CONTROL APPLICATIONS OF NONLINEAR PROGRAMMING AND OPTIMIZATION

Edited by

G. DI PILLO

IFAC

CONTROL APPLICATIONS OF NONLINEAR PROGRAMMING AND OPTIMIZATION

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G. DI PILLO

Dipartimento di Informatica e Sistemistica, University of Rome "La Sapienza", Rome, Italy

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PREFACE

This volume contains a selection of papers presented at the Workshop on Control Applications of Nonlinear Programming and Optimization held in Capri, Italy, during 11-14 June 1985.

The purpose of the Workshop was to exchange ideas and information on the applications of optimization and nonlinear programming techniques to real life control problems, to investigate new ideas that arise from this exchange and to look for advances in nonlinear programming and optimization theory which are useful in solving modern control problems.

The Workshop benefited of the sponsorship of the International Federation of Automatic Control (IFAC) through the Committees on Theory and on Mathematics of Control. It was the fifth IFAC Workshop on the subject.

The attendance to the Workshop was of fiftyfive experts from sixteen countries. Four invited and twentysix contributed papers were presented and discussed; invited speakers were A.E. Bryson, Jr., R. Bulirsch, H.J. Kelley and J.L. Lions.

The scientific program of the Workshop covered various aspects of the optimization of control systems and of the numerical solution of optimization problems; specific applications concerned the optimization of aircraft trajectories, of mineral and metallurgical processes, of wind tunnels, of nuclear reactors; computer aided design of control systems was also considered in some papers.

The scientific program was arranged by an International Committee chaired by Angelo Miele (USA), with other members being G.Di Pillo (Italy), F.M. Kirillova (USSR) D.Q. Mayne (UK), N. Olhoff (Denmark), B.L. Pierson (USA), H.E. Rauch (USA), A. Ruberti (Italy), R.W.H. Sargent (UK) and K.H. Well (FRG).

All contributed papers included in this volume have been reviewed; thanks are due, for their contribution in the reviewing procedure, to R. Bulirsch, J.L. de Jong, P. Fleming, H.J. Kelley, F.M. Kirillova, J.L. Lions, D.Q. Mayne, A. Miele, H.J. Oberle, B.L. Pierson, A.L.Tits, K.H. Well and F. Zirilli.

Finally, it was a great pleasure for me to have served as chairman of the organizing committee.

Gianni Di Pillo

CONTENTS

ON THE ORTHOGONAL COLLOCATION AND MATHEMATICAL PROGRAMMING APPROACH FOR STATE CONSTRAINED OPTIMAL CONTROL PROBLEMS

O. E. Abdelrahman* and B. M. Abuelnasr**

**Department of Mathematics and Computer Sciences, Zagazig University, Zagazig, Egypt*
***Department of Computer Sciences and Automatic Control, Faculty of Engineering, University of Alexandria, Egypt*

Abstract. The orthogonal collocation approach is now well known to solve, effectively, the state constrained optimal control problems. Mathematical programming technique was also used as an effective tool to construct the optimal trajectories. In this paper, a study is done on the efficiency and accuracy requirements of the combined orthogonal collocation and mathematical programming approach, as regarding the employed optimization algorithm, and the number of orthogonal collocation points. It is shown, by experimentation with numerical examples that Fletcher-Powell optimization algorithm is much more faster to produce convergence than Fletcher-Reeves algorithm. The efficiency can be a ratio of six-to-one. The results are compared with an alternative approach to solve the same problem. It is shown that the present algorithm is less costly than the alternative approach, although requiring more computation time. The choice is then a compromise one. As the number of orthogonal points increases, the resulting solutions are more accurate, but the convergence speed decreases. Experimentation with N, shows a save of five-to-one in computing time can be achieved with almost the same cost function. Finally, it is shown, by a numerical example, that uniformly distributed collocation points result in non-optimal solutions, which also violate the problem constraints. It is a numerical proof of the superiority of the orthogonal collocation approach.

