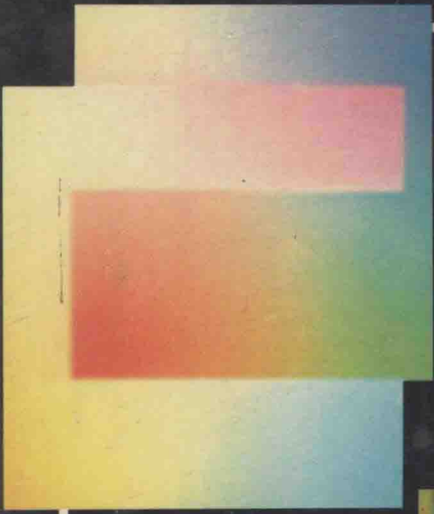
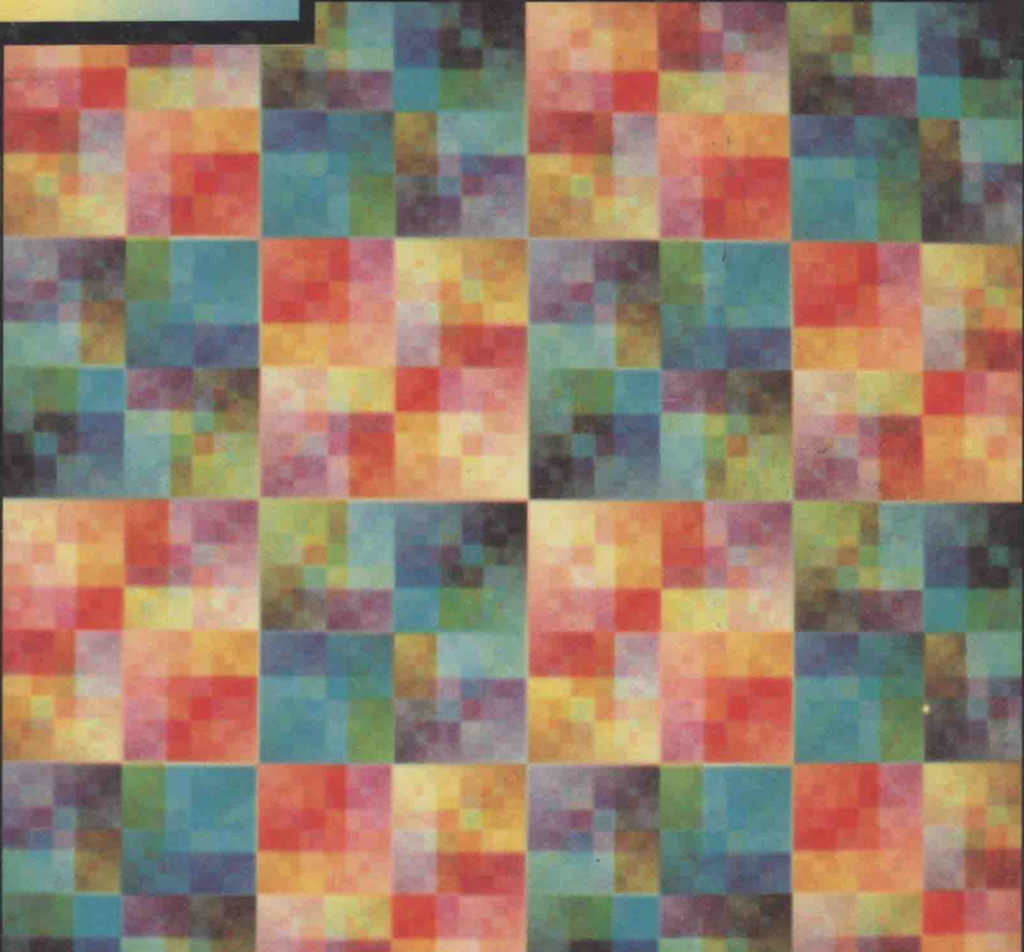


FUNDAMENTAL DEVELOPMENTS *of* COMPUTER-AIDED GEOMETRIC MODELING



Edited by LES PIEGL

Fundamental Developments of Computer-Aided Geometric Modeling

Edited by
Les A. Piegl
University of South Florida

ACADEMIC PRESS

Harcourt Brace & Company, Publishers
London San Diego New York
Boston Sydney Tokyo Toronto

Academic Press Limited
24–28 Oval Road
London NW1 7DX

United States Edition published by
Academic Press Inc.
San Diego, CA 92101

This book is printed on acid-free paper.

