HANDBOOKS IN ECONOMICS

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Editors

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NORTH-HOLLAND
AMSTERDAM • NEW YORK • OXFORD • TOKYO

HANDBOOK OF MATHEMATICAL ECONOMICS

VOLUME III

Edited by

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1986

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PREFACE TO THE HANDBOOK

The field of mathematical economics

Mathematical economics includes various applications of mathematical concepts and techniques to economics, particularly economic theory. This branch of economics traces its origins back to the early nineteenth century, as noted in the historical introduction, but it has developed extremely rapidly in recent decades and is continuing to do so. Many economists have discovered that the language and tools of mathematics are extremely productive in the further development of economic theory. Simultaneously, many mathematicians have discovered that mathematical economic theory provides an important and interesting area of application of their mathematical skills and that economics has given rise to some important new mathematical problems, such as game theory.

Purpose

The *Handbook of Mathematical Economics* aims to provide a definitive source, reference, and teaching supplement for the field of mathematical economics. It surveys, as of the late 1970's, the state of the art of mathematical economics. Bearing in mind that this field is constantly developing, the Editors believe that now is an opportune time to take stock, summarizing both received results and newer developments. Thus all authors were invited to review and to appraise the current status and recent developments in their presentations. In addition to its use as a reference, the Editors hope that this Handbook will assist researchers and students working in one branch of mathematical economics to become acquainted with other branches of this field. Each of the chapters can be read independently.

Organization

The Handbook includes 29 chapters on various topics in mathematical economics, arranged into five parts: *Part 1* treats *Mathematical Methods in Economics*, including reviews of the concepts and techniques that have been most useful for the mathematical development of economic theory. *Part 2* elaborates on *Mathematical Approaches to Microeconomic Theory*, including consumer, producer, oligopoly, and duality theory. *Part 3* treats *Mathematical Approaches to Competi-*

PART 4

MATHEMATICAL APPROACHES TO WELFARE ECONOMICS

ISBN North-Holland for this volume 0 444 86128 9
ISBN North-Holland for this set 0 444 86054 1

Publishers
ELSEVIER SCIENCE PUBLISHERS B.V.
P.O. Box 1991
1000 BZ Amsterdam
The Netherlands

Sole distributors for the USA and Canada
ELSEVIER SCIENCE PUBLISHING COMPANY INC.
52 Vanderbilt Avenue
New York, NY 10017, USA

Library of Congress Cataloging in Publication Data

Main entry under title:

Handbook of mathematical economics.

(Handbooks in economics ; bk. 1)
Includes bibliographies and index.
1. Economics, Mathematical--Addresses, essays, lectures. I. Arrow, Kenneth Joseph, 1921- II. Intriligator, Michael D. III. Series.
HB135.H357 330'.01'51 81-2820
ISBN 0-444-86054-1 (U.S. : set)

PRINTED IN THE NETHERLANDS

INTRODUCTION TO THE SERIES

The aim of the *Handbooks in Economics* series is to produce Handbooks for various branches of economics, each of which is a definitive source, reference, and teaching supplement for use by professional researchers and advanced graduate students. Each Handbook provides self-contained surveys of the current state of a branch of economics in the form of chapters prepared by leading specialists on various aspects of this branch of economics. These surveys summarize not only received results but also newer developments, from recent journal articles and discussion papers. Some original material is also included, but the main goal is to provide comprehensive and accessible surveys. The Handbooks are intended to provide not only useful reference volumes for professional collections but also possible supplementary readings for advanced courses for graduate students in economics.

tive Equilibrium, including such aspects of competitive equilibrium as existence, stability, uncertainty, the computation of equilibrium prices, and the core of an economy. *Part 4* covers *Mathematical Approaches to Welfare Economics*, including social choice theory, optimal taxation, and optimal economic growth. *Part 5* treats *Mathematical Approaches to Economic Organization and Planning*, including organization design and decentralization.

Level

All of the topics presented are treated at an advanced level, suitable for use by economists and mathematicians working in the field or by advanced graduate students in both economics and mathematics.

Acknowledgements

Our principal acknowledgements are to the authors of chapters in the *Handbook of Mathematical Economics*, who not only prepared their own chapters but also provided advice on the organization and content of the Handbook and reviewed other chapters.

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SOCIAL CHOICE THEORY*

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1. Social welfare functions

1.1. Distant origins

The origins of social choice theory can be traced to two rather distinct sources, and it so happens that the theory is nearly in a position to celebrate the bicentenary of each of its two origins. One source is the study of normative analysis in terms of personal welfare (extensively explored in modern welfare economics), and the origins of this, through utilitarianism, can certainly be traced at least to Jeremy Bentham (1789). The other is the mathematical theory of elections and committee decisions, which is comfortably traced to Borda (1781) and Condorcet (1785). The influences of these two different origins will become clear as the modern developments in social choice theory are reviewed.

No approach to welfare economics has received as much support over the years as utilitarianism. If $U_i(\cdot)$ is the utility function of person i defined for each person $i=1,\dots,n$, over the set X of alternative social states, then on the utilitarian approach any state x is at least as good as another y, denoted xRy, if and only if $\sum_{i=1}^{n} U_i(x) \geq \sum_{i=1}^{n} U_i(y)$.