Keywords. Orthogonal collocation; mathematical programming; optimization algorithms; convergence speed; state constrained problems.

INTRODUCTION

State constrained optimal control problems pose a challenging two point boundary value problem(TPBVP). Different approaches exist which solve the resulting TPBVP. The orthogonal collocation approach, as a method of approximating functions, is used to construct the problem solutions with good to excellent accuracies(Oh and Luss, 1977, and Abdelrahman, 1980). Combined with mathematical programming, the orthogonal collocation was used to solve the state constrained optimal control problem(Abuelnasr and Abdelrahman, 1981). The emphasis on just getting a numerically programmed solution without examining the optimization algorithm, which finally gives the required solution, sometimes result in non efficient solutions, as far as computation time is concerned. In this paper, a look at two optimization algorithms, namely that of Fletcher-Reeves and Fletcher-Powell (Kuester and Mize, 1973), is shown to give a comparatively large efficiency. Also, we look at the approximating method of the orthogonal collocation. It is found possible to obtain almost optimal solutions with a reasonable number of collocation points. The orthogonality of the collocation points is also shown to be the right choice for approximating the solution of problem, as an otherwise choice based on non-orthogonal collocating points will give erroneous results.

STATEMENT OF THE PROBLEM AND ITS SOLUTION

Given the state description of the dynamic system as

$$\dot{x} = f(x,u,t) \quad , \quad x(0) = x_0 \qquad (1)$$

where $x(t)$ is an nx1 state vector, $u(t)$ is an rx1 control vector, and $f(x,u,t)$ is an nx1 vector function of x,u,and t. The control vector $u(t)$ is assumed unconstrained.

It is required to minimize, with respect to u, the cost functional

$$J(u) = \int_0^{t_f} L(x,u,t) \ dt, \qquad (2)$$

subject to the differential constraints of Eq.(1), and the state inequality constraint

$$s(x,t) \leq 0 \qquad (3)$$

The solution of the above posed problem, using the orthogonal collocation approach is well known(Abuelnasr and Abdelrahman, 1981). Here, we give a brief outline of the steps which will lead finally to the posed problem solution. Thus, the solution of the problem will consist of three stages. The first stage formulates a TPBVP for the following unconstrained optimization problem.

Minimize J(u), given by Eq.(2), with respect to u , subject to the differential constraints given by Eq.(1). This is done by defining the Hamiltonian of the problem

$$H(x,u,\lambda,t) = L(x,u,t) + \lambda^T f(x,u,t), \quad (4)$$

where $\lambda(t)$ is an nx1 adjoint vector, known also as Lagrange multiplier vector, and $(\ )^T$ denotes vector transposition. The following canonic equations and the necessary condition of optimality will result

$$\dot{x} = \frac{\partial H}{\partial \lambda}, \quad \dot{\lambda} = -\frac{\partial H}{\partial x} \quad (5)$$

$$\frac{\partial H}{\partial u} = 0 \quad (6)$$

Using Eq.(6) in Eq.(5) will give the following two sets of equations

$$\dot{x} = f(x,\lambda,t), \quad x(0) = x_0$$
$$\dot{\lambda} = g(x,\lambda,t), \quad \lambda(t_f) = 0, \quad (7)$$

while the optimal control, u(t) is obtained from Eq.(6). In Eqs.(7), g is an nx1 vector function of x, λ, and t. Also, the solution of Eqs.(7) poses a TPBVP.

The second stage is the transformation of the TPBVP, obtained from the first stage, into a corresponding set of algebraic equations by using the collocation of x(t) and $\lambda(t)$ over the time interval $(0,t_f)$ (see Appendix I). Good to excellent accuracies can be achieved using collocation points chosen as the zeros of transformed Legendre polynomials(A transformed Legendre polynomial is a Legendre polynomial defined on (0,1)).

Denote the set of algebraic equations by

$$F(y) = 0, \quad (8)$$

where y is a vector of order 2nx(N+1), n being the problem dimension, and N is the number of interior collocation points. The details of getting Eq.(8) is illustrated in Appendix I(the case of unconstrained optimal control problem is illustrated , where the state equations are linear and the cost is a quadratic in x and u).

