Differential Analysis

Papers presented at the Bombay Colloquium, 1964, by

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REPORT

An International Colloquium on Differential Analysis was held at the Tata Institute of Fundamental Research, Bombay, on 7-14 January 1964. The Colloquium was a closed meeting of experts and of others seriously interested in differential analysis. It was attended by 23 members, and 26 other participants, from France, India, Japan, the Netherlands, Sweden, Switzerland, the United Kingdom, and the United States.

The Colloquium was jointly sponsored, and financially supported, by the International Mathematical Union, the Sir Dorabji Tata Trust, and the Tata Institute of Fundamental Research. An Organizing Committee consisting of Professor K. Chandrasekharan (Chairman), Professor K. G. Ramanathan, Professor M. S. Narasimhan, Professor Raghavan Narasimhan, Professor G. de Rham, and Professor D. Montgomery was in charge of the scientific programme. Professor de Rham and Professor Montgomery acted as representatives of the Union on the Organizing Committee. The purpose of the Colloquium was to discuss recent developments in some aspects of (i) the theory of differential equations, (ii) analysis in the large and differential geometry, and (iii) differential topology.

The following nineteen mathematicians accepted invitations to address the Colloquium :

Professor M. F. Atiyah, Professor R. Bott, Professor L. Gårding, Professor L. Hörmander, Professor J. J. Kohn, Professor B. Malgrange, Professor Y. Matsushima, Professor J. W. Milnor, Professor D. Montgomery, Professor C. B. Morrey, Jr., Professor J. K. Moser, Professor M. S. Narasimhan, Mr. M. S. Raghunathan, Professor G. de Rham, Professor C. S. Seshadri, Professor S. Smale, Professor D. C. Spencer, Professor R. Thom and Professor A. Van de Ven.

Professor M. Morse, who was unable to accept the invitation to attend the Colloquium, sent in a paper.

The Colloquium met in closed sessions. Eighteen lectures were given. Each lecture lasted fifty minutes, followed by discussions. Informal lectures and discussions continued during the week, outside the official programme.

The social programme during the Colloquium week included a ballet and dinner on 7 January; a show of cultural films on 8 January; a performance of Indian music on the Veena, and on the Sitar, on 9 January; a performance of classical Indian dances on 10 January; an excursion to Elephanta on 12 January; and a violin recital followed by a dinner on 13 January.

CONTENTS

CONDITIONED DIFFERENTIABLE ISOTOPIES

By WILLIAM HUEBSCH† and MARSTON MORSE‡

1. Introduction. The theorems on differentiable isotopies found in recent papers, such as [6], [5] and the "Reduction Theorems" in §3 of [1] are inadequate for the purpose of proving some of the more recent theorems in differential topology. In particular the principal theorem concerning the elimination of a pair of critical points, as stated in [2], seems to require deeper Reduction Theorems and differentiable isotopies. Theorem 1.3 of this paper is one such theorem. This paper will establish Theorem 1.3 with an appropriate background.

We refer to a euclidean n-space E_n with rectangular coordinates $x_1, \ldots, x_n$. The point $x = (x_1, \ldots, x_n)$ can be considered a vector with components equal to the respective coordinates of x. Let $\|x\|$ be the magnitude of x conceived as a vector. Corresponding to a prescribed positive constant ρ set

$$D_\rho = \{x \in E_n \mid \|x\| < \rho\}. \tag{1.0}$$

Given a subset Y of E_n set $E_n - Y = {}^cY$. Let $\mathbf{0}$ denote the origin in E_n. Let R denote the axis of reals.

For simplicity all differentiable mappings used in this paper will be differentiable of class C^∞. It is clear that this condition could be greatly relaxed.

DEFINITION. *A differentiable mapping of E_n onto E_n which leaves $\mathbf{0} \cup {}^cD_\rho$ point-wise invariant, will be termed a mapping with domain of identity $\mathbf{0} \cup {}^cD_\rho$.*

DEFINITION. *Two diffeomorphisms whose domains of definition include $\mathbf{0}$, will be said to be $\mathbf{0}$-related if their restrictions to some neighborhood of the origin are identical.*

† Work of Huebsch supported in part by the National Science Foundation under NSF-G19884.

‡ Work of Morse supported by the Air Force Office of Scientific Research under AF-AFOSR-63-357.