It is clear that utilitarianism uses cardinality and interpersonal comparability of personal utilities. Both these practices received severe reprimand in the 1930's,[1]

*The first version of this paper was written during 1978–79. While the paper has been now revised, I have not tried to bring it "up to date" regarding more recent publications. (There are some references to later publications, but most of these were in fact available in pre-print form earlier.) My greatest debt is to Kenneth Arrow, Michael Dummett and Peter Hammond for extremely helpful comments and suggestions on the earlier version of this paper. I have also benefited greatly from the comments of Brian Barry, Charles Blackorby, Julian Blau, Graciela Chichilnisky, Peter Coughlin, Bhaskar Dutta, Alan Feldman, Wulf Gaertner, Louis Gevers, Geoffrey Heal, Michael Intriligator, Jocelyn Kynch, Tapas Majumdar, John Muellbauer, Prasanta Pattanaik, Robert Pollak, Ariel Rubinstein, Maurice Salles, David Schmeidler, Margaret Sjöberg, Steven Slutsky, Kotaro Suzumura, and H. P. Young.

[1]The most influential attack came from Robbins (1932).

Handbook of Mathematical Economics, vol. III, edited by K.J. Arrow and M.D. Intriligator

with the rebuke drawing sustenance from a single-minded concern with basing utility information on non-verbal behaviour only, dealing with choices in the absence of risk. It thus appeared that social welfare must be based on just the n-tuple of ordinal, interpersonally non-comparable, individual utilities. This informational restriction would, of course, make the traditional utilitarian approach – and a great many other procedures – unworkable.

This "informational crisis" is important to bear in mind in understanding the form that the origin of modern social choice theory took. In fact, with the binary relation of preference replacing the utility function as the primitive of consumer theory, it made sense to characterize the exercise as one of deriving a social preference ordering R from the n-tuple of individual orderings $\{R_i\}$ of social states.

The other source, dealing primarily with election methods, had in any case the tradition of concentrating on the information given by an n-tuple of individual orderings – reliant on an informational framework that was much less ambitious than utilitarianism. Borda, Condorcet, Dodgson (Lewis Carroll), Nanson and others had pursued various results of voting, and had discussed the superiority of some voting systems over others.[2] Economists did not, however, take much notice of this literature, or of the problem studied in them, until the "informational crisis" sent them searching for other methods.

The union produced modern social choice theory. The big bang that characterized the beginning took the form of an "impossibility theorem", viz. Arrow's (1950, 1951) "General Possibility Theorem". It appeared that some conditions that look mild – and are indeed satisfied comfortably by utilitarianism when translated into its cardinal interpersonally comparable framework (see Section 6) – cannot be fulfilled by *any* rule whatsoever that has to base the social ordering on n-tuples of individual orderings. This theorem, which had a profound impact on the way modern social choice theory developed, will be discussed in Section 2.[3]

1.2. The Bergson–Samuelson social welfare function

The concept of a *social welfare function* was first introduced by Bergson (1938). This was defined in a very general form indeed: as a real-valued function $W(\cdot)$, determining social welfare, "the value of which is understood to depend on all the variables that might be considered as affecting welfare" (p. 417). If the relevant information about the social states in set X can be obtained, then such a social welfare function – swf for short – might as well be thought to be a real-valued

[2] For an excellent account of the literature, see Black (1948, 1958).

[3] Another important contribution to the early development of modern social choice theory was Kenneth May's axiomatization of the majority rule [see May (1952, 1953)].

function defined on X. If the issue of numerical representation is not emphasized, this really amounts to an ordering R of X.

While the idea of a social welfare function came from Bergson, the uses to which such a swf can be put were definitively investigated by Samuelson (1947). His exercises made use of many criteria that a swf may be required to satisfy.[4] One of them is the old Pareto criterion. This can be defined in many forms, and since the differences will turn out to be of some significance, we might as well seize this opportunity of distinguishing between them (though not all these versions were, in fact, used by Samuelson).

Let P and I be the asymmetric and symmetric factors of the social preference relation R ("at least as good as"), standing respectively for "strictly better than" and "indifferent to". And let the corresponding individual preference relation and its asymmetric and symmetric factors for any person i be R_i, P_i and I_i, respectively. The different versions of the Pareto Principle may now be stated. The following are all defined with the universal quantifier $\forall x, y \in X$ ("for all x, y in X"):

Condition P (weak Pareto principle)
$(\forall i: xP_i y) \Rightarrow xPy$.

Condition P° (Pareto indifferent rule)
$(\forall i: xI_i y) \Rightarrow xIy$.

*Condition P** (strong Pareto principle)
$(\forall i: xR_i y) \Rightarrow xRy$. And if to the antecedent is added $\exists i$: $xP_i y$, then the consequence is xPy.

It is obvious that Condition P* implies both Conditions P and P°, but is not implied by them even jointly.

If a swf satisfies Condition P*, we shall call it a *Pareto-inclusive* swf. It may be remarked that, given the form in which Bergson defined a swf, it may or may not be possible to check whether it is Pareto-inclusive or not, since there is no obligation to specify the individual preferences in defining a Bergson $W(\cdot)$. However, from the motivating discussion of Bergson (1938, 1948) and more so from the operations chosen by Samuelson to demonstrate the use of such a swf, it appears that the intention is to take note of individual preferences at least to the extent of being Pareto-inclusive.[5]

[4] For excellent examples of application and use of Bergson–Samuelson and social welfare functions, see Dasgupta and Heal (1979), Atkinson and Stiglitz (1981), and Dasgupta (1982).

