

**Second Edition**

**An Introduction  
to Probability Theory  
and its  
Applications**

**Volume 2**

# An Introduction to Probability Theory and Its Applications

WILLIAM FELLER (1906-1970)

*Eugene Higgins Professor of Mathematics*

*Princeton University*

VOLUME II

SECOND EDITION

John Wiley & Sons, Inc.

New York · London · Sydney · Toronto

# An Introduction to Probability Theory and Its Applications

WILLIAM FELLER (1906-1970)

Professor of Mathematics  
University of Minnesota

WILLIAM FELLER (1906-1970)

Professor of Mathematics  
University of Minnesota

Copyright © 1966, 1971 by John Wiley & Sons, Inc.

All rights reserved. Published simultaneously in Canada.

No part of this book may be reproduced by any means,  
nor transmitted, nor translated into a machine  
language without the written permission of the  
publisher.

Library of Congress Catalogue Card Number: 57-10805

ISBN 0-471-257095

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

## Preface to the First Edition

AT THE TIME THE FIRST VOLUME OF THIS BOOK WAS WRITTEN (BETWEEN 1941 and 1948) the interest in probability was not yet widespread. Teaching was on a very limited scale and topics such as Markov chains, which are now extensively used in several disciplines, were highly specialized chapters of pure mathematics. The first volume may therefore be likened to an all-purpose travel guide to a strange country. To describe the nature of probability it had to stress the mathematical content of the theory as well as the surprising variety of potential applications. It was predicted that the ensuing fluctuations in the level of difficulty would limit the usefulness of the book. In reality it is widely used even today, when its novelty has worn off and its attitude and material are available in newer books written for special purposes. The book seems even to acquire new friends. The fact that laymen are not deterred by passages which proved difficult to students of mathematics shows that the level of difficulty cannot be measured objectively; it depends on the type of information one seeks and the details one is prepared to skip. The traveler often has the choice between climbing a peak or using a cable car.

In view of this success the second volume is written in the same style. It involves harder mathematics, but most of the text can be read on different levels. The handling of measure theory may illustrate this point. Chapter IV contains an informal introduction to the basic ideas of measure theory and the conceptual foundations of probability. The same chapter lists the few facts of measure theory used in the subsequent chapters to formulate analytical theorems in their simplest form and to avoid futile discussions of regularity conditions. The main function of measure theory in this connection is to justify formal operations and passages to the limit that would never be questioned by a non-mathematician. Readers interested primarily in practical results will therefore not feel any need for measure theory.

To facilitate access to the individual topics the chapters are rendered as self-contained as possible, and sometimes special cases are treated separately ahead of the general theory. Various topics (such as stable distributions and renewal theory) are discussed at several places from different angles. To avoid repetitions, the definitions and illustrative examples are collected in



chapter VI, which may be described as a collection of introductions to the subsequent chapters. The skeleton of the book consists of chapters V, VIII, and XV. The reader will decide for himself how much of the preparatory chapters to read and which excursions to take.

Experts will find new results and proofs, but more important is the attempt to consolidate and unify the general methodology. Indeed, certain parts of probability suffer from a lack of coherence because the usual grouping and treatment of problems depend largely on accidents of the historical development. In the resulting confusion closely related problems are not recognized as such and simple things are obscured by complicated methods. Considerable simplifications were obtained by a systematic exploitation and development of the best available techniques. This is true in particular for the proverbially messy field of limit theorems (chapters XVI–XVII). At other places simplifications were achieved by treating problems in their natural context. For example, an elementary consideration of a particular random walk led to a generalization of an asymptotic estimate which had been derived by hard and laborious methods in risk theory (and under more restrictive conditions independently in queuing).

I have tried to achieve mathematical rigor without pedantry in style. For example, the statement that  $1/(1 + \xi^2)$  is the characteristic function of  $\frac{1}{2}e^{-|x|}$  seems to me a desirable and legitimate abbreviation for the logically correct version that the function which at the point  $\xi$  assumes the value  $1/(1 + \xi^2)$  is the characteristic function of the function which at the point  $x$  assumes the value  $\frac{1}{2}e^{-|x|}$ .

I fear that the brief historical remarks and citations do not render justice to the many authors who contributed to probability, but I have tried to give credit wherever possible. The original work is now in many cases superseded by newer research, and as a rule full references are given only to papers to which the reader may want to turn for additional information. For example, no reference is given to my own work on limit theorems, whereas a paper describing observations or theories underlying an example is cited even if it contains no mathematics.<sup>1</sup> Under these circumstances the index of authors gives no indication of their importance for probability theory. Another difficulty is to do justice to the pioneer work to which we owe new directions of research, new approaches, and new methods. Some theorems which were considered strikingly original and deep now appear with simple proofs among more refined results. It is difficult to view such a theorem in its historical perspective and to realize that here as elsewhere it is the first step that counts.

