Second Edition

Elementary Applied Partial Differential Equations

with Fourier Series and Boundary Value Problems

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19

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Prentice-Hall International, Inc.

Library of Congress Cataloging-in-Publication Data

Haberman, Richard, (date)
Elementary applied partial differential equations.

Bibliography: p.
Includes index.
1. Differential equations, Partial. 2. Fourier series. 3. Boundary value problems. I. Title.
QA377.H27 1987 515.3'53 86-17051
ISBN 0-13-252891-6

Editorial/production supervision and
interior design: Maria McColligan
Manufacturing buyer: John B. Hall

A Division of Simon & Schuster
Englewood Cliffs, New Jersey 07632

Printed in the Republic of Singapore

10 9 8 7 6 5 4 3 2

ISBN 0-13-252891-6 01

Prentice-Hall International, Inc., *London*
Prentice-Hall of Australia Pty. Limited, *Sydney*
Editora Prentice-Hall do Brasil, Ltda., *Rio de Janeiro*
Prentice-Hall Canada Inc., *Toronto*
Prentice-Hall Hispanoamericana, S.A., *Mexico*
Prentice-Hall of India Private Limited, *New Delhi*
Prentice-Hall of Japan, Inc., *Tokyo*
Prentice-Hall of Southeast Asia Pte. Ltd., *Singapore*
Prentice-Hall, *Englewood Cliffs, New Jersey*

Preface

This text discusses elementary partial differential equations in the engineering and physical sciences. It is suited for courses whose titles include Fourier series, orthogonal functions, or boundary value problems. It may also be used in courses on Green's functions or transform methods.

Simple models (heat flow, vibrating strings and membranes) are emphasized. Equations are formulated carefully from physical principles, motivating most mathematical topics. Solution techniques are developed patiently. Mathematical results frequently are given physical interpretations. Proofs of theorems (if given at all) are presented after explanations based on illustrative examples. Numerous exercises of varying difficulty form an essential part of this text. Answers are provided for those exercises marked with a star (*).

Standard topics such as the method of separation of variables, Fourier series, and orthogonal functions are developed with considerable detail. In addition, there is a variety of clearly presented topics, such as differentiation and integration of Fourier series, zeros of Sturm-Liouville eigenfunctions, Rayleigh quotient, multidimensional eigenvalue problems, Bessel functions for a vibrating circular membrane, eigenfunction expansions for nonhomogeneous problems, Green's functions, Fourier and Laplace transform solutions, method of characteristics, and numerical methods. Some optional advanced material of interest is also included (for example, asymptotic expansion of large eigenvalues, calculation of perturbed frequencies using the Fredholm alternative, and the dynamics of shock waves).

The text has evolved from the author's experiences teaching this material to different types of students at various institutions (M.I.T., U.C.S.D., Rutgers, Ohio State, and S.M.U.). Prerequisites for the reader are calculus and elementary ordinary differential equations. (These are occasionally reviewed in the text, where necessary.) For the beginning student, the core material for a typical course consists of most of Chapters 1–6. This will usually be supplemented by a few other topics. The text is somewhat flexible for an instructor, since most sections in Chapters 7–13 only depend on Chapters 1–6. Chapter 10 is an exception, since it requires Chapters 8 and 9.

Most of the first edition remains. Revised derivations, which are intended to be clearer, are presented for heat flow and the vibrations of strings and membranes. Some of the more advanced mathematical theory on the convergence of a Fourier series has been deleted, so that there is room for a later section that formulates the partial differential equations of traffic flow. Although only a few new exercises have been added, answers are now provided for a substantially increased number of exercises.

My object has been to explain clearly many elementary aspects of partial differential equations as an introduction to this vast and important field. The student, after achieving a certain degree of competence and understanding, can use this text as a reference, but should be prepared to refer to other books cited in the bibliography for additional material.

Finally, it is hoped that this text enables the reader to find enjoyment in the study of the relationships between mathematics and the physical sciences.