[5] In using utility for such social criteria (Pareto optimality, equality, justice, etc.), one source of ambiguity is the possibility of defining them *either* in terms of ex post utilities, *or* in terms of ex ante utilities. On this see Starr (1973). Also Hammond (1983).

Sometimes a Bergson–Samuelson social welfare function is described as "individualistic". There is an ambiguity in this expression which is worth clarifying since it has been the source of some confusion. A swf can be individualistic in the sense of reflecting the preferences of all the individuals in the society taken together *when such preferences do not conflict*, in ranking any pair of social states. In this sense, an individualistic swf is simply a Pareto-inclusive swf. There is a second interpretation, which makes social welfare W a function of the vector of individual utilities u irrespective of the non-utility characteristic of the social states from which the utilities emanate: $W = W(u)$; see Samuelson (1947, pp. 228–229, 246), Graaff (1957, pp. 48–54), and Bergson (1948, p. 418), among others. This is a version of a condition of "neutrality" (sometimes called "welfarism"), which relates closely to Arrow's result (Section 2.1), and which will be further examined in Sections 6 and 9. In effect, it asserts the neutrality of the social ranking towards non-utility features, which can then affect the social ranking *only through* their influence on individual utilities, or preferences. It is easily checked that neither does Pareto-inclusiveness imply this condition of neutrality, nor the converse, and these two interpretations of individualism are, thus, completely independent of each other.

Finally, none of the conditions that Samuelson imposed on a swf for his exercises happened to specify how the social ordering might alter if different n-tuples of individual orderings were considered. If any n-tuple of individual preference orderings is called a "profile", then his exercises – and those considered by Bergson – were all "single profile" problems (see Section 9).

1.3. *The Arrow social welfare function*

Arrow (1951) defined a social welfare function – henceforth SWF (to be distinguished from the Bergson–Samuelson swf) – as a functional relation specifying one social ordering R for any given n-tuple of individual orderings $\{R_i\}$, one ordering for each person,

$$R = f(\{R_i\}). \tag{1.1}$$

Note that if a Bergson–Samuelson swf is defined as a social ordering R, then an Arrow SWF is a function the *value* of which would be a Bergson–Samuelson swf. Arrow's exercise, in this sense, is concerned with the way of arriving at a Bergson–Samuelson swf. Alternatively, if the Bergson–Samuelson swf is taken as a function $W(\cdot)$, defined over a particular profile of individual ordinal utilities, then a Bergson–Samuelson swf fits into the form (1.1). The Arrow exercise can, then, be seen as a way of extending the set of single-profile formulations into one consistent multiple-profile function, specifying correspondences between the respective values of R (or parts thereof) for different n-tuples $\{R_i\}$.

Arrow proceeded to impose a variety of conditions that a reasonable SWF could be expected to satisfy. One of them deals specifically with the multiple-profile characteristics of a SWF: the independence of irrelevant alternatives. For stating this condition, Arrow used the notion of a choice function for the society, $C(\cdot)$, which was defined with respect to the binary relation R, satisfying what is sometimes called the "Condorcet condition" [Condorcet (1785)].[6] For all subsets S of X,

$$C(S) = [x | x \in S \;\&\; \forall y \in S\colon xRy]. \tag{1.2}$$

Condition I (independence of irrelevant alternatives)
For any two n-tuples $\{R_i\}$ and $\{R'_i\}$ in the domain of f, and for any $S \subseteq X$, with the choice functions $C(\cdot)$, and $C'(\cdot)$ corresponding to $f(\{R_i\})$ and $f(\{R'_i\})$, respectively,

$$[\forall i\colon (\forall x, y \in S\colon xR_i y \Leftrightarrow xR'_i y)] \Rightarrow C(S) = C'(S).$$

This condition requires that as long as individual preferences remain the same over a subset S of X, the social choice from that subset should also remain the same.

The property of independence can also be considered in purely relational terms as well, without invoking a choice function at all.[7]

Condition I^2 (pairwise relational independence)
The restriction of the social preference relation over any pair $\{x, y\}$ is a function of the n-tuple of restrictions of individual preferences over that pair,

$$R|^{\{x,y\}} = f^{x,y}(\{R_i|^{\{x,y\}}\}). \tag{1.3}$$

[6] The "Condorcet condition" is sometimes defined specifically for the majority relation only, which relates to Condorcet's (1785) own original concern. On these issues, see Black (1958), Fishburn (1973a), and Young (1977).