<sup>1</sup> This system was used also in the first volume but was misunderstood by some subsequent writers; they now attribute the methods used in the book to earlier scientists who could not have known them.

## ACKNOWLEDGMENTS

Thanks to the support by the U.S. Army Research Office of work in probability at Princeton University I enjoyed the help of J. Goldman, L. Pitt, M. Silverstein, and, in particular, of M. M. Rao. They eliminated many inaccuracies and obscurities. All chapters were rewritten many times and preliminary versions of the early chapters were circulated among friends. In this way I benefited from comments by J. Elliott, R. S. Pinkham, and L. J. Savage. My special thanks are due to J. L. Doob and J. Wolfowitz for advice and criticism. The graph of the Cauchy random walk was supplied by H. Trotter. The printing was supervised by Mrs. H. McDougal, and the appearance of the book owes much to her.

WILLIAM FELLER

October 1965

At a number of places the exposition was simplified by streamlined (and sometimes new) arguments. Some new material has been incorporated into the text.

While writing the first edition I was haunted by the fear of an excessively long volume. Unfortunately, this led me to spend futile months in shortening the original text and economizing on displays. This damage has now been repaired, and a great effort has been spent to make the reading easier. Occasional repetitions will also facilitate a direct access to the individual chapters and make it possible to read certain parts of this book in conjunction with Volume I.

Concerning the organization of the material, see the introduction to the first edition (repeated here) starting with the second paragraph.

I am grateful to many readers for pointing out errors or omissions. I especially thank D. A. Hoggal, of Chicago, for an exhaustive and penetrating list of errata and for suggestions covering the entire book.

January 1970  
Princeton, N.J.

WILLIAM FELLER

THE MANUSCRIPT HAD BEEN FINISHED AT THE TIME OF THE AUTHOR'S DEATH but no proofs had been received. I am grateful to the publisher for providing a proofreader to compare the print against the manuscript and for compiling the index. J. Goldman, A. Grunbaum, H. McKean, L. Pitt, and A. Pittenger divided the book *among* themselves to check on the mathematics. Every mathematician knows what an incredible amount of work that entails. *I express my deep gratitude to these men and extend my heartfelt thanks for their labor of love.*

May 1970

CLARA N. FELLER

I have tried to achieve mathematical rigor without pedantry in style. For example, the statement that  $1/(1+z^2)$  is the characteristic function of  $1/2$  seems to me a desirable and legitimate abbreviation for the logically correct version that the function which at the point  $z$  assumes the value  $1/(1+z^2)$  is the characteristic function of the function which at the point  $z$  assumes the value  $1/2$ .

I fear that the brief historical remarks and citations do not render justice to the many authors who contributed to probability, but I have tried to give credit wherever possible. The subject of probability has in many cases superseded by newer research, and as a rule I have given only to papers to which the reader may want to turn for further information. For example, no reference is given to my own work on  $h$ -processes, whereas a paper describing observations or theories underlying an example is cited even if it contains no mathematics. Under these circumstances the index of authors gives no indication of their importance for probability theory. Another difficulty is to do justice to the pioneer work to which we owe new directions of research and new methods. Some theorems which were considered obvious at the time and which now stand with simple proofs among those which are obvious, have been included in the index in its historical perspective. It is not always clear where it is the first step that counts.

This volume was written in a time when the author was still a young man, and it is possible that they now contain the errors of youth. The author is grateful to those who could and have known them.

## Introduction

THE CHARACTER AND ORGANIZATION OF THE BOOK REMAIN UNCHANGED, BUT the entire text has undergone a thorough revision. Many parts (Chapter XVII, in particular) have been completely rewritten and a few new sections have been added. At a number of places the exposition was simplified by streamlined (and sometimes new) arguments. Some new material has been incorporated into the text.

While writing the first edition I was haunted by the fear of an excessively long volume. Unfortunately, this led me to spend futile months in shortening the original text and economizing on displays. This damage has now been repaired, and a great effort has been spent to make the reading easier. Occasional repetitions will also facilitate a direct access to the individual chapters and make it possible to read certain parts of this book in conjunction with Volume 1.

Concerning the organization of the material, see the introduction to the first edition (repeated here), starting with the second paragraph.

I am grateful to many readers for pointing out errors or omissions. I especially thank D. A. Hejhal, of Chicago, for an exhaustive and penetrating list of errata and for suggestions covering the entire book.

