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LEONARD I. SCHIFF, *Consulting Editor*

## METHODS OF THEORETICAL PHYSICS

# METHODS OF THEORETICAL PHYSICS

*Philip M. Morse*

PROFESSOR OF PHYSICS  
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

*Herman Feshbach*

ASSOCIATE PROFESSOR OF PHYSICS  
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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## METHODS OF THEORETICAL PHYSICS

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