The components of y are those of x(t) and $\lambda(t)$ at the interior collocation points, i.e., y can be written as

$$y = y_x + y_\lambda, \quad (9)$$

where y_x and y_λ are , respectively, the approximations of x(t) and $\lambda(t)$, using orthogonal collocation. Also, y can be partitioned into two components, y_C and y_{NC}, where y_C is an mx(N+1) vector of constrained state components, m being the dimension of the constrained variables in the inequality (3), while y_{NC} is a (2n-m)x(N+1) vector of unconstrained components and the adjoint vector at (N+1) points.

Finally, the third stage solves a constrained optimization problem by using the technique of mathematical programming. Thus

(i) The inequality constraint(3) is transformed into an equality constraint by introducing a slack variable $\alpha(t)$, to obtain

$$s(x,t) + 0.5\alpha^2(t) = 0 \quad (10)$$

The equality (10) is then evaluated at the N interior collocation points chosen as the zeros of transformed Legendre polynomials to give

$$G(y_C,\alpha) = 0, \quad (11)$$

where G is an mx(N+1) vector.

(ii) Construct a cost function, CF, as follows

$$CF = \sum_{i=1}^{2n(N+1)} F_i^2(y) \quad (12)$$

where F_i is the ith component of F.

(iii) Minimize the cost function, CF, subject to the equality constraint given by Eq.(11), using a suitable optimization algorithm.

NUMERICAL EXAMPLES

The examples introduced below are extracted from the control literature(Jacobson and Lele, 1969). A comparison of this work with other authors work can thus be done to evaluate the presented algorithms.

Example 1.
consider

$$\dot{x}_1 = x_2, \quad x_1(0) = 0$$
$$\dot{x}_2 = -x_2 + u, \quad x_2(0) = -1 \quad (13)$$

with the following performance index to be minimized with respest to u

$$J(u) = \int_0^{t_f} (x_1^2 + x_2^2 + 0.005\,u^2)\,dt, \quad (14)$$

subject to the inequality constraint

$$x_1(t) - 8(t-0.5)^2 + 0.5 \leq 0 \quad (15)$$

The inequality(15) is of the second order, which means it has to be differentiated twice to obtain the control u(t).

<u>Formulation of the Solution</u>
The technique of mathematical programming and orthogonal collocation of section two will be used to solve the example. Thus, first, the unconstrained TPBVP is formulated as follows

$$\dot{x}_1 = x_2, \quad x_1(0) = 0$$
$$\dot{x}_2 = -x_2 - 100\lambda_2, \quad x_2(0) = -1$$
$$\dot{\lambda}_1 = -2x_1, \quad \lambda_1(1) = 0 \quad (16)$$
$$\dot{\lambda}_2 = -2x_2 - \lambda_1 + \lambda_2, \quad \lambda_2(1) = 0$$

while the optimal control, u(t), is given by

$$u = -100\lambda_2$$

Then, introduce the slack variable, $\alpha(t)$, to inequality(15) to obtain the equality

$$x_1(t) - 8(t-0.5)^2 + 0.5 + 0.5\alpha^2(t) = 0 \quad (17)$$

By following the procedure of the last section, a set of algebraic equations are obtained from Eqs.(16) and Eq.(17). The optimization algorithms of Fletcher-Reeves and Fletcher-Powell are then applied to obtain the problem solution. Table 1 shows a comparison of the speed of convergence of the obtained solutions, using the two algorithms. For comparison purposes, the corresponding results of Jacobson and Lele(1969) are included in Table 1.