Richard Haberman

Contents

1

Heat Equation

1.1 INTRODUCTION

We wish to discuss the solution of elementary problems involving partial differential equations, the kinds of problems that arise in various fields of science and engineering. A **partial differential equation** (PDE) is a mathematical equation containing partial derivatives, for example,

$$\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} = 0. \tag{1.1.1}$$

We could begin our study by determining what functions $u(x, t)$ satisfy (1.1.1). However, we prefer to start by investigating a physical problem. We do this for two reasons. First, our mathematical techniques probably will be of greater interest to you when it becomes clear that these methods analyze physical problems. Second, we will actually find that physical considerations will motivate many of our mathematical developments.

Many diverse subject areas in engineering and the physical sciences are dominated by the study of partial differential equations. No list could be all-inclusive. However, the following examples should give you a feeling for the type of areas that are highly dependent on the study of partial differential equations: acoustics, aerodynamics, elasticity, electrodynamics, fluid dynamics, geophysics (seismic wave propagation), heat transfer, meteorology, oceanography, optics, petroleum engineering, plasma physics (ionized liquids and gases), quantum mechanics.

We will follow a certain philosophy of applied mathematics in which the analysis of a problem will have three stages:

1. Formulation
2. Solution
3. Interpretation

We begin by formulating the equations of heat flow describing the transfer of thermal energy. Heat energy is caused by the agitation of molecular matter. Two basic processes take place in order for thermal energy to move: conduction and convection. **Conduction** results from the collisions of neighboring molecules in which the kinetic energy of vibration of one molecule is transferred to its nearest neighbor. Thermal energy is thus spread by conduction even if the molecules themselves do not move their location appreciably. In addition, if a vibrating molecule moves from one region to another, it takes its thermal energy with it. This type of movement of thermal energy is called **convection**. In order to begin our study with relatively simple problems, we will study heat flow only in cases in which the conduction of heat energy is much more significant than its convection. We will thus think of heat flow primarily in the case of solids, although heat transfer in fluids (liquids and gases) is also primarily by conduction if the fluid velocity is sufficiently small.

1.2 DERIVATION OF THE CONDUCTION OF HEAT IN A ONE-DIMENSIONAL ROD

Thermal energy density. We begin by considering a rod of constant cross-sectional area A oriented in the x-direction (from $x = 0$ to $x = L$) as illustrated in Fig. 1.2.1. We temporarily introduce the amount of thermal energy per unit volume as an unknown variable and call it the **thermal energy density**:

$$\boxed{e(x, t) \equiv \text{thermal energy density.}}$$

We assume that all thermal quantities are constant across a section; the rod is one-dimensional. The simplest way this may be accomplished is to insulate perfectly the lateral surface area of the rod. Then no thermal energy can pass through the lateral surface. The dependence on x and t corresponds to a situation in which the rod is not uniformly heated; the thermal energy density varies from one cross section to another.

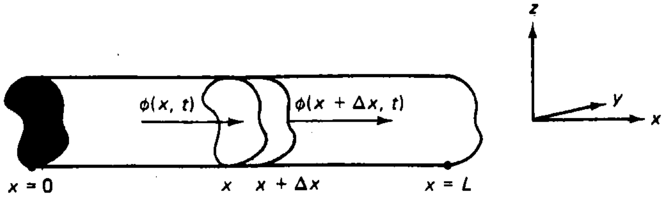

Figure 1.2.1 One-dimensional rod with heat energy flowing into and out of a thin slice.

Heat energy. We consider a thin slice of the rod contained between x and $x + \Delta x$ as illustrated in Fig. 1.2.1. If the thermal energy density is constant throughout the volume, then the total energy in the slice is the product of the thermal energy density and the volume. In general, the energy density is not constant. However, if Δx is exceedingly small, then $e(x, t)$ may be approximated as a constant throughout the volume so that

$$\boxed{\text{heat energy} = e(x, t)A\,\Delta x,}$$

since the volume of a slice is $A\,\Delta x$.