[7] Given the binary specification of the choice function for society, as in (1.2), it is easily checked that this relational independence condition I^2 is exactly equivalent to Arrow's choice-functional independence condition I. In the proofs that will be presented in Section 2.1, Condition I^2 will be used, because it simplifies matters, and makes Arrow's theorem entirely relation-theoretic. However, Conditions I and I^2 are not *generally* equivalent. When the choice function for the society cannot be represented by a binary relation (to be investigated in Section 4 below), Condition I can be used without implying Condition I^2, and vice versa. Indeed, in a purely relation-theoretic framework with the use of Condition I^2, it need not even be assumed that a choice function for the society exists. Similarly, in a purely choice-oriented framework, the relation-theoretic notions (including Condition I^2) can be entirely dispensed with. Finally, it is possible to define the binary relation of social preference just in terms of choices over pairs, i.e. xRy if and only if $x \in C(\{x, y\})$, which will make Condition I^2 strictly weaker than Arrow's choice-functional independence condition I. This avenue, which will be explored in Section 4.2, will be useful in interpreting some recent results on collective rationality, and in demonstrating that Arrow had proved a more general result than he had claimed.

2. Arrow's impossibility theorem

2.1. The general possibility theorem

Arrow's General Possibility Theorem asserts the inconsistency of some mild-looking conditions imposed on social welfare functions, viz. Conditions P and I as defined in the last section and the following two additional ones.[8]

Condition U (unrestricted domain)
The domain of the SWF, i.e. $f(\cdot)$ defined by (1.1), includes all (logically) possible n-tuples of individual orderings of X.

Condition D (non-dictatorship)
There is no individual i such that for all preference n-tuples in the domain of $f(\cdot)$, for each ordered pair $x, y \in X$, $xP_i y \Rightarrow xPy$.

Denoting the set of individuals in the society as H, and the cardinality of the set X of social states as $\#X$, the General Possibility Theorem can be stated thus.

General possibility theorem (GPT)
If H is finite and $\#X \geq 3$, then there is no SWF satisfying Conditions U, I, P and D.

This result has been the prime mover in getting the discipline of social choice theory started, and though recently the focus has somewhat shifted from impossibility theorems to other issues, there is no doubt that Arrow's formulation of the social choice problem in presenting the GPT laid the foundations of social choice theory. In seeking a demonstration of the GPT, Condition I^2 can be used rather than Condition I to get a fully relation-theoretic statement, which can be used with or without the further assumption of binary choice.

Pair relational general possibility theorem (GPT*)
If H is finite and $\#X \geq 3$, then there is no SWF satisfying Conditions U, I^2, P and D.

In establishing the General Possibility Theorem, it is convenient to go via two lemmas that are of interest in themselves. We shall call a set G of *persons* a "group" G (but – beware – no "group theory" is involved!). We define a group G of persons "decisive" over the ordered pair $\{x, y\}$, denoted $\bar{D}_G(x, y)$, if and only if xPy whenever $xP_i y$ for all i in G. Group G is "almost decisive" over that ordered pair, denoted $D_G(x, y)$, if and only if xPy whenever $xP_i y$ for all i in G and $yP_i x$ for all i not in G, i.e. for all i in $H - G$. Obviously decisiveness implies almost decisiveness, but in general not vice versa.

[8] This version of the GPT was presented in the 2nd edition of Arrow's book [Arrow (1963, pp. 96–100)]. An error in Arrow's (1950, 1951) original presentation was noted and rectified by Blau (1957).

The first lemma applies not merely to SWFs, but to a broader class of transformations from individual preferences to social preferences, relaxing the requirement of full transitivity of social preference relation R.

Transitivity
R is transitive on X if and only if $\forall x, y, z \in X$, $(xRy \ \& \ yRz) \Rightarrow xRz$.

Quasi-transitivity
R is quasi-transitive on X if and only if $\forall x, y, z \in X$, $(xPy \ \& \ yPz) \Rightarrow xPz$.

Acyclicity
R is acyclic on X if and only if there is no cycle of strict preference: that is, no subset $(x_1, x_2, \dots, x_k)$ of X such that $x_1Px_2, x_2Px_3, \dots, x_{k-1}Px_k$, and x_kPx_1.

Obviously, in this framework, transitivity implies quasi-transitivity, but not vice versa, and quasi-transitivity implies acyclicity but not vice versa. Where the line of "collective rationality" is to be drawn depends partly on what use is to be made of the social preference relation R. While we have for the moment kept aside the question of whether or not to base social choice entirely on a binary relation – as given by (1.2) – it is relevant to note that for a reflexive and complete preference relation, acyclicity is the *necessary and sufficient* condition for the choice set $C(S)$, as defined by (1.2), to be non-empty for every non-empty, finite subset S of X.[9] We may call a choice function that has a non-empty $C(S)$ for every non-empty, finite $S \subseteq X$, a "finitely complete choice function".

If it is required that the binary relation of social preference should provide a minimally sufficient basis for a finitely complete choice function, then it is natural to confine the range of the function $f(\cdot)$ to preference relations that are reflexive, complete and *acyclic*. Such a function will be called a *social decision function* SDF. If the range is further restricted to reflexive, complete and *quasi-transitive* preference relations, then $f(\cdot)$ will be called a *quasi-transitive social decision function* QSDF (a transferred epithet to be sure, but it need not cause any confusion). If the range is further restricted to preference relations that are reflexive, complete and *transitive*, then we are back to the case of Arrow's social welfare functions SWF. It is trivial that a SWF is a QSDF, and a QSDF is a SDF, but in general not vice versa.

