Second Edition

An Introduction to Probability Theory and its Applications

Volume 2

An Introduction to Probability Theory and Its Applications

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VOLUME II

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Preface to the First Edition

AT THE TIME THE FIRST VOLUME OF THIS BOOK WAS WRITTEN (BETWEEN 1941 and 1948) the interest in probability was not yet widespread. Teaching was on a very limited scale and topics such as Markov chains, which are now extensively used in several disciplines, were highly specialized chapters of pure mathematics. The first volume may therefore be likened to an all-purpose travel guide to a strange country. To describe the nature of probability it had to stress the mathematical content of the theory as well as the surprising variety of potential applications. It was predicted that the ensuing fluctuations in the level of difficulty would limit the usefulness of the book. In reality it is widely used even today, when its novelty has worn off and its attitude and material are available in newer books written for special purposes. The book seems even to acquire new friends. The fact that laymen are not deterred by passages which proved difficult to students of mathematics shows that the level of difficulty cannot be measured objectively; it depends on the type of information one seeks and the details one is prepared to skip. The traveler often has the choice between climbing a peak or using a cable car.

In view of this success the second volume is written in the same style. It involves harder mathematics, but most of the text can be read on different levels. The handling of measure theory may illustrate this point. Chapter IV contains an informal introduction to the basic ideas of measure theory and the conceptual foundations of probability. The same chapter lists the few facts of measure theory used in the subsequent chapters to formulate analytical theorems in their simplest form and to avoid futile discussions of regularity conditions. The main function of measure theory in this connection is to justify formal operations and passages to the limit that would never be questioned by a non-mathematician. Readers interested primarily in practical results will therefore not feel any need for measure theory.

To facilitate access to the individual topics the chapters are rendered as self-contained as possible, and sometimes special cases are treated separately ahead of the general theory. Various topics (such as stable distributions and renewal theory) are discussed at several places from different angles. To avoid repetitions, the definitions and illustrative examples are collected in chapter VI, which may be described as a collection of introductions to the subsequent chapters. The skeleton of the book consists of chapters V, VIII, and XV. The reader will decide for himself how much of the preparatory chapters to read and which excursions to take.

Experts will find new results and proofs, but more important is the attempt to consolidate and unify the general methodology. Indeed, certain parts of probability suffer from a lack of coherence because the usual grouping and treatment of problems depend largely on accidents of the historical development. In the resulting confusion closely related problems are not recognized as such and simple things are obscured by complicated methods. Considerable simplifications were obtained by a systematic exploitation and development of the best available techniques. This is true in particular for the proverbially messy field of limit theorems (chapters XVI–XVII). At other places simplifications were achieved by treating problems in their natural context. For example, an elementary consideration of a particular random walk led to a generalization of an asymptotic estimate which had been derived by hard and laborious methods in risk theory (and under more restrictive conditions independently in queuing).

I have tried to achieve mathematical rigor without pedantry in style. For example, the statement that $1/(1 + \xi^2)$ is the characteristic function of $\frac{1}{2}e^{-|x|}$ seems to me a desirable and legitimate abbreviation for the logically correct version that the function which at the point ξ assumes the value $1/(1 + \xi^2)$ is the characteristic function of the function which at the point x assumes the value $\frac{1}{2}e^{-|x|}$.

I fear that the brief historical remarks and citations do not render justice to the many authors who contributed to probability, but I have tried to give credit wherever possible. The original work is now in many cases superseded by newer research, and as a rule full references are given only to papers to which the reader may want to turn for additional information. For example, no reference is given to my own work on limit theorems, whereas a paper describing observations or theories underlying an example is cited even if it contains no mathematics.[1] Under these circumstances the index of authors gives no indication of their importance for probability theory. Another difficulty is to do justice to the pioneer work to which we owe new directions of research, new approaches, and new methods. Some theorems which were considered strikingly original and deep now appear with simple proofs among more refined results. It is difficult to view such a theorem in its historical perspective and to realize that here as elsewhere it is the first step that counts.

[1] This system was used also in the first volume but was misunderstood by some subsequent writers; they now attribute the methods used in the book to earlier scientists who could not have known them.