Conservation of heat energy. The heat energy between x and $x + \Delta x$ changes in time due only to heat energy flowing across the edges (x and $x + \Delta x$) and heat energy generated inside (due to positive or negative sources of heat energy). No heat energy changes are due to flow across the lateral surface, since we have assumed that the lateral surface is insulated. The fundamental heat flow process is described by the word equation

$$\boxed{\begin{matrix}\text{rate of change}\\ \text{of heat energy}\\ \text{in time}\end{matrix} = \begin{matrix}\text{heat energy flowing}\\ \text{across boundaries}\\ \text{per unit time}\end{matrix} + \begin{matrix}\text{heat energy generated}\\ \text{inside per unit time.}\end{matrix}}$$

This is called **conservation of heat energy**. For the small slice, the rate of change of heat energy is

$$\frac{\partial}{\partial t}\,[e(x, t)A\,\Delta x],$$

where the partial derivative $\partial/\partial t$ is used because x is being held fixed.

Heat flux. Thermal energy flows to the right or left in a one-dimensional rod. We introduce the **heat flux**

$$\boxed{\phi(x, t) = \begin{matrix}\text{heat flux (the amount of thermal energy } \textit{per unit}\\ \textit{time} \text{ flowing to the right } \textit{per unit surface area}\text{).}\end{matrix}}$$

If $\phi(x, t) < 0$, it means that heat energy is flowing to the left. Heat energy flowing per unit time across the boundaries of the slice is $\phi(x, t)A - \phi(x + \Delta x, t)A$, since the heat flux is the flow per unit surface area and it must be multiplied by the surface area. If $\phi(x, t) > 0$ and $\phi(x + \Delta x, t) > 0$, as illustrated in Fig. 1.2.1, then the heat energy flowing per unit time at x contributes to an increase of the heat energy in the slice, whereas the heat flow at $x + \Delta x$ decreases the heat energy.

Heat sources. We also allow for internal sources of thermal energy:

$$\boxed{Q(x, t) = \text{heat energy per unit volume generated per unit time,}}$$

perhaps due to chemical reactions or electrical heating. $Q(x, t)$ is approximately constant in space for a thin slice, and thus the total thermal energy generated per unit time in the thin slice is approximately $Q(x, t)A\,\Delta x$.

Conservation of heat energy (thin slice). The rate of change of heat energy is due to thermal energy flowing across the boundaries and internal sources:

$$\frac{\partial}{\partial t}[e(x, t)A\,\Delta x] \approx \phi(x, t)A - \phi(x + \Delta x, t)A + Q(x, t)A\,\Delta x. \qquad (1.2.1)$$

Equation (1.2.1) is not precise because various quantities were assumed approximately constant for the small cross-sectional slice. We claim that (1.2.1) becomes increasingly accurate as $\Delta x \to 0$. Before giving a careful (and mathematically rigorous) derivation, we will just attempt to explain the basic ideas of the limit process, $\Delta x \to 0$. In the limit as $\Delta x \to 0$, (1.2.1) gives no interesting information, namely, $0 = 0$. However, if we first divide by Δx and then take the limit as $\Delta x \to 0$, we obtain

$$\frac{\partial e}{\partial t} = \lim_{\Delta x \to 0} \frac{\phi(x, t) - \phi(x + \Delta x, t)}{\Delta x} + Q(x, t), \qquad (1.2.2)$$

where the constant cross-sectional area has been canceled. We claim that this result is exact (with no small errors), and hence we replace the $\approx$ in (1.2.1) by $=$ in (1.2.2). In this limiting process, $\Delta x \to 0$, t is being held fixed. Consequently, from the definition of a partial derivative,