A CIP record for this book is available from the British Library
ISBN 0-12-554765-X

Typeset by P&R Typesetters Limited,
Salisbury, Wiltshire, UK
Printed and bound in Great Britain by the Bath Press, Avon

Contributors

Andrew P. Armit May Cottage, 21 Greenside, Waterbeach, Cambridge CB5 9HW, UK

Robert E. Barnhill Department of Computer Science and Engineering, Arizona State University, Tempe, AZ 85287-5406, USA

Pierre E. Bézier 12 Avenue Gourgaud, 75017 Paris, France

Carl de Boor Mathematics Research Center, University of Wisconsin-Madison, 610 Walnut Street, Madison, WI 53705, USA

Ian C. Braid Three-Space Limited, 70 Castle Street, Cambridge CB3 0AJ, UK

Paul de Casteljau 4 Avenue du Commerce, 7800 Versailles, France

Maurice G. Cox DITC, Building 93, National Physical Laboratory, Teddington, Middlesex TW11 0LW, UK

Charles M. Eastman Graduate School of Architecture and Urban Planning, University of California, Los Angeles, 405 Hilgard Avenue, Los Angeles, CA 90024-1467, USA

James C. Ferguson BBN Systems and Technologies Corporation, Advanced Simulation Division, 14100 South East 36th Street, Bellevue, WA 98006, USA

William J. Gordon Department of Mathematics and Computer Science, Drexel University, Philadelphia, PA 19104, USA

Robert W. Mann Department of Mechanical Engineering, Massachusetts Institute of Technology, Room 3-144, Cambridge, MA 02139, USA

Michael J. Pratt Design Research Center, Rensselaer Polytechnic Institute, 110 8th Street, Troy, NY 12180, USA

Aristides A. G. Requicha Computer Science Department, University of Southern California, Los Angeles, CA 90089-0782, USA

Richard F. Riesenfeld Department of Computer Science, University of Utah, Salt Lake City, UT 84112, USA

David F. Rogers Aerospace Engineering Department, Department of the Navy, United States Naval Academy, Annapolis, MD 21402-5042, USA

Douglas T. Ross SofTech, Inc., 460 Totten Pond Road, Waltham, MA 02154, USA

Malcolm A. Sabin FEGS Limited, 5 Coles Lane, Oakington, Cambridge CB4 5BA, UK

Herbert B. Voelcker Sibley School of Mechanical & Aerospace Engineering, Cornell University, Grumman & Upson Halls, Ithaca, NY 14853-7501, USA

. . ., we show modern methods for geometry,
so constructed and organized that geometric manipulations
can be performed in a way natural to the computer,
and can yield results that are natural to man.

Steven Anson Coons

Preface

Les Piegl

Welcome to *Fundamental Developments of Computer-Aided Geometric Modeling:* a collection of papers written by pioneers of Computer-Aided Design. This book is for every computer professional who is interested in the scientific basics as well as in the historical/evolutionary aspects of computer geometry.

This book is neither a text book nor a history book. It combines science and history. The reader will find enough material to understand computational geometric methods as used in CAD/CAM, and at the same time the book gives an exposure to how things were discovered from the late 1950s. Computer-Aided Geometric Modeling has helped shape the future of computer supported design. The contributors of this volume had vision, knowledge and dedication to promote progress. Please accept this book as a reflection of their dedication to the betterment of the contemporary society.

HOW THIS BOOK CAME TO BE

Five years ago, Jenny Hayes, former Editor of the journal *Computer-Aided Design*, was soliciting contributions from Advisory Editors to the upcoming 20th anniversary issue. *Computer-Aided Design* was founded in 1968, and it was time to celebrate its 20 years of success. As a fresh member of the board, I was pleased to offer a paper, and to suggest that it would be interesting to pay tribute to pioneering developments in CAD. The field was so young that almost all the pioneers were still alive, and it sounded like a good idea to edit a special issue with contributions from the pioneers of Computer-Aided Design. Jenny liked the idea, however, she argued that a special issue would be too small a forum, and suggested putting a book together instead. After a couple of quick letter exchanges, I was given an offer to edit an anniversary volume on the pioneering works in CAD. Being completely green in book publishing, and having no idea how difficult it was to edit such a volume, I accepted the job. What followed after that was a series of refinements of the initial project. It was clear that one book was insufficient to cover the entire spectrum of CAD, so we restricted our attention to the geometric aspects of Computer-Aided Design, commonly referred to as *Computer-Aided Geometric Modeling* (other names used are Geometric Modeling, Computer-Aided Geometric Design (CAGD), Computational Geometry, etc. – you pick your favorite name). A series of discussions with well-known people followed my initial contact, and lots of modifications were made on the initial draft. By mid-1988, the scope of the project was more or less finalized, and the first chapter was completed (although in French). It took us four more years to bring the book to completion.

Looking back at the five years I spent on this book, I have all kinds of memories ranging from sweet ones to not so sweet ones. I have learned a lot about people, about publishers, and about managing a project in which different people are involved from different countries on different continents. I hand this book over to the reader with great sentiments, and in the hope that he/she will enjoy reading it at least as much as we enjoyed doing it. This book is unique in the sense that everything was written by the inventors themselves weaving their personalities into the subject. It also provides descriptions on all the methods used widely in Computer-Aided Design and Manufacturing.