TABLE 1 Results of Two Optimization Algorithms

Algorithm	Number of Iterations	Cost Function, J
Fletcher-Reeves*	466	0.7045
Fletcher-Powell*	76	0.6742
*N=7, the number of collocation points		
Jacobson and Lele(1969)	16	0.75

It is observed, by looking at Table 1, that the second algorithm, namely Fletcher-Powell is much more efficient than Fletcher-Reeves. The efficiency can be measured in terms of the ratio of the number of iterations to produce convergence. In the table, Fletcher-Powell is six times more efficient than Fletcher-Reeves. By comparison, the Jacobson and Lele approach produces results, which are more efficient , although giving a slightly higher cost. A common feature of all methods used in Table 1, is the use of conjugate gradients to search for the minimum of the objective function. Besides, Jacobson and Lele used the conjugate gradient in the function space, which proved to be more efficient than the normal conjugate gradient(Lasdon, Mitter, and Waren, 1967). The wide variation in the number of iterations in the first two lines of the table, is that Fletcher-Powell is a second order method, which produces quadratic convergence(see Appendix II, where a matrix H is used to accelerate convergence); while Fletcher-Reeves is a first order method, which gives linear convergence.

Another point to be discussed, i.e. the effect of orthogonality of collocation points on the approximation of the solution of the problem. For this purpose, example 1 is re-solved, but using collocation points not based on the zeros of orthogonal polynomials. The points are chosen on an equal interval basis. The obtained results are shown in Table 2.

TABLE 2 Results of Two Sets of Collocation Points

N=7	Cost Function J
1. Collocation points are zeros of Legendre polynomials*	0.7045
2. Collocation points distributed equally on (0,1)*	1.0411

* Fletcher-Reeves algorithm is implemented

The resulting trajectories for the choices in Table 2 are plotted in Fig. 1 . Table 1 and Fig. 1 show a non-optimal solution for the equal interval collocation points, which motivates the use of the orthogonal collocation method for approximating the problem solutions.

Example 2.
Same as example 1, except that this case treats a constraint of the first order, given by

$$x_2(t) - 8(t-0.5)^2 + 0.5 \leq 0 \qquad (18)$$

In this example, N, the number of orthogonal collocation points is given values of 4,5,6,and 7. The results are illustrated in Table 3 and Fig. 2.

TABLE 3 Results of Varying N on the Convergence Speed and the Optimal Cost Function

N	Number of Iterations	Cost Function, J
4	149	0.134408
5	112	0.145040
6	119	0.145990
7	505	0.136334
Note: A Fletcher-Reeves algorithm is implemented in this example		
Jacobson and Lele(1969) (Using the Conjugate Gradient Method)	16	0.164

The table shows, at first, a decrease, then an increase in the convergence iteration cycles. This phenomena is due to the interplay between the truncation and the round-off error. As N increases, the accuracy increases, and consequently the truncation error decreases. But, the roundoff error increases by increasing N. There is, thus a value of N at which there is a minimum for the combined truncation and roundoff errors. The figure also confirms the above claims. The optimum cost function, J, is not much sensitive to the variation of N. Thus, based on the number of iterations and the associated graphs in Fig. 2, an optimum value for N can be selected. The presented values of the table show that N=5 is an optimum choice. For comparison, the results of Jacobson and Lele are included. The same observations and comments concerning the number of iterations and accuracy will apply as before. In addition, the increase of N, is assosiated with an increase of the number of equations to be solved, in the form 2n(N+1); and thus more computation time is needed for convergence. But, a gain in accuracy is evident as shown in the third column of Table 3. The advantage of the conjugate gradient in the function space is also noted.

CONCLUSIONS

Looking at the results in the tables and their associated figures, several conclutions can be reached. The orthogonal collocation and mathematical programming is presented as an alternative approach to solve the state constrained optimal control problem. It seems appropriate to be compared to other approaches, as was shown in Table 1, and Table 3, as far accuracy is concerned. It is also concluded that the conjugate gradient in the function space produces faster convergence than the normal conjugate gradient, as used in Fletcher-Reeves and Fletcher-Powell algorithms. The last conclsion accounted for the relatively smaller iterations in Table 1 and Table 3 of Jacobson and Lele results. The final conclusion of the paper is a strong preference to the use of orthogonal collocation other than any other method of approximating the variables of the problem.