Theorem 3.1 of [1] can be reformulated as follows. {Cf. [5] Lemma 8.1, and [6] Lemma 3.2.}

THEOREM 1.1. *Let X be an open neighborhood of* $\mathbf{0}$, *and let* $x \to f(x)$ *be a sense-preserving diffeomorphism of X into E_n which leaves* $\mathbf{0}$ *invariant.*

Corresponding to a prescribed positive constant ρ there exists a diffeomorphism of E_n onto E_n, $\mathbf{0}$*-related to f, with domain of identity* $\mathbf{0} \cup {}^cD_\rho$.

To state an extension of Theorem 1.1 we recall a definition.

DEFINITION. *An isotopy H. Let X be an open subset of E_n. A diffeomorphism h of X into E_n will be said to be differentiably isotopic to a diffeomorphism k of X into E_n if there exists a differentiable mapping:*

$$H: X \times R \to E_n; \ (x, t) \to H(x, t) \tag{1.1}$$

such that each partial mapping:

$$x \to H(x, t) = H^t(x) \ (\text{introducing } H^t) \tag{1.2}$$

is a diffeomorphism of X into E_n, and if $H^t = h$ for $t \leqslant 0$, and $H^t = k$ for $t \geqslant 1$. We then term H a differentiable isotopy of h into k, and H^t the t-section of H.

The following extension of Theorem 3.1 of [1] is a consequence of Theorem 1.3 of this paper.

THEOREM 1.2. *Let X and f be given as in Theorem 1.1. Corresponding to a prescribed positive constant ρ there exists a diffeomorphism h of E_n onto E_n,* $\mathbf{0}$*-related to f, with domain of identity* $\mathbf{0} \cup {}^cD_\rho$, *admitting a differentiable isotopy H into the identity, such that each section H^t of H is a diffeomorphism of E_n onto E_n with domain of identity* $\mathbf{0} \cup {}^cD_\rho$.

In this paper a differentiable m-manifold Σ_m, $0 < m \leqslant n$, "in E_n" is a differentiable manifold which is *regular* and *proper* in E_n. Σ_m is proper in the sense that its topology is induced by that of E_n; it is regular in E_n if each point in Σ_m has a neighborhood N, relative to Σ_m, such that rectangular coordinates in E_n of an arbitrary point $q \in N$ are functions of class C^∞ of some m of these coordinates of q.

Definition. *An indicatrix of Σ_m at $\mathbf{0}$. Suppose that Σ_m meets the origin $\mathbf{0}$. An ordered set of m linearly independent vectors tangent to Σ_m at $\mathbf{0}$ will be called an indicatrix of Σ_m at $\mathbf{0}$. Two indicatrices of Σ_m at $\mathbf{0}$ are termed equivalent if one can be deformed, as a linearly independent ordered set of m vectors tangent to Σ_m at $\mathbf{0}$, into the other. Nonequivalent indicatrices are termed opposite.*

Definition. *The f-image of an indicatrix. Let f be a diffeomorphism into E_n of a neighborhood X of $\mathbf{0}$. If $\Sigma_m \subset X$, $f(\Sigma_m)$ is well-defined. Let*

$$(w) = (w(1), \dots, w(m))$$

be an ordered set of m contravariant vectors which define an indicatrix of Σ_m at $\mathbf{0}$. The contravariant image under f of the vectors $w(1), \dots, w(m)$, is a set

$$(w') = (w'(1), \dots, w'(m))$$

of vectors tangent to the manifold $f(\Sigma_m)$ at $\mathbf{0}$ which serves as an indicatrix of $f(\Sigma_m)$ at $\mathbf{0}$. We term (w') the f-image of (w).

It is clear that f maps equivalent indicatrices into equivalent indicatrices.

Definition. *Relative similarity of indicatrices. Let r and s be positive integers such that $r + s = n$. Let M_r, M_r^* and L_s be differentiable manifolds in E_n with dimensions r, r and s, respectively. Suppose that*

$$M_r \cap L_s = \mathbf{0}, \quad M_r^* \cap L_s = \mathbf{0},$$

and that M_r and L_s have no tangent vector in common at $\mathbf{0}$, nor M_r^* and L_s. Let

$$(w) = (w(1), \dots, w(r)) \tag{1.3}$$

$$(w^*) = (w^*(1), \dots, w^*(r)) \tag{1.4}$$

be indicatrices of M_r and M_r^*, respectively, at $\mathbf{0}$. Let

$$(\lambda) = (\lambda(1), \dots, \lambda(s)) \tag{1.5}$$

be an arbitrary indicatrix of L_s at $\mathbf{0}$.