WHO AND WHAT IS IN THIS BOOK?

There are numerous techniques used to represent and process geometric information using a computer. Some of them are useful, and many of them are considered as mathematical or computational plums only. Most of them have become parts of standard university curriculums in CAD and/or in Graphics, and have initiated substantial research activities in many fields including mathematics, computer science, engineering, art, business, geography; just to name a few.

We start the book with two French pioneers, Paul de Casteljau and Pierre Bézier. Working independently for two French car companies, Citroën and Renault, they discovered an approximation method for free-form curve and surface design. Although company policies and language barriers did not allow de Casteljau to publicize his results, his contribution is equally as important as those who published extensively. De Casteljau writes about polar forms, whereas Bézier gives an overview of the first years of CAD/CAM, and describes the mathematical basis of Renault's UNISURF system.

B-splines were investigated thoroughly by mathematicians way before the advent of computers in engineering design. Two remarkable mathematicians, Carl de Boor and Maurice Cox, have developed algorithms and mathematical tools for their efficient evaluation and processing. The basics of B-splines are presented by de Boor, and algorithms for spline curves and surfaces are given by Cox.

The application of B-splines to Computer-Aided Design was investigated by Richard Riesenfeld. Riesenfeld gives an overview of B-spline curves and surfaces as used in design. He also shares his experience of building a modeler called *Alpha_1* as a research testbed.

Interpolation is another way to represent free-form entities. The chapters by James Ferguson, William Gordon and Robert Barnhill discuss curve and surface techniques based on blending function methods. Ferguson describes the use of Hermite interpolation for curve, surface and hypersurface definition. Gordon's chapter is the first open account of the so-called 'Gordon' surface used at General Motors. Coons' patches and convex combinations are covered by Barnhill, who researched the field over the past 20 years.

The above techniques are used to represent one single entity, i.e. a curve, a surface or a volume. To define more complicated (realistic) objects, the collection of many simple geometric entities is required. The new entity is termed as *solid,* and is dealt with by Ian Braid, Charles Eastman, Herbert Voelcker and Aristides Requicha. There are a number of ways to define a solid, and there are different aspects of solid modeling. Braid overviews boundary modeling where solids are described by their boundary surfaces. Eastman's chapter describes architectural design and research testbeds developed at Carnegie Mellon University. Voelcker and Requicha are well known for their research in Constructive Solid Geometry (CSG), and for the solid modelers developed at the University of Rochester. The history of their research accomplishments at the University of Rochester is described in their chapter.

Malcolm Sabin has made contributions to many branches of geometric design. In his chapter he gives a tour of his 20 years of investigations into many aspects of computer-aided geometry.

Computer-Aided Design, at least its geometric modeling aspects, relies heavily on geometric methods. All kinds of geometric tools, including those of classical analytic geometry, projective geometry, differential geometry and descriptive geometry, are used to define and process shape information. Michael Pratt surveys geometric tools, including those of classical geometry as well as modern computer geometry, used in the CAD, as well as in the CAM process.

Inventing a design tool is not the end of the line in the engineering design process. A computer system has to be built in such a way that it is user friendly enough to be useful for non-computer hackers. No matter how sophisticated the mathematical scheme is, it is completely useless if nobody can make it part of a user-understandable CAD system. Andrew Armit takes us on an excursion into the world of several systems he has built over the last 25 years. He describes two of his better known systems, Multipatch and Multiobject, and follows by describing more recent and, of course, more sophisticated systems.

Graphical communication between the computer and the user, and post-processing of the results for numerical control milling, are two very important ingredients in the complete design process. The information should be brought into the database, and the results should be interpreted in such a way that they are natural to the computer as well as to the user. After the design has been completed and accepted by the designer, the non-existent object has to be turned into a real one by, say, carving it out from some kind of material. The chapter by David Rogers is about interactive computer graphics and numerical control as used at the US Naval Academy's CAD and Graphics Laboratory.

The last two chapters are by two MIT

pioneers, Robert Mann and Douglas Ross. Mann, a close friend and colleague of Steven Coons, describes research activities in Computer-Aided Design at MIT from the late 1950s through to the mid 1960s. Ross concludes the book with a philosophy of Computer-Aided Design as seen from a perspective resulting from 30 years' experience in the field.

REGRETS

I regret that I was not able to convince Bertram Herzog and Robin Forrest to contribute to this volume. I feel their names should be mentioned, and they should be credited for the significant work they have accomplished.

APOLOGIES

I offer my apology to those whom I have left out, or whose contributions just could not fit into this volume. I am sure that the reviewer or the reader will find someone in the computer geometry area whose contribution he/she feels is important to warrant inclusion in this book. I think to give credit to everybody, one really needs to edit an encyclopedia. Well, I was not prepared to edit an encyclopedia.