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APPENDIX I

The orthogonal Collocation Approach for Unconstrained TPBVP

This appendix is a summary of the necessary steps followed to solve an unconstrained TPBVP, using orthogonal collocation . The procedure is taken from Oh and Luss(1977). Thus, consider the following problem, a system is described by

$$\dot{x} = f(x,u,t) \quad , \; x(0)= x_o \tag{19}$$

where x,u,and f have been defined before, and x_o is the initial state vector. It is required to find the optimal control function, u(t), which minimizes

$$J(u)= \int_0^{t_f} L(x,u,t)\, dt \; , \tag{20}$$

subject to the differential constraint of Eq.(19). The steps of solving the above posed problem are as follows.

(1)Define the Hamiltonian,H

$$H(x,u, \lambda,t)=L(x,u,t) + \lambda^T(t)f(x,u,t), \tag{21}$$

From H, obtain the following canonic equations

$$\begin{aligned} \dot{x}&=f(x, \lambda ,t) \quad , \; x(0)=x_o \\ \dot{\lambda}&=g(x, \lambda, t) \quad , \; \lambda(t_f)= 0 \end{aligned} \tag{22}$$

where λ is obtained from

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \quad , \tag{23}$$

and the optimal control,u, is obtained from the necessary condition

$$\frac{\partial H}{\partial u} = 0 \tag{24}$$

The set of equations(22) constitutes a TPBVP.

(2)Choose a number of interior collocation points,N,as the N zeros of a transformed Legendre polynomial. Then,expand x(t) and λ(t) into a finite power series in t as follows

$$\begin{aligned} x(t)&=\sum_{j=1}^{N+2} c_j \; t^{j-1} \\ \lambda(t)&=\sum_{j=1}^{N+2} d_j \; t^{j-1} \quad , \end{aligned} \tag{25}$$

for $0 \leq t \leq t_f$. The unknown constants c_j, d_j; j=1,2,...,N+2 can be determined by satisfying Eqs.(25)at t=0,at $t=t_f$, and at N interior collocation points. By imposing that Eqs.(22)are satisfied at $t=0, t_1, t_2, \ldots, t_N, t_{N+1}=t_f$, then 2n(N+1) equations will result. By using the initial condition on x(t) and the final condition on λ(t), another set of 2n equations is obtained.Then, in all, 2n(N+2) algebraic equations can be formed in the unknowns c_j, d_j, j=1,2,...,N+2.

(3)Depending on the form of f and L,the resulting set of 2n(N+2) algebraic equations can be solved, giving the unknown coefficients in x(t) and λ(t). If f is linear and L is quadratic functions of x and u,then Eqs.(22) become linear in x and λ . In this case, the unknown coefficients can be obtained by solving the following matrix equation

$$AC = B \quad , \tag{26}$$

where $C = [c_1, c_2, \ldots, c_{N+2}, d_1, d_2, \ldots, d_{N+2}]^T$, is a 2n(N+2)vector of coefficients, A is a known square matrix of dimension 2n(N+2), and B is a known 2n(N+2) vector. If A is invertible, then C is obtained from

$$C = A^{-1}B \quad , \tag{27}$$

where A^{-1} is inverse matrix of A.

(4)Once C is obtained, hence c_j, d_j; j=1,2,....N+2. By substituting C in Eqs.(25), approximations of x(t), and λ(t), using N interior collocation points, can be obtained. Also, the optimal vector,u(t), can be approximated by using Eqs.(24) and the obtained value of C.

APPENDIX II

Fletcher-Reeves and Fletcher-Powell Optimization Algorithms

The purpose of these algorithms is to find a local minimum of unconstrained function of more than one variable. Both algorithms use conjugate gradients to generate the necessary changes in the function variables Thus, consider the unconstrained minimization of the following function

$$F(x_1, x_2, \ldots, x_p),$$

using,first, the Fletcher-Reeves algorithm. The steps proceed as follows

(1)A starting point is selected,i.e., $x_1^{(0)}, x_2^{(0)}, \ldots, x_p^{(0)}$ are chosen.