[9] For infinite sets, acyclicity would require supplementation by other conditions for guaranteeing the existence of a best element. This supplementation has been investigated in different ways. See Herzberger (1973), Smith (1974), Bergstrom (1975), Suzumura (1976a), Birchenhall (1977), Mukherji (1977), and Walker (1977). See also Aizerman, Zavalishin and Piatnitsky (1976), and Aizerman and Malishevski (1980).

Field expansion lemma
For any quasi-transitive social decision function (QSDF) satisfying Conditions U, I^2 and P, with $\#X \geq 3$, if any group is almost decisive over any ordered pair of social states, then it is decisive over every ordered pair of social states,

$$[\exists x, y \in X: D_G(x, y)] \Rightarrow [\forall a, b \in X: \overline{D}_G(a, b)].$$

To see clearly how this works, it may be useful to consider the case of four distinct states x, y, a and b. Let the preference ordering – in strict decreasing order – of every i in G be a, x, y, b and let everyone not in G strictly prefer a to x, y to b, and y to x, leaving the ordering of a and b completely unspecified. By the weak Pareto principle aPx and yPb. Further, since $D_G(x, y)$, clearly xPy. Thus, by quasi-transitivity, aPb. By Condition I^2, this must depend only on the individual orderings of a and b, of which – in fact – only the orderings of those in G have been specified. Hence $\overline{D}_G(a, b)$.

By virtue of the Field Expansion Lemma, there is no difference between a group being almost decisive over some ordered pair and being decisive over every ordered pair. Let such a group be called a decisive group.

Group contraction lemma
For any social welfare function (SWF) satisfying Conditions U, I^2 and P, with $\#X \geq 3$, if any group G, with $\#G > 1$, is decisive, then so is some proper subset of that group.

To prove this, partition a decisive group G into two non-empty proper subsets G_1 and G_2, respectively. Let the preference orderings of the three groups be the following in strict descending order, over some triple $\{x, y, z\}$: G_1: x, y, z; G_2: y, z, x; $H - G$: z, x, y. By the decisiveness of G, it follows that yPz. Clearly, either xPz, or zRx, by the completeness of R. Hence it follows from yPz that xPz or yPx, by the transitivity of R. Hence either G_1 is almost decisive over $\{x, z\}$, or G_2 is almost decisive over $\{y, x\}$. By the Field Expansion Lemma, either G_1 or G_2 is, thus, decisive.

Now Arrow's General Possibility Theorem (GPT).

*Proof of GPT**
By the weak Pareto principle, the group of all persons H is decisive. By the Group Contraction Lemma, we can go on persistently eliminating some members in each contraction, still leaving the rest decisive. Since H is finite, this would lead ultimately to some individual being a dictator.

Proof of GPT
By (2), Condition I implies I^2. Hence GPT* entails GPT.

2.2. Variants

In the original version of the General Possibility Theorem, Arrow (1950, 1951) had not used the Pareto principle, and had used instead a pair of conditions which he had called "citizens' sovereignty" and "positive association". The former is a requirement of "non-imposition", asserting that social preference should not be imposed from outside irrespective of the preferences of the members of the community, while the latter is, in fact, a weak condition of "monotonicity" [see Murakami (1968)], requiring non-negative response of social preference to individual preferences. Let $R = f(\{R_i\})$ and $R' = f(\{R'_i\})$.

Condition NI (non-imposition)
For no pair of social states $\{x, y\}$ it is true that xRy for every possible n-tuple $\{R_i\}$ in the domain of $f(\cdot)$.

Condition M (weak monotonicity)
For any two n-tuples $\{R_i\}$ and $\{R'_i\}$, for a given social state x, if for all individuals i, for all states y, $xI_i y \Rightarrow xR'_i y$, and $xP_i y \Rightarrow xP'_i y$, and for all states a and b both distinct from x, $aR_i b \Leftrightarrow aR'_i b$, then $xPy \Rightarrow xP'y$.

The original version of the impossibility theorem [Arrow (1950, 1951)] was concerned with showing the irreconcilability of Conditions U, I, M, NI and D for any SWF. In fact, for a SWF, Conditions U, I, M and NI together imply the weak Pareto principle, and thus the earlier version would be a corollary of the GPT presented in Arrow (1963), and thus of the GPT*.

There was, however, another difference in the original presentation of Arrow (1950, 1951). A weaker domain condition was used, requiring only that the domain of $f(\cdot)$ must include all n-tuples of individual orderings consistent with ordering a particular triple $\{x, y, z\}$ in any way whatsoever (but not necessarily ordering the whole X in any way). This proved insufficient for the impossibility result, and required strengthening as Blau (1957) showed, and hence the domain requirement was tightened to Condition U.[10] An alternative way of obtaining the impossibility is found in leaving the domain condition in its weaker form, while strengthening the non-dictatorship condition by ruling out *local* dictators over the specified triple $\{x, y, z\}$ on which individual preferences could be freely varied according to the weaker version of the domain condition.[11]

[10] The logical problem was absent in one of the earlier versions of Arrow's theorem [viz. Arrow (1952)], which did not, however, go into "field expansion" beyond a triple. Blau's contributions (1957, 1971, 1972, 1976) have brought out the "neutrality" implications of Arrow's framework for social choice by clarifying the full "field expansion" consequences of that framework.