ACKNOWLEDGEMENTS

It is my great pleasure to acknowledge the help of many people. First I thank Jenny Hayes for the idea of editing this book. If it was not for Jenny, this book would not have been written. Her encouragement at the beginning of the project kept me going even if I had only a slim chance to complete what I had in mind. Jenny, wherever you are, may God bless your heart!

I thank my contributors for their support and co-operation throughout the life of this project. It was a great pleasure working with top notch people who were not only professionals in their scientific fields at the highest degree, but who treated me as a professional colleague despite the fact that I could not even come close to them. Thank you all for understanding the importance of this project, for the excellent chapters, and for the personal encouragement and support that I needed at times.

Special thanks are due to Jack Gregory and Michel Menard for translating de Casteljau's chapter from French to English. Jeff Stevens provided illustrative material for Gordon's chapter, for which I am particularly grateful.

Les Piegl
June, 1992
Tampa, Florida

Contents

1 Polar Forms for Curve and Surface Modeling as used at Citroën

Paul de Casteljau

Originating in 1958 at Citroën, the methodology of mathematical definition of shapes (curves and surfaces) has been taught at the drawing school of Citroën since 1960 [1]. The shapes are defined with the aid of pilot points called poles, which give a good idea of the shape of the curve. From an algebraic point of view, the poles are equivalent to segments or patches of parametric polynomial forms, and possess all of their properties. They allow great flexibility by a progressive deformation of the shape under the influence of the corresponding pole. At the beginning, the limited performance of the computing power available at that time forced us to use simple poles, without considering the continuity between segments.

The word 'pole' comes from the triple repetition of 'pol' in the expression 'Inter*pol*ation of *pol*ynomials with *pol*ar forms'. The construction of shapes from the poles is completely independent of the computation of the poles from sampled data. The theory is pretty much algebraic, and is related to integral calculus, quadratic minimization and orthogonal function theory. Those notions can create more difficulties than they solve, and therefore we do not enter into their details here.

I started within an exceptional team, spirited by M. de la Boixiere, who, in the pure tradition of the Pilgrims (in the sense of caring only for efficiency) was investigating industrialization by numerical control. To the great amazement of the engineers, he constructed, *ex nihilo*, high performance machines previously unavailable at that time. For example, he produced a pneumatic paper tape reader, a stepper motor, a variable speed tape recorder (its impulses were produced by a car distributor!), an electrical power cabinet, a five axis milling machine, etc. Nothing could discourage him. Only one detail remained: the calculation of the pieces to be machined. The objective was a forged or cast shape, which was believed to be mathematically well defined.

Later, my interest turned to car bodies; the designers were astonished and scandalized. Was it some kind of joke? It was considered nonsense to represent a car body mathematically. It was enough to please the eye, the word accuracy had no meaning, and wind tunnel testing was done after the car was already in production. Between orthogonal cross-sections of a piece, no one would admit a 0.8 mm discrepancy. Worse, the difference between the right and left fender of a car reached the centimeter mark! Far from being discouraged, I persisted in my desire to make a test. From then on, there was no doubt, I was considered insane . . .

The first surface computed with an electromechanical machine was the size of a postage stamp. It was defined with 350 points of the cutter location offset surface, and the noise of the printer annoyed the people nearby! When M. de la Boixière saw the mound of paper tapes, he became doubtful and claimed: "That will generate a million useless cubics!" Prudently, Citroën designated a draftsman, working currently with the author, who digitized the tool path on a 20:1 drawing so as to achieve acceptable results.

In the end, it was the mathematician who comforted the draftsman. The idea of numerically representing the car body was taken seriously. Citroën bought a Burrough's E101 computer (128 program steps, a memory of about 220, about seven multiplications a second, and it required 5 kW of power!). This allowed more ambitious computations, such as the hood, the dash board, and also 'the flying saucer', a dream of an extinct aviation firm (see Figure 1.1).

Editor's note: the terminology used by the author is different from that used in several well-known textbooks such as I. D. Faux and M. J. Pratt, *Computational Geometry for Design and Manufacture* (Ellis Horwood, Chichester, 1989). The 'parameter sequence' is the *knot vector*, 'Poles' or 'Generalized Poles' are the *control points*, sometimes called *de Boor points*, and 'Simple Poles' are the *Bézier control points*.