January 1970  
Princeton, N.J.

WILLIAM FELLER



## Abbreviations and Conventions

*Iff* is an abbreviation for *if and only if*.

*Epoch*. This term is used for points on the time axis, while time is reserved for intervals and durations. (In discussions of stochastic processes the word "times" carries too heavy a burden. The systematic use of "epoch," introduced by J. Riordan, seems preferable to varying substitutes such as moment, instant, or point.)

*Intervals* are denoted by bars:  $\overline{a, b}$  is an open,  $\overline{a, b}$  a closed interval; half-open intervals are denoted by  $\overline{a, b}$  and  $\overline{a, b}$ . This notation is used also in higher dimensions. The pertinent conventions for vector notations and order relations are found in V,1 (and also in IV,2). The symbol  $(a, b)$  is reserved for pairs and for points.

$\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^r$  stand for the line, the plane, and the  $r$ -dimensional Cartesian space.

**1** refers to volume one, Roman numerals to chapters. Thus **1**; XI,(3.6) refers to section 3 of chapter XI of volume **1**.

► indicates the end of a proof or of a collection of examples.

$n$  and  $\mathcal{N}$  denote, respectively, the normal density and distribution function with zero expectation and unit variance.

$O, o$ , and  $\sim$ . Let  $u$  and  $v$  depend on a parameter  $x$  which tends, say, to  $a$ . Assuming that  $v$  is positive we write

$$\left. \begin{array}{l} u = O(v) \\ u = o(v) \\ u \sim v \end{array} \right\} \quad \text{if} \quad \frac{u}{v} \quad \left\{ \begin{array}{l} \text{remains bounded} \\ \rightarrow 0 \\ \rightarrow 1. \end{array} \right.$$

$\int f(x) U\{dx\}$ . For this abbreviation see V,3.

Regarding Borel sets and Baire functions, see the introduction to chapter V.

# Contents

## CHAPTER

### I THE EXPONENTIAL AND THE UNIFORM DENSITIES

1. Introduction	1
2. Densities. Convolutions	3
3. The Exponential Density	8
4. Waiting Time Paradoxes. The Poisson Process	11
5. The Persistence of Bad Luck	15
6. Waiting Times and Order Statistics	17
7. The Uniform Distribution	21
8. Random Splittings	25
9. Convolutions and Covering Theorems	26
10. Random Directions	29
11. The Use of Lebesgue Measure	33
12. Empirical Distributions	36
13. Problems for Solution	39

## CHAPTER

### II SPECIAL DENSITIES. RANDOMIZATION

1. Notations and Conventions	45
2. Gamma Distributions	47
*3. Related Distributions of Statistics	48
4. Some Common Densities	49
5. Randomization and Mixtures	53
6. Discrete Distributions	55

\* Starred sections are not required for the understanding of the sequel and should be omitted at first reading.

7. Bessel Functions and Random Walks . . . . .	58
8. Distributions on a Circle . . . . .	61
9. Problems for Solution . . . . .	64

CHAPTER

III DENSITIES IN HIGHER DIMENSIONS. NORMAL DENSITIES AND PROCESSES . . . . .	66
1. Densities . . . . .	66
2. Conditional Distributions. . . . .	71
3. Return to the Exponential and the Uniform Distributions . . . . .	74
*4. A Characterization of the Normal Distribution . . . . .	77
5. Matrix Notation. The Covariance Matrix . . . . .	80
6. Normal Densities and Distributions . . . . .	83
*7. Stationary Normal Processes . . . . .	87
8. Markovian Normal Densities. . . . .	94
9. Problems for Solution . . . . .	99

CHAPTER

IV PROBABILITY MEASURES AND SPACES. . . . .	103
1. Baire Functions . . . . .	104
2. Interval Functions and Integrals in $\mathbb{R}^n$ . . . . .	106
3. $\sigma$ -Algebras. Measurability . . . . .	112
4. Probability Spaces. Random Variables . . . . .	115
5. The Extension Theorem . . . . .	118
6. Product Spaces. Sequences of Independent Variables. . . . .	121
7. Null Sets. Completion . . . . .	125

CHAPTER

V PROBABILITY DISTRIBUTIONS IN $\mathbb{R}^n$ . . . . .	127
1. Distributions and Expectations . . . . .	128
2. Preliminaries . . . . .	136
3. Densities . . . . .	138
4. Convolutions . . . . .	143

