Adams' phenomenon. What resulted was a reformulation which can be proved just using the existence of an algebraic

[2] Although the Hasse-Minkowski theorem on quadratic forms should do this.

construction of the finite cohomology of an algebraic variety (etale theory).

This picture which can only be described in the context of the p-adic integers is the following – in the p-adic context the theory of vector bundles *in each dimension* has a natural group of symmetries.

These symmetries in the $(n-1)$ dimensional theory provide canonical fibre homotopy equivalence in the n dimensional theory which more than prove the assertion of Adams. In fact each orbit of the action has a well defined (unstable) fibre homotopy type.

The symmetry in these vector bundle theories is the Galois symmetry of the roots of unity homotopy theoretically realized in the 'Čech nerves' of algebraic coverings of Grassmannians.

The symmetry extends to K-theory and a dense subset of the symmetries may be identified with the "isomorphic part of the Adams operations". We note however that this identification is not essential in the development of consequences of the Galois phenomena. The fact that certain complicated expressions in exterior powers of vector bundles give good operations in K-theory is more a testament to Adams' ingenuity than to the ultimate naturality of this viewpoint.

The Galois symmetry (because of the K-theory formulation of the signature invariant) extends to combinatorial theory and even topological theory (because of the triangulation theorems of Kirby-Siebenmann). This symmetry can be combined with the periodicity of geometric topology to extend Adams' program in several ways –

i) the homotopy relation implied by conjugacy under the action of the Galois group holds in the topological theory and is also *universal* there.

ii) an explicit calculation of the effect of the Galois group on the topology can be made –

for vector bundles E the signature invariant has an analytical description,

$$\triangle_E \text{ in } K_C(E)\,,$$

and the topological type of E is measured by the effect of the Galois group on this invariant.

One consequence is that two different vector bundles which are fixed by elements of finite order in the Galois group are also topologically distinct. For example, at the prime 3 the torsion subgroup is generated by complex conjugation – thus any pair of non isomorphic vector bundles are topologically distinct at 3.

The periodicity alluded to is that in the theory of fibre homotopy equivalences between PL or topological bundles (see Chapter 6 - Normal Invariants).

For odd primes this theory is isomorphic to K-theory, and geometric periodicity becomes Bott periodicity. (For non-simply connected manifolds the periodicity finds beautiful algebraic expression in the surgery groups of C. T. C. Wall.)

To carry out the discussion of Chapter 6 we need the works of the first five chapters.

The main points are contained in chapters 3 and 5.

In chapter 3 a description of the p-adic completion of a homotopy type is given. The resulting object is a homotopy type with the extra structure[3] of a compact topology on the contravariant functor it determines.

The p-adic types one for each p can be combined with a rational homotopy type (Chapter 2) to build a classical homotopy type.

One point about these p-adic types is that they often have symmetry which is not apparent or does not exist in the classical context. For example in Chapter 4 where p-adic spherical fibrations are discussed, we find from the extra symmetry in $\mathbb{C}\mathbb{P}^\infty$, p-adically completed, one can construct a theory of principal spherical fibrations (one for each divisor of $p-1$).

Another point about p-adic homotopy types is that they can be naturally constructed from the Grothendieck theory of etale cohomology in algebraic geometry. The long chapter 5 concerns this etale theory which we explicate using the Čech like construction of Lubkin. This construction has geometric appeal and content and should yield many applications in geometric homotopy theory.[4]

[3]which is "intrinsic" to the homotopy type in the sense of interest here.

[4]The study of homotopy theory that has geometric significance by geometrical qua homotopy theoretical methods.

To form these p-adic homotopy types we use the inverse limit technique of Chapter 3. The arithmetic square of Chapter 3 shows what has to be added to the etale homotopy type to give the classical homotopy type.[5]

We consider the Galois symmetry in vector bundle theory in some detail and end with an attempt to analyze "real varieties". The attempt leads to an interesting topological conjecture.

Chapter 1 gives some algebraic background and preparation for the later Chapters. It contains the examples of profinite groups in topology and algebra that concern us here.

In part II[6] we study the prime 2 and try to interpret geometrically the structure in Chapter 6 on the manifold level. We will also pursue the idea of a localized manifold – a concept which has interesting examples from algebra and geometry.

Finally, we acknowledge our debt to John Morgan of Princeton University – who mastered the lion's share of material in a few short months with one lecture of suggestions. He prepared an earlier manuscript on the beginning Chapters and I am certain this manuscript would not have appeared now (or in the recent future) without his considerable efforts.

Also, the calculations of Greg Brumfiel were psychologically invaluable in the beginning of this work. I greatly enjoyed and benefited from our conversations at Princeton in 1967 and later.