Fundamental Developments of Computer-Aided Geometric Modeling
ISBN 012-554765-X

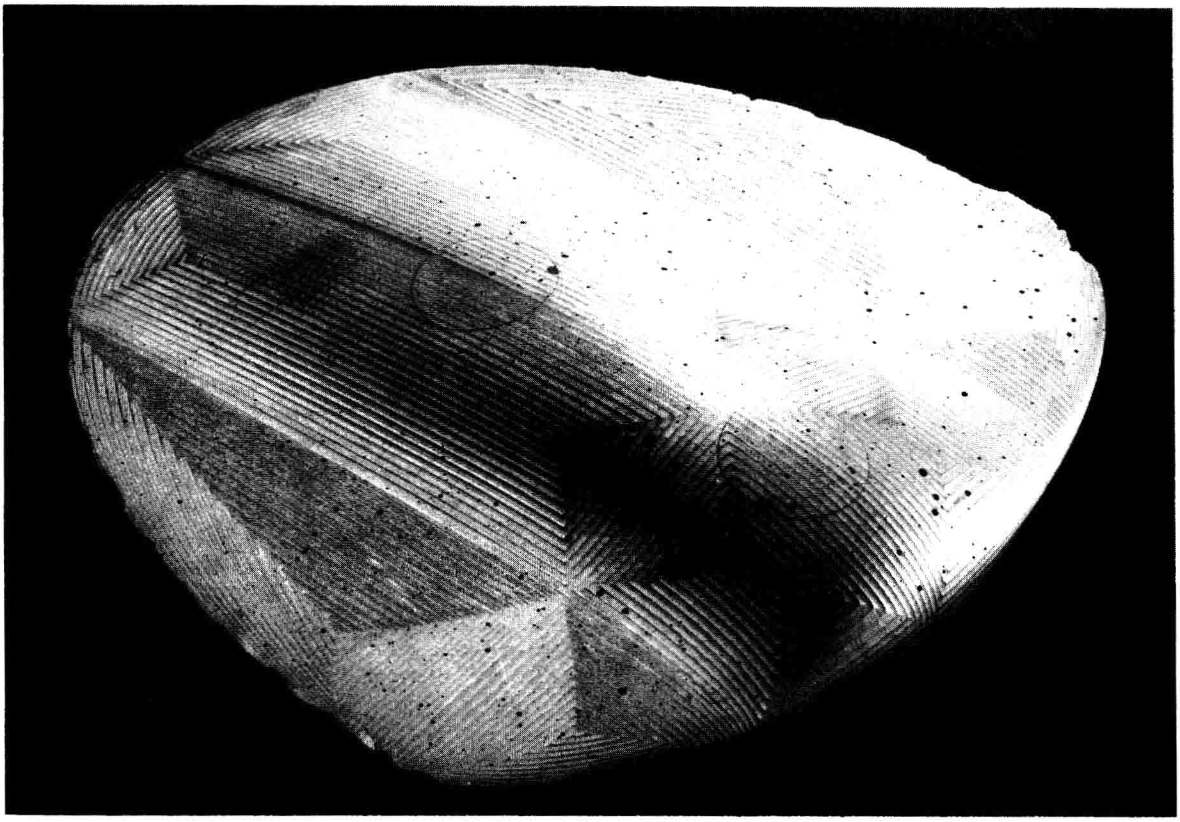


Figure 1.1 A premiere model from 1962: the 'Flying Saucer'. Notice the spiral machining marks and the rectangular and triangular patches.

Being convinced, the management decided to use this technique inside the company in 1963, not without some resentment and jealousy, and this allowed the full drawing of a car, the Citroën GS, for the first time in history. This generated a system at Citroën to compute the shape (SPAC-CAR-G), and another to produce the machining information (SPAC-CAR-U).

DEFINITION OF POLAR FORMS

Parametric curves

Assume that we are given the points $\mathbf{P}_i$, called poles, and a set of polynomial functions $\phi_i(t)$, $i = 0, \ldots, n$. The form

$$\mathbf{P} = \sum_{i=0}^{n} \phi_i(t)\mathbf{P}_i \tag{1}$$

defines a parametric curve. To make this definition useful practically, we require the polynomials to sum to 1 ($\sum_i \phi_i(t) = 1$), i.e. for a fixed t, $\phi_i(t)$ are the barycentric coordinates of $\mathbf{P}$ with respect to the poles $\mathbf{P}_i$.

The vector $\mathbf{OP}$ can be obtained as the sum of vectors proportional to the vectors $\mathbf{OP}_i$, for any given origin $\mathbf{O}$, and therefore $\mathbf{O}$ is omitted in later discussion. Expression (1) is valid for each coordinate, or projection on any line or axis, and can be extended to several parameters to generate a surface patch.

Example

$$\phi_i(t, u) = C_{n,i}\, t^i u^{n-i} = \binom{n}{i} t^i u^{n-i}$$

$$= (t + u)^n, \quad \text{where } t + u = 1.$$

Two types of surfaces are of practical importance:

- rectangular with $T + U = 1$. Each polynomial is obtained from $(t + u)^n (T + U)^N$.
- triangular with $t + u + v = 1$. The polynomials are of the form $t^\alpha u^\beta v^\gamma$ (with $\alpha + \beta + \gamma = n$) generated by $(t + u + v)^n$.

Without loss of generality, we may assume that t, u, v, T and U vary between 0 and 1.