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.} \qquad (1.2.3)$$

Conservation of heat energy (exact). An alternative derivation of conservation of heat energy has the advantage of our not being restricted to small slices. The resulting approximate calculation of the limiting process ($\Delta x \to 0$) is avoided. We consider any *finite* segment (from $x = a$ to $x = b$) of the original one-dimensional rod (see Fig. 1.2.2). We will investigate the conservation of heat energy in this region. The total heat energy is $\int_a^b e(x, t)\,dx$, the sum of the contributions of the infinitesimal slices. Again it changes only due to heat energy flowing through the side edges ($x = a$ and $x = b$) and heat energy generated inside the region, and thus (after canceling the constant A)

$$\boxed{\frac{d}{dt}\int_a^b e\,dx = \phi(a, t) - \phi(b, t) + \int_a^b Q\,dx.} \qquad (1.2.4)$$

Figure 1.2.2 Heat energy flowing into and out of a finite segment of a rod.

Technically, an ordinary derivative d/dt appears in (1.2.4) since $\int_a^b e\,dx$ depends only on t, not also on x. However,

$$\frac{d}{dt}\int_a^b e\,dx = \int_a^b \frac{\partial e}{\partial t}\,dx,$$

if a and b are constants (and if e is continuous). This holds since inside the integral the ordinary derivative now is taken keeping x fixed, and hence it must be replaced by a partial derivative. Every term in (1.2.4) is now an ordinary integral if we notice that

$$\phi(a, t) - \phi(b, t) = -\int_a^b \frac{\partial \phi}{\partial x}\,dx,$$

(this* being valid if ϕ is continuously differentiable). Consequently,

$$\int_a^b \left(\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q\right) dx = 0.$$

This integral must be zero *for arbitrary a and b*; the area under the curve must be zero for arbitrary limits. This is possible only if the integrand itself is identically zero.† Thus, we rederive (1.2.3) as

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.} \tag{1.2.5}$$

Equation (1.2.4), the **integral conservation law**, is more fundamental than the differential form (1.2.5). Equation (1.2.5) is valid in the usual case in which the physical variables are continuous.

A further explanation of the minus sign preceding $\partial\phi/\partial x$ is in order. For example, if $\partial\phi/\partial x > 0$ for $a \leq x \leq b$, then the heat flux ϕ is an increasing function of x. The heat is flowing greater to the right at $x = b$ than at $x = a$ (assuming that $b > a$). Thus (neglecting any effects of sources Q), the heat energy must decrease between $x = a$ and $x = b$, resulting in the minus sign in (1.2.5).

Temperature and specific heat. We usually describe materials by their **temperature**,

$$\boxed{u(x, t) = \text{temperature},}$$

not their thermal density. Distinguishing between the concepts of temperature

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* This is one of the fundamental theorems of calculus.

† Most proofs of this result are inelegant. Suppose that $f(x)$ is continuous and $\int_a^b f(x)\,dx = 0$ for arbitrary a and b. We wish to prove $f(x) = 0$ for all x. We can prove this by assuming that there exists a point x_0 such that $f(x_0) \neq 0$ and demonstrating a contradiction. If $f(x_0) \neq 0$ and $f(x)$ is continuous, then there exists some region near x_0 in which $f(x)$ is of one sign. Pick a and b to be in this region, and hence $\int_a^b f(x)\,dx \neq 0$ since $f(x)$ is of one sign throughout. This contradicts the statement that $\int_a^b f(x)\,dx = 0$, and hence it is impossible for $f(x_0) \neq 0$. Equation (1.2.5) follows.

and thermal energy is not necessarily a trivial task. Only in the mid-1700s did the existence of accurate experimental apparatus enable physicists to recognize that it may take different amounts of thermal energy to raise two different materials from one temperature to another larger temperature. This necessitates the introduction of the **specific heat** (or heat capacity):

$$\boxed{c = \text{specific heat (the heat energy that must be supplied to a unit mass of a substance to raise its temperature one unit).}}$$