[11] See Murakami (1961, 1968) and Pattanaik (1971).

A great many other variations in the theme of Arrow's impossibility theorem have been explored in the literature. Some of the variations will come up in the discussion of specific issues in later sections, and here I shall confine myself to a few remarks only. Recently Kelly (1978) has provided an excellent account of the main lines of development since Arrow's pioneering contribution.[12]

First, some versions of the result use neither the Pareto principle nor any condition of non-negative responsiveness. Consider the following requirement, which rules out "reverse dictators".

Non-suppression (NS)
There is no individual i such that for every preference n-tuple in the domain of $f(\cdot)$, for each ordered pair $x, y \in X$, $xP_i y \Rightarrow yPx$.

Conditions U, I, NI, NS and D are inconsistent for a social welfare function [see Wilson (1972b) and Binmore (1975); for related results, see Murakami (1968), Hansson (1969a, 1969b), Wilson (1972a), Fishburn (1974a), Binmore (1976), Monjardet (1979), and Kim and Roush (1980a)]. In fact, given unrestricted domain and independence of irrelevant alternatives, the possibilities that are open are (i) dictatorship, (ii) reverse dictatorship, and (iii) collective impotence. Either one person's strict preferences are fully reflected in social rankings of all pairs (positively or negatively), or not even everyone put together can influence social preference over some pair. The weak Pareto principle eliminates collective impotence as well as reverse dictatorship, leaving us only with the possibility of dictatorship (as in the version of the GPT presented in the last subsection).

Second, when the set of individuals is infinitely large, Arrow's conditions are mutually consistent, even though the permitted decision procedures are not very attractive [see Fishburn (1970b), Hansson (1972, 1976), Kirman and Sondermann (1972), Brown (1974), Schmitz (1977), and Armstrong (1980)]. There is, however, no "approximate" consistency for "very large" communities, and the impossibility result continues to hold exactly for all finite communities no matter how large, so that the practical relevance of the consistency possibility may not be very great. Furthermore, the Field Expansion Lemma and the Group Contraction Lemma both continue to hold for infinitely large communities and decisive sets can be endlessly curtailed, effectively disenfranchising nearly everybody [leading to such "limit" concepts as the existence of "invisible dictators", to use Kirman and Sondermann's (1972) description].

Third, McManus (1975, 1978, 1982, 1983) has investigated important issues of inter-taste consistency and inter-profile welfare comparisons, continuity conditions imposed on social welfare evaluation, and related matters [see also Inada (1964a) on an earlier study with a bearing on these issues]. He has provided both impossibility results and positive possibility theorems involving various combinations of these conditions. He has also provided reasons for not requiring the "independence" conditions, making positive possibilities that much easier.

Fourth, Chichilnisky (1976, 1980a, 1982a, 1982b) has established a set of important impossibility results without the use of the "independence" condition. For a class of social aggregation problems satisfying unanimity (a weak version of the Pareto principle) and anonymity, she shows the absence of continuous rules of transforming n-tuples of individual preferences into social preferences. Continuity too is a condition of "inter-profile consistency", but of a very different sort from "independence". She has investigated various general properties of individual and social choice [Chichilnisky (1979, 1980a, 1980b, 1981, 1983)], and also explored the possibilities of generalizing her original impossibility results by systematic relaxation of specific restrictions, such as non-satiation, preference ordinality, etc. [Chichilnisky (1980c, 1982a, 1983)].

Fifth, the formulation of social choice problems can be broadened by bringing in lotteries on alternatives [see Zeckhauser (1969), Shepsle (1970), Niemi and Weisberg (1972), Fishburn (1972b, 1973a, 1975b), Intriligator (1973, 1979), Nitzan (1975), Barbera (1979), Kalai and Megiddo (1980), Machina and Parks (1981), Coughlin and Nitzan (1981, 1983), and Heiner and Pattanaik (1983)]. This opens up new possibilities. If the problem is reformulated as demanding a lottery over *social preferences* (rather than over the alternatives to be chosen), based on n-tuples of individual orderings of social states, then Arrow-like impossibilities re-emerge in the form of arbitrary distribution of power (the exclusion of which would appear to be reasonable); see Barbera and Sonnenschein (1978), Bandyopadhyay, Deb and Pattanaik (1979), McLenan (1980), and Heiner and Pattanaik (1983).

Sixth, another variation that has been recently investigated is the eschewal of the assumption of completeness of the social preference. Arrow's impossibility result can be adapted for such an extended framework with only a little loss of power [Barthelemy (1983) and Weymark (1983)]. These analyses are, in fact, closely related to results dealing with admitting social intransitivity (see Section 3 below), since intransitivity can be given the particular form of dropping completeness.

Finally, many variations of the way of setting up the problem of social choice will be examined in some detail in the following sections: admitting non-transitive social preference (Section 3); admitting non-binary social choice (Section 4); seeking the acceptable rather than the best (Section 5); enriching the input of utility information (Section 6); restricting the domain of social choice procedures (Section 8); and weakening the independence condition and enriching the use of non-utility information (Section 9). While the focus very often will not be on the

[12] See also Murakami (1968), Pattanaik (1971), Fishburn (1973a), Brams (1976), and Plott (1976). There is also an important Russian book, viz. Mirkin (1974), with an English translation (1979). On closely related matters, see also Blin (1973), Brams (1976), Gottinger and Leinfellner (1978), Pattanaik (1978), Laffont (1979), Mueller (1979), Feldman (1980a), Kim and Roush (1980a), Moulin (1983), Suzumura (1983a), and Peleg (1984).

specific issue of avoiding Arrow's impossibility result, the implications of these different approaches for that problem will be, inter alia, clarified.