We say that the indicatrices (w) *and* (w^*) *at* $\mathbf{0}$ *are similar relative to* L_s *if the two ensembles of vectors*

$$(\lambda(1), \ldots, \lambda(s) : w(1), \ldots, w(r)) \tag{1.6}$$

$$(\lambda(1), \ldots, \lambda(s) : w^*(1), \ldots, w^*(r)) \tag{1.7}$$

are equivalent as indicatrices of E_n *at* $\mathbf{0}$.

The property of (w) and (w^*) being similar relative to L_s, is independent of the choice of (λ) as an indicatrix of L_s at $\mathbf{0}$, and of the choice of (w) and (w^*) in equivalence classes of (w) and (w^*) respectively.

Theorem 1.3 is the principal theorem of this paper.

Data in Theorem 1.3. Let X be an open neighborhood of $\mathbf{0}$ in E_n, and L_s and M_r differentiable manifolds in X such that

$$M_r \cap L_s = \mathbf{0}, \tag{1.8}$$

with $r + s = n$ and $0 < s < n$. Suppose moreover that M_r and L_s have no tangent in common at $\mathbf{0}$.

THEOREM 1.3. *Let* f *be a sense-preserving diffeomorphism of* X *into* E_n, *leaving* $\mathbf{0}$ *fixed, and such that* (a_1) *and* (a_2) *are satisfied.*

(a_1) $L_s \cap f(M_r) = \mathbf{0}$ *and there is no tangent line common to* L_s *and* $f(M_r)$ *at* $\mathbf{0}$.

(a_2) *If* (w) *is an indicatrix of* M_r *at* $\mathbf{0}$, *and if* (w^*) *is the indicatrix of* $f(M_r)$ *at* $\mathbf{0}$ *which is the* f*-image of* (w), *then* (w) *and* (w^*) *are similar relative to* L_s.

Corresponding to a prescribed positive constant ρ *there then exists a diffeomorphism* h *of* E_n *onto* E_n, $\mathbf{0}$*-related to* f, *with domain of identity* $\mathbf{0} \cup {}^cD_\rho$, *with*

$$L_s \cap h(M_r) = \mathbf{0} \tag{1.9}$$

and such that there exists a differentiable isotopy H *of* h *into the identity on* E_n *each section* H^t *of which is a diffeomorphism of* E_n *onto* E_n *with domain of identity* $\mathbf{0} \cup {}^cD_\rho$.

The proof of Theorem 1.3 will be completed in §5.

Methods. In proving Theorem 1.3 we shall rely on two special types of diffeomorphisms of E_n onto E_n termed *ξ-diffeomorphisms* and *perispherical* diffeomorphisms. They will be sense-preserving and leave $\mathbf{0}$ invariant.

A fundamental condition on ξ-diffeomorphisms will be that they "deviate" from the identity in a measured way to be defined in §2. These ξ-diffeomorphisms have been used in [1] in proving Theorem 3.1. However they do not seem adequate in proving Theorem 1.3 of the present paper.

The major difficulty in proving Theorem 1.3 arises from the problem of choosing the diffeomorphism h so that (1.9) of Theorem 1.3 is satisfied, as well as the other conditions on h and H in Theorem 1.3. There are many choices of h such that the conditions of Theorem 1.2 are satisfied, but condition (1.9) of Theorem 1.3 is not satisfied. *Perispherical* diffeomorphisms aid in defining the diffeomorphism h and homotopy H so that *all* conditions on h in Theorem 1.3 are satisfied.

We close this section by recalling some useful definitions.

A product of two isotopies. Let P and Q be differentiable isotopies whose sections P^t and Q^t are diffeomorphisms of E_n onto E_n. If $P^1 = Q^0$, a differentiable isotopy, $W = QP$, termed the *product* of P and Q, is defined by setting

$$\begin{aligned} W^t &= P^0 && (t \leqslant 0) \\ W^t &= P^{2t} && (0 < t \leqslant \tfrac{1}{2}) \\ W^t &= Q^{2t-1} && (\tfrac{1}{2} < t \leqslant 1) \\ W^t &= Q^1 && (t \geqslant 1). \end{aligned}$$

So defined W is a differentiable isotopy of P^0 into Q^1 as one readily shows.