Parameter sequences

We consider the non-decreasing series of parameters

$$t_0 \leqslant t_1 \leqslant \ldots \leqslant t_i \leqslant t_{i+1} \leqslant \ldots \leqslant t_f \tag{2}$$

with the assumption that the number of equal parameters is less than or equal to the degree of the curve. The insertion of one (or more) new t value(s) into the interval $[t_i, t_{i+1}]$ $(t_i \neq t_{i+1})$ will generate one (or more) new sequence(s). This operation is called *parameter insertion*. In the discussions that follow, we assume that t denotes either t_i, a constant t_α, or the variable t.

Symmetric function of the parameters

Any function $f(t_1, t_2, \ldots, t_n)$ which is invariant when two variables t_i and t_j are exchanged is called a *symmetric function*. Newton demonstrated that we can express this function with the *elementary symmetric functions* (see [6] for details):

$$f(t_1, t_2, \ldots, t_n) = \sum_{i=0}^{n} g_i(x_1, x_2, \ldots, x_n)$$

where

$$\begin{aligned} x_1 &= t_1 + t_2 + \ldots + t_n \\ x_2 &= t_1t_2 + t_1t_3 + \ldots + t_{n-1}t_n \\ &\vdots \\ x_n &= t_1t_2 \ldots t_n \end{aligned} \tag{3}$$

These functions can be obtained by expressing

$$\begin{aligned} &(1 + t_1)(1 + t_2) \ldots (1 + t_n) \\ &\quad = S_{n,0} + S_{n,1} + S_{n,2} + \ldots + S_{n,n}, \; S_{n,0} = 1 \end{aligned} \tag{4}$$

One can easily verify, by introducing the new factor $(1 + t)$, the following recurrence formula:

$$S_{n+1,p} = S_{n,p} + tS_{n,p-1} \tag{5}$$

Example

Introduce the fourth variable t (which can be t_0 or t_4) to the sequence t_1, t_2, t_3 to get

$$\begin{aligned} S_{4,0} &= 1 \\ S_{4,1} &= t_1+t_2+t_3 + t(1) \\ S_{4,2} &= t_1t_2+t_1t_3+t_2t_3 + t(t_1+t_2+t_3) \\ S_{4,3} &= t_1t_2t_3 + t(t_1t_2+t_1t_3+t_2t_3) \\ S_{4,4} &= t(t_1t_2t_3) \end{aligned} \tag{6}$$

Operations on poles

Let us assume that $\phi_i(t)$ in equation (1) are symmetric functions expressed in terms of elementary symmetric functions (see (3)) with variables extracted from (2) (the number of variables is equal to the degree). We can consider two types of operation:

1. *Insertion:* obtained by inserting a parameter into a given sequence. This creates two sequences skewed by one unit from one another

   $$\begin{array}{l} |\longleftarrow n \text{ values} \longrightarrow| \\ t_i \leqslant t_{i+1} \leqslant \ldots \leqslant t_{j-1} \leqslant t_j \\ \quad\;\; |\longleftarrow n \text{ values} \longrightarrow| \end{array} \tag{7}$$

   These two sequences will be abbreviated by s' and s''.

2. *Difference:* the barycentric form

   $M = uA + tB$ with

   $t + u = 1$ or $u = 1 - t$,

can be expressed as $M = A + \mathbf{t}AB$, where $AB = B - A$. We can do the same on the barycentric form (1) using the sequences s' and s'' (see below).

Multilinear function of degree n

A function E, which is the function of elementary symmetric functions, can be expressed as follows (see (5)):

$$E(t_{i+1},\ldots,t_{j-1},\mathbf{t}) = K_{i+1,\ldots,j-1} + \mathbf{t}Q_{i+1,\ldots,j-1} \qquad (8)$$

In case t has the values t_i and t_j, and E denotes a polar form of degree n, we have

$$P_{s'} = K + t_i Q$$
$$P_{s''} = K + t_j Q$$

where K and Q stand for $K_{i+1,\ldots,j-1}$, and $Q_{i+1,\ldots,j-1}$, respectively, and s' and s'' are the sequences obtained from (7). K and Q can be expressed as

$$K = \frac{t_j P_{s'} - t_i P_{s''}}{t_j - t_i} = \Delta P_{i+1,\ldots,j-1}$$
$$Q = \frac{P_{s''} - P_{s'}}{t_j - t_i} = \delta P_{i+1,\ldots,j-1} \qquad (9)$$

Δ and δ are two true operators applied on the polar forms giving multilinear forms of degree $n-1$ with indices as the common indices. We can reiterate these operators to discover their commutative property, i.e. $\Delta\delta = \delta\Delta$. Consequently, we demonstrated the formula

$$E(t_{i+1},\ldots,t_{j-1},\mathbf{t}) = \Delta E_{i+1,\ldots,j-1} + \mathbf{t}\delta E_{i+1,\ldots,j-1}, \quad t_i \leqslant t \leqslant t_j \qquad (10)$$

This algorithm is known as the Cox–de Boor algorithm used for index insertion. A similar formula can be obtained using the second differences $\Delta\Delta E$, $\delta\Delta E = \Delta\delta E$ and $\delta\delta E$, and step-by-step we can compute the multilinear form E.