3. Non-transitive social preference

3.1. *Quasi-transitivity*

There has been speculation for some time as to whether the impossibility results of the type pioneered by Arrow could be avoided by weakening the requirement of collective rationality. There have been broadly two approaches to this question. One retains the Arrovian focus on a social preference relation R, but weakens the consistency requirement of R from the full dose of transitivity to milder conditions. The other dispenses with the notion of social preference as such and formulates the problem in choice functional terms. In this section the use of the first approach is discussed, while the second approach will be taken up in Section 4.

In establishing Arrow's theorem, two lemmas were used in the last section. The Field Expansion Lemma requires no more than quasi-transitivity of social preference, while the Group Contraction Lemma cannot be derived from quasi-transitivity alone, and was, in fact, established by using full transitivity of social preference. The latter result is crucial to deriving dictatorship from Arrow's Conditions U, P and I (or I^2), and if that result is nullified by relaxing the requirement of consistency of social preference to quasi-transitivity only, the Arrow impossibility result will fail to hold. On the other hand, quasi-transitivity is more than sufficient for generating a finitely complete choice function from a reflexive and complete social preference relation. Thus the avoidance of the Arrow impossibility result can be shown to exist strictly within the limits of Arrow's search for a preference-based social choice procedure satisfying Conditions U, P, I and D [see Sen (1969, 1970a) and Schick (1969)]. A simple example of such a procedure is a social decision function that yields the "Pareto-extension rule", with x being socially preferred to y if and only if everyone prefers x to y, while x and y being socially "indifferent" if either they are Pareto-indifferent or Pareto-non-comparable [Sen (1969); for an axiomatic examination of the Pareto-extension rule, see Pollak (1979)]. The unattractiveness of such a social decision procedure (despite its providing a formal route to escape the Arrow impossibility) led to the question as to whether or not the Arrow conditions were in an important sense "too weak" rather than "too strong" [Sen (1969)].

The Pareto-extension rule gives everyone a "veto", and if anyone prefers x to y strictly, he can guarantee that x is socially at least as good as y. Allan Gibbard showed in an unpublished paper [discussed in Sen (1970a)] that the existence of a veto is a necessary result of resolving the Arrow problem through weakening the transitivity of social preference to quasi-transitivity. Define a person i as "semi-decisive" over some ordered pair $\{x, y\}$ if $x P_i y$ implies $x R y$. A person has a veto if and only if he is semi-decisive over every ordered pair. A SDF is called oligarchic if and only if there is a unique group G of persons such that G is decisive and every member of G has a veto.

Quasi-transitive oligarchy theorem
If H is finite and $\#X \geq 3$, then any QSDF satisfying Conditions U, P and I^2 must be oligarchic.

Just like the Field Expansion Lemma, which continues to hold, it is possible to establish a "Veto-Field Expansion Lemma" asserting that any person who is almost semi-decisive over some ordered pair must be semi-decisive over all ordered pairs, i.e. must have a veto. (Almost-semi-decisiveness of i over x, y is defined as the requirement that $x P_i y$ *and*, for all $j \neq i$, $y P_j x$ must together imply $x R y$.) Now take a smallest decisive group G of persons, which must exist by the weak Pareto principle and the finiteness of H. Split G into any unit set $\{i\}$ consisting of one person i and the rest $G-\{i\}$. Assume the following preference orderings (shown in strict descending order) over a triple x, y, z: $G-\{i\}: x, y, z$; $\{i\}: y, z, x$; and $H-G: z, x, y$. By the decisiveness of G, we have $y P z$. By G being a smallest decisive group, $G-\{i\}$ cannot be decisive. But if $x P z$, then it will be almost decisive over this ordered pair, and thus by the Field Expansion Lemma, must be decisive. So $z R x$. If we now have $x P y$, this together with $y P z$ and $z R x$ will contradict quasi-transitivity. Hence $y R x$. But then i is almost semi-decisive over some ordered pair, and thus by the Veto-Field Expansion Lemma has a veto. This can be shown for every member of G. The proof is completed by noting that no group other than a superset of G can be decisive since every member of G has a veto.[13]

The replacement of transitivity by quasi-transitivity has translated the possibility of dictatorship to oligarchy with veto powers, and while the existence of vetoers may be less unattractive than that of a dictator, it is unappetizing enough not to provide a grand resolution of the Arrow problem.

In fact, even the dictatorship result reappears if the conditions imposed are supplemented by the requirement of "positive responsiveness" – a stricter version of the weak monotonicity condition (M) defined earlier. Positive responsiveness is defined below in a framework that incorporates independence of irrelevant alternatives. Denote $R = f(\{R_i\})$, and $R' = f(\{R'_i\})$.