5. Symmetrization. . . . .	148
6. Integration by Parts. Existence of Moments . . . . .	150
7. Chebyshev's Inequality . . . . .	151
8. Further Inequalities. Convex Functions . . . . .	152
9. Simple Conditional Distributions. Mixtures . . . . .	156
*10. Conditional Distributions. . . . .	160
*11. Conditional Expectations. . . . .	162
12. Problems for Solution . . . . .	165

CHAPTER

VI. A SURVEY OF SOME IMPORTANT DISTRIBUTIONS AND PROCESSES	169
1. Stable Distributions in $\mathcal{R}^1$ . . . . .	169
2. Examples . . . . .	173
3. Infinitely Divisible Distributions in $\mathcal{R}^1$ . . . . .	176
4. Processes with Independent Increments . . . . .	179
*5. Ruin Problems in Compound Poisson Processes . . . . .	182
6. Renewal Processes . . . . .	184
7. Examples and Problems . . . . .	187
8. Random Walks. . . . .	190
9. The Queuing Process . . . . .	194
10. Persistent and Transient Random Walks . . . . .	200
11. General Markov Chains . . . . .	205
*12. Martingales. . . . .	209
13. Problems for Solution . . . . .	215

CHAPTER

VII. LAWS OF LARGE NUMBERS. APPLICATIONS IN ANALYSIS . . . . .	219
1. Main Lemma and Notations . . . . .	219
2. Bernstein Polynomials. Absolutely Monotone Functions . . . . .	222
3. Moment Problems . . . . .	224
*4. Application to Exchangeable Variables . . . . .	228
*5. Generalized Taylor Formula and Semi-Groups . . . . .	230
6. Inversion Formulas for Laplace Transforms . . . . .	232



*7. Laws of Large Numbers for Identically Distributed Variables . . . . .	234
*8. Strong Laws . . . . .	237
*9. Generalization to Martingales . . . . .	241
10. Problems for Solution . . . . .	244

## CHAPTER

VIII THE BASIC LIMIT THEOREMS . . . . .	247
1. Convergence of Measures . . . . .	247
2. Special Properties . . . . .	252
3. Distributions as Operators . . . . .	254
4. The Central Limit Theorem . . . . .	258
*5. Infinite Convolutions . . . . .	265
6. Selection Theorems . . . . .	267
*7. Ergodic Theorems for Markov Chains . . . . .	270
8. Regular Variation . . . . .	275
*9. Asymptotic Properties of Regularly Varying Functions . . . . .	279
10. Problems for Solution . . . . .	284

## CHAPTER

IX INFINITELY DIVISIBLE DISTRIBUTIONS AND SEMI-GROUPS . . . . .	290
1. Orientation . . . . .	290
2. Convolution Semi-Groups . . . . .	293
3. Preparatory Lemmas . . . . .	296
4. Finite Variances . . . . .	298
5. The Main Theorems . . . . .	300
6. Example: Stable Semi-Groups . . . . .	305
7. Triangular Arrays with Identical Distributions . . . . .	308
8. Domains of Attraction . . . . .	312
9. Variable Distributions. The Three-Series Theorem . . . . .	316
10. Problems for Solution . . . . .	318

## CHAPTER

X	MARKOV PROCESSES AND SEMI-GROUPS	321
1.	The Pseudo-Poisson Type	322
2.	A Variant: Linear Increments	324
3.	Jump Processes	326
4.	Diffusion Processes in $\mathbb{R}^1$	332
5.	The Forward Equation. Boundary Conditions	337
6.	Diffusion in Higher Dimensions	344
7.	Subordinated Processes	345
8.	Markov Processes and Semi-Groups	349
9.	The "Exponential Formula" of Semi-Group Theory	353
10.	Generators. The Backward Equation	356

## CHAPTER

XI	RENEWAL THEORY	358
1.	The Renewal Theorem	358
2.	Proof of the Renewal Theorem	364
*3.	Refinements	366
4.	Persistent Renewal Processes	368
5.	The Number $N_t$ of Renewal Epochs	372
6.	Terminating (Transient) Processes	374
7.	Diverse Applications	377
8.	Existence of Limits in Stochastic Processes	379
*9.	Renewal Theory on the Whole Line	380
10.	Problems for Solution	385

## CHAPTER

XII	RANDOM WALKS IN $\mathbb{R}^1$	389
1.	Basic Concepts and Notations	390
2.	Duality. Types of Random Walks	394
3.	Distribution of Ladder Heights. Wiener-Hopf Factorization	398
3a.	The Wiener-Hopf Integral Equation	402