Remark

If we repeat the operators Δ and δ n times, we get the algebraic coefficients of the polynomial (see example below). Under a change of the parameter $t = k\theta + h$, the poles, which have a geometrical nature, will be invariant, and therefore we can restrict ourselves to the interval $[0, 1]$, or sometimes to $[-1, 1]$, to simplify calculations.

Example

Consider a degree 3 curve. We use the following notations: **P** and **A** denote the vectors **OP** and **OA** with **O** as the origin, and **B**, **C** and **D** are difference vectors, e.g. **B** points from **A** (the startpoint) to the next pole:

$$\mathbf{P}(t_1, t_2, \mathbf{t}) = \mathbf{A} + [(t_1 + t_2) + \mathbf{t}]\mathbf{B} + [t_1t_2 + \mathbf{t}(t_1 + t_2)]\mathbf{C} + \mathbf{t}(t_1t_2)\mathbf{D} \qquad (11)$$

$$= [\mathbf{A} + (t_1 + t_2)\mathbf{B} + t_1t_2\mathbf{C}] + \mathbf{t}[\mathbf{B} + (t_1 + t_2)\mathbf{C} + t_1t_2\mathbf{D}] \qquad (12)$$

From equation (12) we get the poles P_{012} (for $t = t_0$), and P_{123} (for $t = t_3$), called Riesenfeld points (see Figure 1.2). Similarly, for $t = t_4$ and $t = t_5$ we obtain P_{234} and P_{345}. Using formula (9), we have

$$\delta P_{12} = \frac{t_3 P_{012} - t_0 P_{123}}{t_3 - t_0} = \mathbf{A} + (t_1 + t_2)\mathbf{B} + t_1t_2\mathbf{C}$$
$$\Delta P_{12} = \frac{P_{123} - P_{012}}{t_3 - t_0} = \mathbf{B} + (t_1 + t_2)\mathbf{C} + t_1t_2\mathbf{D} \qquad (13)$$

which gives $P(t_1, t_2, \mathbf{t}) = \Delta P_{12} + \mathbf{t}\delta P_{12}$ in harmony with (10). Now, computing ΔP_{23} and δP_{23}, we have

$$\Delta\Delta P_2 = \frac{t_3\Delta P_{12} - t_1\Delta P_{23}}{t_3 - t_1} = \mathbf{A} + t_2\mathbf{B}$$

$$\delta\Delta P_2 = \frac{\Delta P_{23} - \Delta P_{12}}{t_3 - t_1} = \mathbf{B} + t_2\mathbf{C} \qquad (14\,\mathrm{a})$$

$$\delta\delta P_2 = \frac{\delta P_{23} - \delta P_{12}}{t_3 - t_1} = \mathbf{C} + t_2\mathbf{D}$$