Condition PR (positive responsiveness)
For any $x, y \in X$, if for all i, $(x P_i y \Rightarrow x P'_i y \ \& \ x I_i y \Rightarrow x R'_i y)$, and for some i, $(x I_i y \ \& \ x P'_i y)$ or $(y P_i x \ \& \ x R'_i y)$, then $x R y \Rightarrow x P' y$.

[13] This theorem was first established by Gibbard, and in different ways by Schwartz (1972), Mas-Colell and Sonnenschein (1972), and Guha (1972). Guha noted a hierarchy of oligarchies with a stricter version of the Pareto principle such that indifference by an oligarchic group would lead to a fresh oligarchy among the rest.

Quasi-transitive positive-responsive dictatorship theorem
If H is finite and $\#X \geq 3$, then there is no QSDF satisfying Conditions U, I^2, P, D and PR.

This theorem, established by Mas-Colell and Sonnenschein (1972), shows that transitivity can be weakened to quasi-transitivity of social preference if a stricter version of the monotonicity requirement is imposed.

3.2. Acyclicity

Quasi-transitivity may also be thought to be too demanding a condition, especially since acyclicity – a weaker requirement than quasi-transitivity – is sufficient for generating a finitely complete choice function based on the binary relation of social preference. Mas-Colell and Sonnenschein (1972) have a veto-result with *acyclicity* as such. (It can, in fact, be shown that the result goes through even with the weaker condition of "triple acyclicity", i.e. no cycles over triples.[14])

Triple-acyclic positive-responsive vetoer theorem
For H finite, $\#H \geq 4$, and $\#X \geq 3$, any SDF (even with the requirement of acyclicity relaxed to triple acyclicity) satisfying Conditions U, I^2, P and PR, must yield someone with veto.

An alternative way of generating the vetoer result is to use the weaker monotonicity condition M (essentially, non-negative responsiveness), but marry it with a requirement of neutrality towards the nature of social states. Combining neutrality with independence (in the form of I^2) and monotonicity (in the weak form) yields the following:

Condition NIM (neutrality, independence cum monotonicity)
For any $x, y, a, b \in X$, if for all i, $xP_i y \Rightarrow aP'_i b$, and $xI_i y \Rightarrow aR'_i b$, then $xPy \Rightarrow aP'b$.

The following theorem was established by Blau and Deb (1977).

Acyclic neutral monotonicity vetoer theorem
If $\#X \geq \#H$, with a finite H, then any SDF satisfying Conditions U and NIM must yield someone with a veto.[15]

[14] See Blair, Bordes, Kelly and Suzumura (1976).
[15] See also Schwartz (1974). The cycle involved in the proof is that of the $(n-1)$-majority rule. On related matters, see Dummett and Farquharsen (1961), Murakami (1968), Craven (1971), Pattanaik (1971), Fishburn (1973a), Ferejohn and Grether (1974), Deb (1976), Blau and Brown (1978), Nakamura (1978), Peleg (1978, 1979b), and Suzumura (1983a).

To establish this, suppose – to the contrary – there is no vetoer. So there is no one who is semi-decisive over all pairs. By the neutrality and monotonicity properties of NIM, there is thus no one who is almost semi-decisive over any pair. (If someone were, then by monotonicity he will be semi-decisive over that pair, and by neutrality a vetoer.) So everyone loses over any pair if unanimously opposed by others. With this in mind, consider the following n-tuple of preference orderings (in descending order) over a subset $\{x_1, x_2, \ldots, x_n\}$ of X, for the n individuals $1, \ldots, n$.

1: $x_1, x_2, \ldots, x_{n-1}, x_n,$

2: $x_2, x_3, \ldots, x_n, x_1,$

⋮

n: $x_n, x_1, \ldots, x_{n-2}, x_{n-1}.$

Clearly, x_1Px_2, x_2Px_3, ..., $x_{n-1}Px_n$, and x_nPx_1. This violation of acyclicity shows the falsity of the contrary hypothesis.

Thus, even acyclicity does not help very much in delivering us from the Arrow problem. A weaker consistency condition combined with other properties leads to a weakening – rather than elimination – of the dictatorship result, in the form of the existence of vetoers. And acyclicity *is* necessary for binary choice using the Condorcet condition.

Recently, Blair and Pollak (1982, 1983) and Kelsey (1982, 1983a, 1983b) have established various extensions of these impossibility results. Blair and Pollak have shown in particular that even without neutrality, some of the sting of the veto power remains in the form of an individual being semi-decisive over $(m-n+1)(m-1)$ pairs of states, where m and n are respectively the numbers of states and individuals. Given the individuals, when larger and larger sets of states – without bound – are considered, the proportion of pairs over which the individual is semi-decisive approaches unity [Blair and Pollak (1982)]. Kelsey (1982, 1983a, 1983b) has established similar – though weaker – arbitrariness of power (semi-decisive or *anti*-semi-decisive) over a large proportion of pairs of states – approaching $\frac{1}{2}$ as more and more states are considered – without neutrality and even without the Pareto principle.

3.3. Semi-transitivity, interval order and generalizations

I turn now to a somewhat different question. From quasi-transitivity to move to acyclicity is an act of weakening. What about the act of strengthening in going from just quasi-transitivity to semi-orders (and similar structures) without moving all the way to full transitivity? Would the Arrow impossibility result hold with full