(2) The direction of steepest descent is determined by determining the following direction vector components(normalized form) at the starting point,

$$M_i^{(k)} = \left\{ \frac{-\frac{\partial F}{\partial x_i}}{\left[\sum_{j=1}^{p} \left(\frac{\partial F}{\partial x_j}\right)^2 \right]^{\frac{1}{2}}} \right\}^{(k)} , i=1,2,\ldots,p,$$

where k=0, for the starting point.

(3)A one dimensional search is then conducted along the direction of steepest descent utilizing the relation,

$$x_{i(new)} = x_{i(old)} + sM_i \ , \ i=1,2,\ldots,p,$$

where s is the distance moved in the $\underline{M}$ directions. When a minimum is obtained along the direction of steepest descent, a new "conjugate direction" is evaluated at the new point with the following normalized components

$$M_i^{(k)} = \frac{-\left(\frac{\partial F}{\partial x_i}\right)^{(k)} + \beta^{(k-1)} M_i^{(k-1)}}{\left[\sum_{i=1}^{p} \left\{ -\frac{\partial F^{(k)}}{\partial x_i} + \beta^{(k-1)} M_i^{(k-1)} \right\}^2 \right]^{\frac{1}{2}}}$$

where i=1,2,...,n, and

$$\beta^{(k-1)} = \frac{\sum_{i=1}^{p} \left[\left(\frac{\partial F}{\partial x_i}\right)^{(k)} \right]^2}{\sum_{i=1}^{p} \left[\left(\frac{\partial F}{\partial x_i}\right)^{(k-1)} \right]^2}$$

(4)A one dimensional search is then conducted in this direction. When a minimum is found, an overall convergence check is made. If convergence is achieved, the procedure terminates. If convergence is not achieved, new "conjugate direction" vector components are evaluated per step(3). This process continues until convergence is achieved or n+1 directions have been reached. If a cycle of n+1 directions have been completed, a new cycle is started consisting of a steepest descent direction(step 2)and n "conjugate directions"(step 3).

The Fletcher-Powell algorithm proceeds as follows

(1)Select a starting point.

(2)Compute a direction of search. In normalized form, this is as follows

$$M_i^{(k)} = \frac{-\sum_{j=1}^{p} H_{i,j} \left(\frac{\partial F}{\partial x_j}\right)^{(k)}}{\left[\sum_{l=1}^{p} \left(\sum_{j=1}^{p} H_{l,j} \frac{\partial F}{\partial x_j} \right)^2 \right]^{\frac{1}{2}}} ,$$

where i=1,2,...,p, k is the iteration index (k=0 at the starting point), M_i are the direction vector components, and $\partial F/\partial x_j$ is the jth component of the gradient vector. $H_{i,j}$ is the i-j element of a symmetric positive definite matrix(pxp), which is initially chosen to be the identity matrix. Thus, the initial direction of search is the path of steepest descent.

(3)A one dimensional search is conducted in the direction chosen by the previous step until a minimum is located using the relation

$$x_{i(new)} = x_{i(old)} + sM_i, \ i=1,2,\ldots,p,$$

where s is the step size in the direction of search.

(4)A convergence check is made. If covergence is achieved, the procedure is terminated. Otherwise, a new search direction is chosen per step(2) except $H^{(k+1)}$ is calculated as follows

$$H^{(k+1)} = H^{(k)} + A^{(k)} - B^{(k)} ,$$

where

$$A^{(k)} = \frac{\Delta x^{(k)} (\Delta x^{(k)})^T}{(\Delta x^{(k)})^T (\Delta G^{(k)})}$$

$$B^{(k)} = \frac{H^{(k)} \Delta G^{(k)} (\Delta G^{(k)})^T H^{(k)}}{(\Delta G^{(k)})^T H^{(k)} \Delta G^{(k)}}$$

$$x^{(k)} = x^{(k+1)} - x^{(k)}$$

$$\Delta G^{(k)} = \left(\frac{\partial F}{\partial x}\right)^{(k+1)} - \left(\frac{\partial F}{\partial x}\right)^{(k)}$$

A new one dimensional search is performed in the new direction. The process is repeated until convergence is obtained.