Deformations of indicatrices represented. For each $t \in R$ let $w^t = (w_1^t, \ldots, w_n^t)$ be a vector in E_n. The mapping $t \to w^t$ is regarded as *continuous* (*differentiable*) if each mapping $t \to w_i^t$, $i = 1, \ldots, n$ of R into R is continuous (differentiable).

(a) For each $t \in R$ and for $0 < m \leqslant n$ let

$$(w^t) = (w^t(1), \ldots, w^t(m))$$

be an ordered set of linearly independent vectors in E_n. The mapping $t \to (w^t)$ is regarded as *continuous* (*differentiable*) if each mapping $t \to w^t(r)$, $r = 1, \ldots, m$ is continuous (differentiable).

(b) The preceding mapping $t \to (w^t)$, if continuous (differentiable), will represent a *continuous* (*differentiable*) *deformation* of the indicatrix (w^0) into the indicatrix (w^1) if

$$(w^t(1), \ldots, w^t(m)) = (w^0(1), \ldots, w^0(m)) \qquad (t \leqslant 0)$$

$$(w^t(1), \ldots, w^t(m)) = (w^1(1), \ldots, w^{(1)}(m)) \qquad (t \geqslant 1).$$

2. ξ-diffeomorphisms. Let $h = (h_1, \ldots, h_n)$ be a differentiable mapping of E_n into E_n. Understanding that $x = (x_1, \ldots, x_n)$, set

$$d_0(h) = \sup_{x \in E_n} \| x - h(x) \|. \tag{2.1}$$

We suppose that $d_0(h)$ is finite. Assuming that the partial derivatives of the mappings h_i, $i = 1, \ldots, n$, are bounded, set

$$d_1(h) = \max_{i,j} \left(\sup_{x \in E_n} | \delta_i^j - \frac{\partial h_i}{\partial x_j}(x) | \right) \tag{2.2}$$

where i and j have the range $1, \ldots, n$ and δ_i^j is a "Kronecker delta". Set

$$d(h) = d_0(h) + d_1(h). \tag{2.3}$$

We term $d(h)$ the 1st-order *deviation of h from the identity.*

The constant ξ. There clearly exists a positive constant ξ so small that a C^1-mapping h of E_n onto E_n for which $d(h) < \xi$ has the property that

$$\frac{3}{2} > \frac{D(h_1, \ldots, h_n)}{D(x_1, \ldots, x_n)} > \frac{1}{2} \quad (x \in E_n). \tag{2.4}$$

So chosen, ξ will be invariable in this paper.

DEFINITION. *A ξ-mapping. A differentiable mapping h of E_n into E_n such that $d(h) < \xi$, and such that h leaves $\mathbf{0} \cup {}^cD_\rho$ point-wise invariant for some positive constant ρ, will be called a ξ-mapping.*

LEMMA 2.1. *A ξ-mapping h is a diffeomorphism of E_n onto E_n.*

We begin by proving (i).

(i) *The mapping h is onto E_n.*

It is readily seen that the set $h(E_n)$ is both open and closed relative to E_n. Since E_n is connected, $h(E_n) = E_n$. Thus (i) is true.

The mapping h is "proper" in the sense that $h^{-1}(K)$ is compact for arbitrary choice of K as a compact subset of $h(E_n) = E_n$. However a proper mapping of E_n onto E_n, which is locally a diffeomorphism, is a diffeomorphism. See Lemma 4.1 [4].

Thus h is a diffeomorphism of E_n onto E_n.

DEFINITION. *Taking account of Lemma 2.1, a ξ-mapping of E_n onto E_n will be referred to as a ξ-diffeomorphism.*

LEMMA 2.2. *A ξ-diffeomorphism k of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^cD_\rho$ admits a differentiable isotopy K into the identity each section K^t of which is a ξ-diffeomorphism with domain of identity $\mathbf{0} \cup {}^cD_\rho$, and such that*

$$d(K^t) \leqslant d(k). \tag{2.5}$$

The mapping μ. In proving this lemma we shall make use of a differentiable mapping μ of R onto $[0, 1]$ such that

$$(0 = \mu(t) \mid t \leqslant 0) \quad (1 = \mu(t) \mid t \geqslant 1). \tag{2.6}$$