ACKNOWLEDGMENTS

Thanks to the support by the U.S. Army Research Office of work in probability at Princeton University I enjoyed the help of J. Goldman, L. Pitt, M. Silverstein, and, in particular, of M. M. Rao. They eliminated many inaccuracies and obscurities. All chapters were rewritten many times and preliminary versions of the early chapters were circulated among friends. In this way I benefited from comments by J. Elliott, R. S. Pinkham, and L. J. Savage. My special thanks are due to J. L. Doob and J. Wolfowitz for advice and criticism. The graph of the Cauchy random walk was supplied by H. Trotter. The printing was supervised by Mrs. H. McDougal, and the appearance of the book owes much to her.

WILLIAM FELLER

October 1965

THE MANUSCRIPT HAD BEEN FINISHED AT THE TIME OF THE AUTHOR'S DEATH but no proofs had been received. I am grateful to the publisher for providing a proofreader to compare the print against the manuscript and for compiling the index. J. Goldman, A. Grunbaum, H. McKean, L. Pitt, and A. Pittenger divided the book *among* themselves to check on the mathematics. Every mathematician knows what an incredible amount of work that entails. *I express my deep gratitude to these men and extend my heartfelt thanks for their labor of love.*

May 1970 CLARA N. FELLER

Introduction

THE CHARACTER AND ORGANIZATION OF THE BOOK REMAIN UNCHANGED, BUT the entire text has undergone a thorough revision. Many parts (Chapter XVII, in particular) have been completely rewritten and a few new sections have been added. At a number of places the exposition was simplified by streamlined (and sometimes new) arguments. Some new material has been incorporated into the text.

While writing the first edition I was haunted by the fear of an excessively long volume. Unfortunately, this led me to spend futile months in shortening the original text and economizing on displays. This damage has now been repaired, and a great effort has been spent to make the reading easier. Occasional repetitions will also facilitate a direct access to the individual chapters and make it possible to read certain parts of this book in conjunction with Volume 1.

Concerning the organization of the material, see the introduction to the first edition (repeated here), starting with the second paragraph.

I am grateful to many readers for pointing out errors or omissions. I especially thank D. A. Hejhal, of Chicago, for an exhaustive and penetrating list of errata and for suggestions covering the entire book.

January 1970
Princeton, N.J.

WILLIAM FELLER

Abbreviations and Conventions

Iff is an abbreviation for *if and only if.*

Epoch. This term is used for points on the time axis, while time is reserved for intervals and durations. (In discussions of stochastic processes the word "times" carries too heavy a burden. The systematic use of "epoch," introduced by J. Riordan, seems preferable to varying substitutes such as moment, instant, or point.)

Intervals are denoted by bars: $\overline{a, b}$ is an open, $\overset{\vdash\!\dashv}{a, b}$ a closed interval; half-open intervals are denoted by $\overset{\!\!-\!\!\dashv}{a, b}$ and $\overset{\vdash\!\!-}{a, b}$. *This notation is used also in higher dimensions.* The pertinent conventions for vector notations and order relations are found in V,1 (and also in IV,2). The symbol (a, b) is reserved for pairs and for points.

$\mathfrak{R}^1, \mathfrak{R}^2, \mathfrak{R}^r$ stand for the line, the plane, and the r-dimensional Cartesian space.

1 refers to volume one, Roman numerals to chapters. Thus **1**; XI,(3.6) refers to section 3 of chapter XI of volume **1**.

▶ indicates the end of a proof or of a collection of examples.

$\mathfrak{n}$ *and* $\mathfrak{N}$ denote, respectively, the normal density and distribution function with zero expectation and unit variance.

O, o, *and* $\sim$. Let u and v depend on a parameter x which tends, say, to a. Assuming that v is positive we write

$$\left.\begin{array}{l} u = O(v) \\ u = o(v) \\ u \sim v \end{array}\right\} \quad \textit{if} \quad \frac{u}{v} \quad \left\{\begin{array}{l} \textit{remains bounded} \\ \to 0 \\ \to 1. \end{array}\right.$$

$f(x)\, U\{dx\}$. For this abbreviation see V,3.

Regarding Borel sets and Baire functions, see the introduction to chapter V.

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* Starred sections are not required for the understanding of the sequel and should be omitted at first reading.