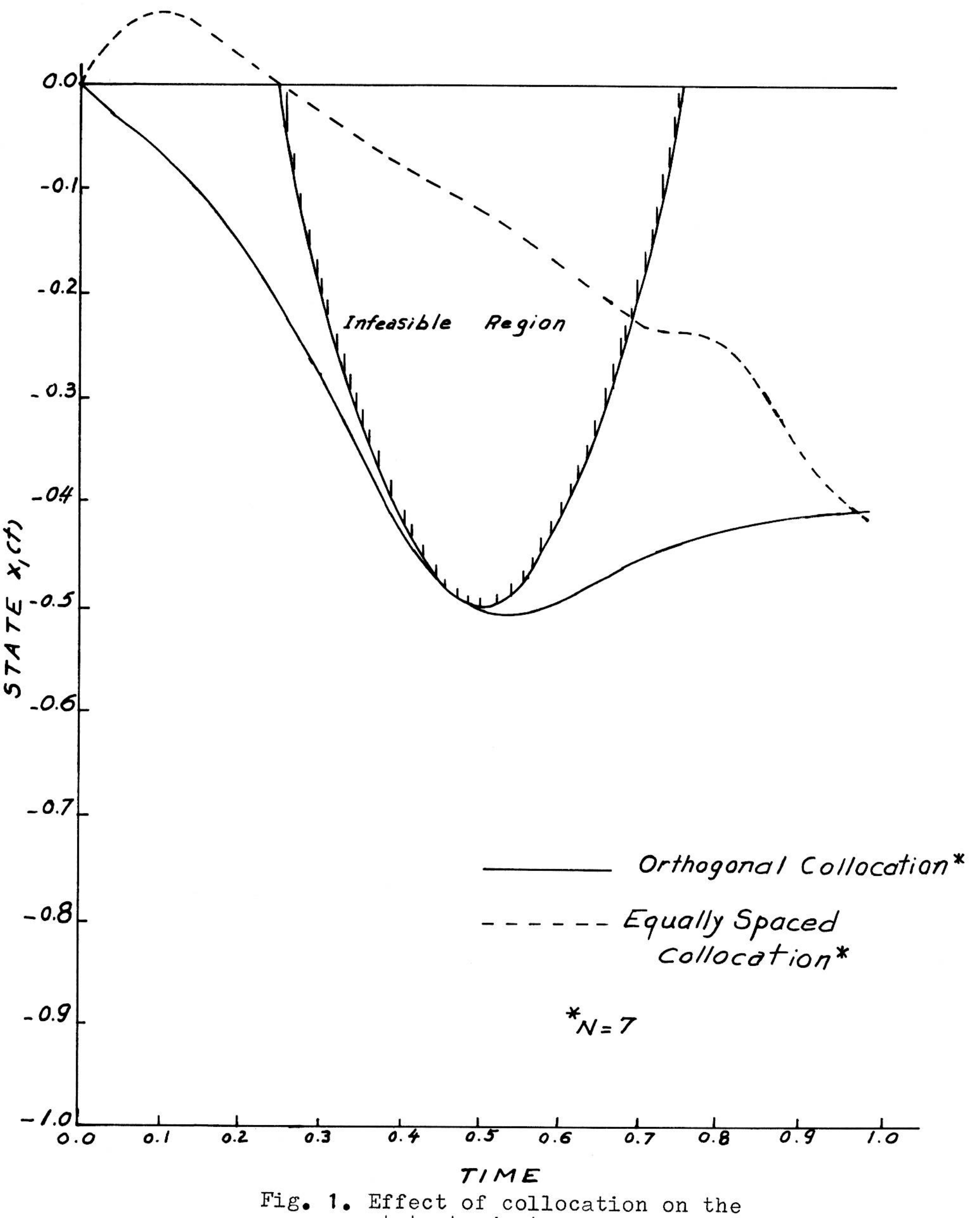

Fig. 1. Effect of collocation on the state trajectory