Given k as in the lemma we define a mapping K of $E_n \times R$ into E_n by setting

$$K(x, t) = (1 - \mu(t))\, k(x) + \mu(t)\, x \quad (x \in E_n,\ t \in R). \tag{2.7}$$

One sees that K is a differentiable mapping of $E_n \times R$ into E_n such that for each t, K^t leaves $\mathbf{0} \cup {}^cD_\rho$ point-wise invariant. Moreover for each t

$$d_0(K^t) = (1 - \mu(t))\, d_0(k), \quad d_1(K^t) = (1 - \mu(t))\, d_1(k)$$

so that (2.5) holds. For each t, $d(K^t) < \xi$, since $d(k) < \xi$ by hypothesis. By Lemma 2.1 then, for each t, K^t is a diffeomorphism of E_n onto E_n.

Finally one sees that K^t, as defined by (2.7), is an isotopy of k into the identity, thereby completing the proof of Lemma 2.2.

Lemma 2.3, below, implies Theorem 1.1 in the special case in which the linear terms at $\mathbf{0}$ in the diffeomorphism f define the identity, that is, in the case in which f is a mapping

$$x \to x + A(x) = M(x) \quad (x \in X) \tag{2.8}$$

in which A is differentiable on X and each component A_i of A has a critical point at the origin.

Lemma 2.3 contains information not conveyed by Theorem 1.1, information useful in proving Theorem 1.3.

LEMMA 2.3. *Corresponding to the above diffeomorphism, $x \to M(x)$, of X into E_n, to any positive constant ρ and any positive constant ϵ, there exists a diffeomorphism k of E_n onto E_n, $\mathbf{0}$-related to M, with domain of identity $\mathbf{0} \cup {}^cD_\rho$ and with $d(k) < \epsilon$.*

Let $t \to \lambda(t)$ be a differentiable mapping of R onto $[0, 1]$ such that

$$(1 = \lambda(t) \mid t \leqslant 1) \quad (0 = \lambda(t) \mid t \geqslant 4).$$

Let σ be a positive constant at most $\rho/2$ such that $\bar{D}_{2\sigma} \subset X$. Denote $\| x \|$ by r. In vector notation, set

$$k(x) = x + \lambda\left(\frac{r^2}{\sigma^2}\right) A(x) \quad (r \leqslant 2\sigma) \tag{2.9}$$

and $k(x) = x$ for $r \geqslant 2\sigma$. Then

$$k(x) = x + A(x) = M(x) \quad (r \leqslant \sigma). \tag{2.10}$$

It is clear that k is a differentiable mapping of E_n into E_n. For $i, j = 1, \ldots, n$ and for $r \leqslant 2\sigma$

$$\left| \delta_j^i - \frac{\partial k_j}{\partial x_i}(x) \right| = \left| \lambda\left(\frac{r^2}{\sigma^2}\right) \frac{\partial A_j}{\partial x_i}(x) + 2\,\lambda'\left(\frac{r^2}{\sigma^2}\right) \frac{x_i}{\sigma} \frac{A_j(x)}{\sigma} \right|. \tag{2.11}$$

The right member of (2.11) is at most $\epsilon/2$ for $r \leqslant 2\sigma$ if σ is sufficiently small. The left member of (2.11) vanishes for $r \geqslant 2\sigma$. Hence $d_1(k) < \epsilon/2$. Moreover

$$d_0(k) \leqslant \max\,(\| A(x) \| \mid \| x \| \leqslant 2\sigma)$$

in accord with (2.9), so that $d_0(k) < \epsilon/2$ if σ is sufficiently small.

Hence $d(k) < \epsilon$ for σ sufficiently small.

By definition k has a domain of identity $\mathbf{0} \cup {}^cD_{2\sigma}$. By Lemma 2.1, k is then a diffeomorphism of E_n onto E_n. Since $\rho \geqslant 2\sigma$, $\mathbf{0} \cup {}^cD_\rho$ is also a domain of identity of k. By (2.10) k is $\mathbf{0}$-related to M.

This completes the proof of Lemma 2.3.