In general, from experiments (and our definition) the specific heat c of a material depends on the temperature u. For example, the thermal energy necessary to raise a unit mass from 0°C to 1°C could be different from that needed to raise the mass from 85°C to 86°C for the same substance. Heat flow problems with the specific heat depending on the temperature are mathematically quite complicated. (Exercise 1.2.1 briefly discusses this situation.) Often for restricted temperature intervals, the specific heat is approximately independent of the temperature. However, experiments suggest that different materials require different amounts of thermal energy to heat up. Since we would like to formulate the correct equation in situations in which the composition of our one-dimensional rod might vary from position to position, the specific heat will depend on x, $c = c(x)$. In many problems the rod is made of one material (a uniform rod), in which case we will let the specific heat c be a constant. In fact, most of the solved problems in this text (as well as other books) correspond to this approximation, c constant.

Thermal energy. The thermal energy in a thin slice is $e(x, t)A\,\Delta x$. However, it is also defined as the energy it takes to raise the temperature from a reference temperature 0° to its actual temperature $u(x, t)$. Since the specific heat is independent of temperature, the heat energy per unit mass is just $c(x)u(x, t)$. We thus need to introduce the **mass density** $\rho(x)$:

$$\boxed{\rho(x) = \text{mass density (mass per unit volume),}}$$

allowing it to vary with x, possibly due to the rod being composed of nonuniform material. The total mass of the thin slice is $\rho A\,\Delta x$. The total thermal energy in any thin slice is thus $c(x)u(x, t) \cdot \rho A\,\Delta x$, so that

$$e(x, t)A\,\Delta x = c(x)u(x, t)\rho A\,\Delta x.$$

In this way we have explained the basic relationship between thermal energy and temperature:

$$\boxed{e(x, t) = c(x)\rho(x)u(x, t).} \tag{1.2.6}$$

This states that the thermal energy per unit volume equals the thermal energy per unit mass per unit degree times the temperature times the mass density (mass per unit volume). When the thermal energy density is eliminated using (1.2.6), conservation of thermal energy, (1.2.3) or (1.2.5), becomes

$$\boxed{c(x)\rho(x)\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.} \tag{1.2.7}$$

Fourier's law. Usually, (1.2.7) is regarded as one equation in two unknowns, the temperature $u(x, t)$ and the heat flux (flow per unit surface area per unit time) $\phi(x, t)$. How and why does heat energy flow? In other words, we need an expression for the dependence of the flow of heat energy on the temperature field. First we summarize certain qualitative properties of heat flow with which we are all familiar:

1. If the temperature is constant in a region, no heat energy flows.
2. If there are temperature differences, the heat energy flows from the hotter region to the colder region.
3. The greater the temperature differences (for the same material), the greater is the flow of heat energy.
4. The flow of heat energy will vary for different materials, even with the same temperature differences.

Fourier (1768–1830) recognized properties 1 through 4 and summarized them (as well as numerous experiments) by the formula

$$\boxed{\phi = -K_0\frac{\partial u}{\partial x},} \tag{1.2.8}$$

known as **Fourier's law of heat conduction**. Here $\partial u/\partial x$ is the derivative of the temperature; it is the slope of the temperature (as a function of x for fixed t); it represents temperature differences (per unit length). Equation (1.2.8) states that the heat flux is proportional to the temperature difference (per unit length). If the temperature u increases as x increases (i.e., the temperature is hotter to the right, $\partial u/\partial x > 0$), then we know (property 2) that heat energy flows to the left. This explains the minus sign in (1.2.8).

We designate the coefficient of proportionality K_0. It measures the ability of the material to conduct heat and is called the **thermal conductivity**. Experiments indicate that different materials conduct heat differently; K_0 depends on the particular material. The larger K_0 is, the greater the flow of heat energy with the same temperature differences. A material with a low value of K_0 would be a poor conductor of heat energy (and ideally suited for home insulation). For a rod composed of different materials, K_0 will be a function of x. Furthermore, experiments show that the ability to conduct heat for most materials is different at different temperatures, $K_0(x, u)$. However, just as with the specific heat c, the dependence on the temperature is often not important in particular problems.