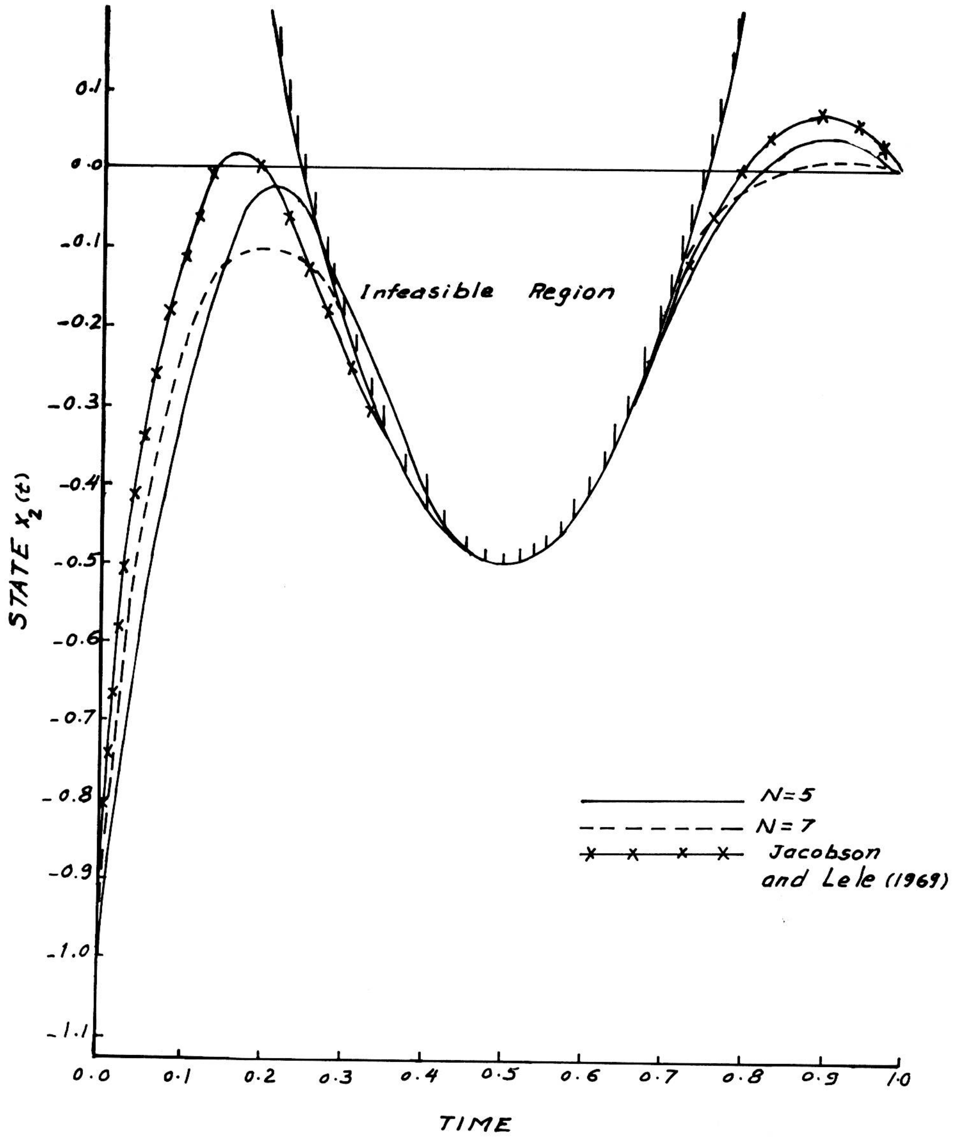

Fig. 2. Effect of the number of interior collocation points, N, on the state trajectory