We return to a diffeomorphism $x \to f(x)$ of Theorem 1.1 and, for $i = 1, \ldots, n$, set

$$g_i(x) = \frac{\partial f_i}{\partial x_j}(\mathbf{0})\, x_j \quad (x \in E_n) \tag{2.12}$$

summing as to j on the range $1, \ldots, n$. Theorem 2.1 below is a corollary of Lemma 2.3. In it we refer to the linear diffeomorphism

$$x \to g(x) = (g_1(x), \ldots, g_n(x)). \tag{2.13}$$

THEOREM 2.1. *Let X be an open neighborhood of $\mathbf{0}$ in E_n and $x \to f(x)$ a sense-preserving diffeomorphism of X into E_n which leaves $\mathbf{0}$ invariant. Corresponding to prescribed positive constants ρ and ϵ, there exists a diffeomorphism k_ϵ of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^cD_\rho$, with $d(k_\epsilon) < \epsilon$, and such that the composite diffeomorphism $g\,k_\epsilon$ of E_n onto E_n is $\mathbf{0}$-related to f.*

NOTE. Theorem 2.1 is also true if f is sense-inverting as our proof shows. We have written Theorem 2.1 as above to preserve the continuity of our development.

PROOF OF THEOREM 2.1. Observe that the mapping

$$x \to (g^{-1}f)(x) = M(x) \quad (x \in X) \tag{2.14}$$

(introducing $M(x)$) has the form

$$x \to M(x) = x + A(x) \tag{2.15}$$

where A has the properties ascribed to A in (2.8). From Lemma 2.3 we then infer the following. There exists a diffeomorphism k_ϵ of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^cD_\rho$, with $d(k_\epsilon) < \epsilon$, and k_ϵ $\mathbf{0}$-related to $g^{-1}f$. It follows that $g\,k_\epsilon$ is $\mathbf{0}$-related to f.

This completes the proof of Theorem 2.1.

Theorem 1.2 *will follow from Theorem* 1.3 as proved in § 5. However a *proof of Theorem* 1.2 can here be sketched as follows.

Lemma 2.2 implies the following. For $0 < \epsilon < \xi$ there exists a differentiable isotopy K_ϵ of k_ϵ (of Theorem 2.1) into the identity such that for each t, K_ϵ^t is a diffeomorphism of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^c D_\rho$.

The mapping g is a linear sense-preserving diffeomorphism of E_n onto E_n leaving $\mathbf{0}$ invariant. One could readily show that there exists a diffeomorphism γ of E_n onto E_n, $\mathbf{0}$-related to g, with domain of identity $\mathbf{0} \cup {}^c D_\rho$, admitting an isotopy Γ into the identity such that for each t, Γ^t is a diffeomorphism of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^c D_\rho$.

For each $\epsilon < \xi$ Theorem 1.2 would be satisfied by the composite diffeomorphism $h_\epsilon = \gamma k_\epsilon$ and by an isotopy H_ϵ of which the section H_ϵ^t is the composite diffeomorphism,

$$H_\epsilon^t = \Gamma^t K_\epsilon^t \quad (t \in R), \tag{2.16}$$

taking h_ϵ and H_ϵ in place of h and H in Theorem 1.2. To verify this one notes that h_ϵ is a diffeomorphism of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^c D_\rho$ and is $\mathbf{0}$-related to f. The isotopy H_ϵ deforms h_ϵ into the identity. Its sections H_ϵ^t are diffeomorphisms of E_n onto E_n with domains of identity $\mathbf{0} \cup {}^c D_\rho$. Theorem 1.2 will thus be satisfied by h_ϵ and H_ϵ in place of h and H for *arbitrary* choice of $\epsilon < \xi$.

However h_ϵ and H_ϵ will not in general satisfy Theorem 1.3 because (1.9) will not in general be satisfied by such an h_ϵ.

The structure of the proof of Theorem 1.3 *is similar to the above. One chooses* k_ϵ *and* K_ϵ *as above, but then chooses* γ *and* Γ *in a special way so that Theorem* 1.3, *including* (1.9), *is satisfied by* h_ϵ *and* H_ϵ, *as defined by* (3.16), *provided* ϵ *is sufficiently small.*

"Perispherical diffeomorphisms" will aid in defining Γ and γ.

3. Perispherical diffeomorphisms. For each positive number c let S_c denote the $(n-1)$-sphere in E_n with center at the origin and radius c.

PERISPHERICAL DIFFEOMORPHISMS DEFINED. A diffeomorphism ζ of E_n onto E_n leaving $\mathbf{0}$ invariant will be termed *perispherical*