[5]Actually it is a beginning.
[6]which was never written (AAR).

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Chapter 1

ALGEBRAIC CONSTRUCTIONS

We will discuss some algebraic constructions. These are localization and completion of rings and groups. We consider properties of each and some connections between them.

Localization

Unless otherwise stated rings will have units and be integral domains.

Let R be a ring. $S \subseteq R - \{0\}$ is a multiplicative subset if $1 \in S$ and $a, b \in S$ implies $a \cdot b \in S$.

DEFINITION 1.1 *If $S \subseteq R - \{0\}$ is a multiplicative subset then*

$$S^{-1}R, \quad \text{“}R \text{ localized away from } S\text{”}$$

is defined as equivalence classes

$$\{r/s \mid r \in R, s \in S\}$$

where

$$r/s \sim r'/s' \quad \text{iff} \quad rs' = r's\,.$$

$S^{-1}R$ is made into a ring by defining

$$[r/s] \cdot [r'/s'] = [rr'/ss'] \text{ and}$$

$$[r/s] + [r'/s'] = \left[\frac{rs' + sr'}{ss}\right].$$

The *localization homomorphism*

$$R \to S^{-1}R$$

sends r into $[r/1]$.

EXAMPLE 1 If $p \subset R$ is a prime ideal, $R - p$ is a multiplicative subset. Define

$$R_p, \text{ “}R \text{ localized at } p\text{”}$$

as $(R-p)^{-1}R$.

In R_p every element outside pR_p is invertible. The localization map $R \to R_p$ sends p into the unique maximal ideal of non-units in R_p.

If R is an integral domain 0 is a prime ideal, and R localized at zero is the field of quotients of R.

The localization of the ring R extends to the theory of modules over R. If M is an R-module, define the localized $S^{-1}R$-module, $S^{-1}M$ by

$$S^{-1}M = M \otimes_R S^{-1}R.$$

Intuitively $S^{-1}M$ is obtained by making all the operations on M by elements of S into isomorphisms.

Interesting examples occur in topology.

EXAMPLE 2 (P. A. Smith, A. Borel, G. Segal) Let X be a locally compact polyhedron with a symmetry of order 2 (involution), T.

What is the relation between the homology of the subcomplex of fixed points F and the “homology of the pair (X, T)”?

Let S denote the (contractible) infinite dimensional sphere with its antipodal involution. Then $X \times S$ has the diagonal fixed point free involution and there is an equivariant homotopy class of maps

$$X \times S \to S$$

(which is unique up to equivariant homotopy). This gives a map

$$X_T \equiv (X \times S)/T \to S/T \equiv \mathbb{R}\,\mathbb{P}^\infty$$

and makes the "equivariant cohomology of (X, T)"

$$H^*(X_T; \mathbb{Z}/2)$$

into an R-module, where

$$R \;=\; \mathbb{Z}_2\,[x] = H^*(\mathbb{R}\,\mathbb{P}^\infty\,; \mathbb{Z}/2)\,.$$

In R we have the multiplicative set S generated by x, and the cohomology of the fixed points with coefficients in the ring $S^{-1}R = R_x = R[x^{-1}]$ is just the localized equivariant cohomology,

$$H^*(F; R_x) \cong H^*(X_T; \mathbb{Z}/2) \text{ with } x \text{ inverted} \equiv H^*(X_T; \mathbb{Z}/2) \otimes_R R_x\,.$$

For most of our work we do not need this general situation of localization. We will consider most often the case where R is the ring of integers and the R-modules are arbitrary Abelian groups.

Let ℓ be a set of primes in $\mathbb{Z}$. We will write "$\mathbb{Z}$ localized at ℓ"

$$\mathbb{Z}_\ell = S^{-1}\,\mathbb{Z}$$

where S is the multiplicative set generated by the primes *not* in ℓ.

When ℓ contains only one prime $\ell = \{p\}$, we can write

$$\mathbb{Z}_\ell = \mathbb{Z}_p$$

since $\mathbb{Z}_\ell$ is just the localization of the integers at the prime ideal p.

Other examples are

$$\mathbb{Z}_{\{\text{all primes}\}} = \mathbb{Z} \;\text{ and }\; \mathbb{Z}_\emptyset = \mathbb{Q} = \mathbb{Z}_0\ .$$

In general, it is easy to see that the collection of $\mathbb{Z}_\ell$'s

$$\{\mathbb{Z}_\ell\}$$

is just the collection of subrings of $\mathbb{Q}$ with unit. We will see below that the tensor product over $\mathbb{Z}$,

$$\mathbb{Z}_\ell \otimes_\mathbb{Z} \mathbb{Z}_{\ell'} \cong \mathbb{Z}_{\ell \cap \ell'}$$