4. Examples . . . . .	404
5. Applications . . . . .	408
6. A Combinatorial Lemma . . . . .	412
7. Distribution of Ladder Epochs . . . . .	413
8. The Arc Sine Laws . . . . .	417
9. Miscellaneous Complements . . . . .	423
10. Problems for Solution . . . . .	425

## CHAPTER

## XIII LAPLACE TRANSFORMS. TAUBERIAN THEOREMS. RESOLVENTS . 429

1. Definitions. The Continuity Theorem . . . . .	429
2. Elementary Properties . . . . .	434
3. Examples . . . . .	436
4. Completely Monotone Functions. Inversion Formulas . . . . .	439
5. Tauberian Theorems . . . . .	442
*6. Stable Distributions . . . . .	448
*7. Infinitely Divisible Distributions . . . . .	449
*8. Higher Dimensions . . . . .	452
9. Laplace Transforms for Semi-Groups . . . . .	454
10. The Hille-Yosida Theorem . . . . .	458
11. Problems for Solution . . . . .	463

## CHAPTER

## XIV APPLICATIONS OF LAPLACE TRANSFORMS . 466

1. The Renewal Equation: Theory . . . . .	466
2. Renewal-Type Equations: Examples . . . . .	468
3. Limit Theorems Involving Arc Sine Distributions . . . . .	470
4. Busy Periods and Related Branching Processes . . . . .	473
5. Diffusion Processes . . . . .	475
6. Birth-and-Death Processes and Random Walks . . . . .	479
7. The Kolmogorov Differential Equations . . . . .	483
8. Example: The Pure Birth Process . . . . .	488
9. Calculation of Ergodic Limits and of First-Passage Times . . . . .	491
10. Problems for Solution . . . . .	495

CHAPTER

<b>XV</b>	<b>CHARACTERISTIC FUNCTIONS</b>	498
1.	Definition. Basic Properties	498
2.	Special Distributions. Mixtures	502
2a.	Some Unexpected Phenomena	505
3.	Uniqueness. Inversion Formulas	507
4.	Regularity Properties	511
5.	The Central Limit Theorem for Equal Components	515
6.	The Lindeberg Conditions	518
7.	Characteristic Functions in Higher Dimensions	521
*8.	Two Characterizations of the Normal Distribution	525
9.	Problems for Solution	526

CHAPTER

<b>XVI*</b>	<b>EXPANSIONS RELATED TO THE CENTRAL LIMIT THEOREM</b>	531
1.	Notations	532
2.	Expansions for Densities	533
3.	Smoothing	536
4.	Expansions for Distributions	538
5.	The Berry-Esséen Theorems	542
6.	Expansions in the Case of Varying Components	546
7.	Large Deviations	548

CHAPTER

<b>XVII</b>	<b>INFINITELY DIVISIBLE DISTRIBUTIONS</b>	554
1.	Infinitely Divisible Distributions	554
2.	Canonical Forms. The Main Limit Theorem	558
2a.	Derivatives of Characteristic Functions	565
3.	Examples and Special Properties	566
4.	Special Properties	570
5.	Stable Distributions and Their Domains of Attraction	574
*6.	Stable Densities	581
7.	Triangular Arrays	583



*8. The Class $L$ . . . . .	588
*9. Partial Attraction. "Universal Laws" . . . . .	590
*10. Infinite Convolutions . . . . .	592
11. Higher Dimensions . . . . .	593
12. Problems for Solution . . . . .	595

## CHAPTER

<b>XVIII APPLICATIONS OF FOURIER METHODS TO RANDOM WALKS</b> . . . . .	598
1. The Basic Identity . . . . .	598
*2. Finite Intervals. Wald's Approximation . . . . .	601
3. The Wiener-Hopf Factorization . . . . .	604
4. Implications and Applications . . . . .	609
5. Two Deeper Theorems . . . . .	612
6. Criteria for Persistency . . . . .	614
7. Problems for Solution . . . . .	616

## CHAPTER

<b>XIX HARMONIC ANALYSIS</b> . . . . .	619
1. The Parseval Relation . . . . .	619
2. Positive Definite Functions . . . . .	620
3. Stationary Processes . . . . .	623
4. Fourier Series . . . . .	626
*5. The Poisson Summation Formula . . . . .	629
6. Positive Definite Sequences . . . . .	633
7. $L^2$ Theory . . . . .	635
8. Stochastic Processes and Integrals . . . . .	641
9. Problems for Solution . . . . .	647

<b>ANSWERS TO PROBLEMS</b> . . . . .	651
--------------------------------------	-----

<b>SOME BOOKS ON COGNATE SUBJECTS</b> . . . . .	655
---	-----

<b>INDEX</b> . . . . .	657
------------------------	-----