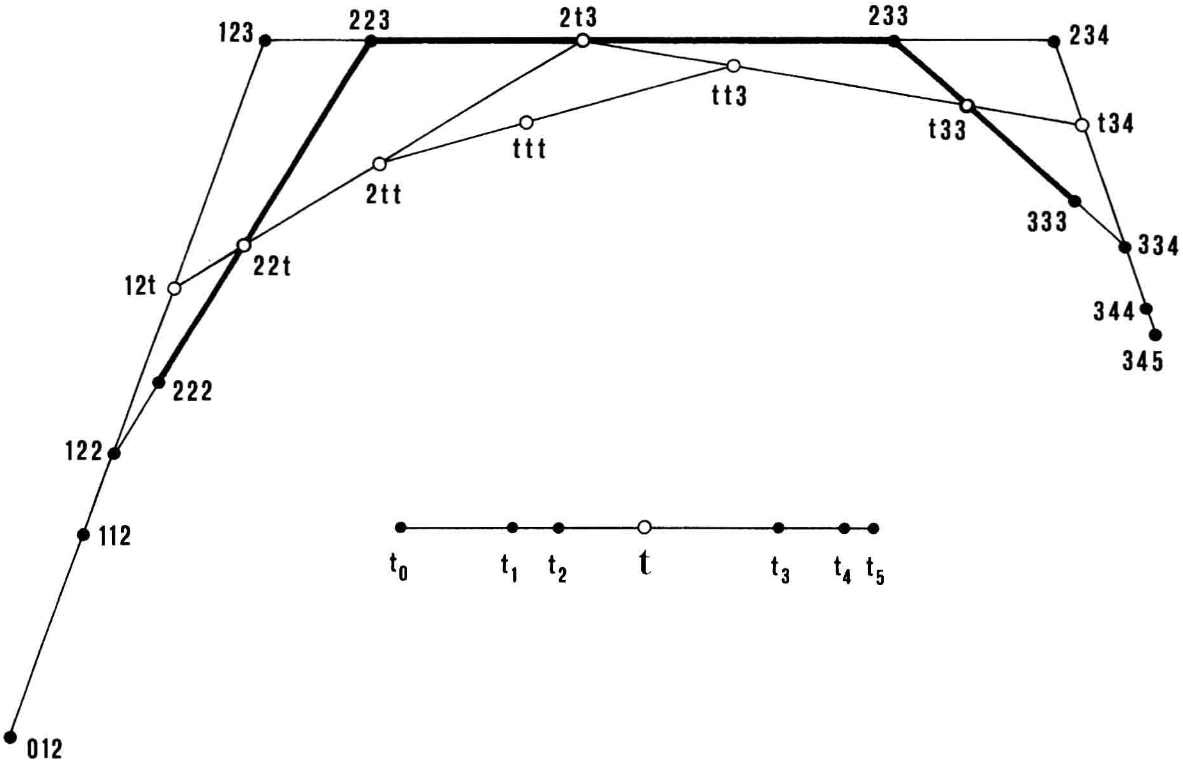


Figure 1.2 Geometric construction of a point for a fixed parameter t. On each segment, only the varying index corresponds to the ratio. The lines joining the poles are drawn in bold.

With P_{345}, we finally get the coefficients of the polynomial

$$\begin{aligned}\Delta\Delta\Delta P &= \mathbf{A}\\ \delta\Delta\Delta P &= \mathbf{B}\\ \delta\delta\Delta P &= \mathbf{C}\\ \delta\delta\delta P &= \mathbf{D}\end{aligned} \tag{14b}$$

Simple poles

This is a very important particular case. The parameters t_{i-1}, t_i and t_{i+1} can be repeated up to a maximum equal to the degree. For simplicity, we assume that $t_i - t_{i-1} = t_{i+1} - t_i$. Without loss of generality, we can take $t_{i-1} = -1$, $t_i = 0$ and $t_{i+1} = 1$. This will help us to compare two consecutive segments at the point $t = 0$: the first is defined by **A**, **B**′, **C**′, **D**′ for $t < 0$, and the second is given by **A**, **B**″, **C**″, **D**″ for $t > 0$. Let us compute the difference table for degree $n = 3$:

	Difference 1	Difference 2	Difference 3
$P_{-1-1-1} = \mathbf{A} - 3\mathbf{B}' + 3\mathbf{C}' - \mathbf{D}'$			
	$\mathbf{B}' - 2\mathbf{C}' + \mathbf{D}'$		
$P_{-1-10} = \mathbf{A} - 2\mathbf{B}' + \mathbf{C}'$		$\mathbf{C}' - \mathbf{D}'$	
	$\mathbf{B}' - \mathbf{C}'$		$\mathbf{D}'$
$P_{-100} = \mathbf{A} - \mathbf{B}'$		$\mathbf{C}'$	
	$\mathbf{B}'$		
$P_{000} = \mathbf{A}$			
	$\mathbf{B}''$		
$P_{001} = \mathbf{A} + \mathbf{B}''$		$\mathbf{C}''$	
	$\mathbf{B}'' + \mathbf{C}''$		$\mathbf{D}''$
$P_{011} = \mathbf{A} + 2\mathbf{B}'' + \mathbf{C}''$		$\mathbf{C}'' + \mathbf{D}''$	
	$\mathbf{B}'' + 2\mathbf{C}'' + \mathbf{D}''$		
$P_{111} = \mathbf{A} + 3\mathbf{B}'' + 3\mathbf{C}'' + \mathbf{D}''$			

(15)

where, for example, P_{001} denotes $P(0, 0, 1)$ obtained from equation (11).

Now the study of continuity is straightforward, and is a function of the number of equal coefficients **B**, **C** and **D** corresponding to the terms, t, t^2, t^3 (with $C_{3,p}$ omitted):

- common point **A** in all cases
- common tangent if in addition $B' = B'' = B$
- continuous curvature if in addition $C' = C'' = C$
- algebraic identity if in addition $D' = D'' = D$.

Although this verification was done for $n = 3$, it can be extended to any degree. If we set all the variables in (11) to t, the multiplication of the difference table by $C_{n,p}t^p$ will give the desired polynomial.

Example

Let us take the case $P_{ttt} = t^3$:

	8				
		4			
	12		2		
		6		1	
arc 2,3	18		3		
		9			
	27				
		9			
	36		3		
arc 3,4		12		1	(16)
	48		4		
		16			
	64				
	...				
	n^3				
		n^2			
	mn^2		n		
arc m,n		mn		1	
$(m = n + 1)$	m^2n		m		
		m^2			
	m^3				

Despite the fact that these tables are longer than the well-known Newton differences, they do not have useless coefficients. The successive steps are the poles of the arc. This kind of table presentation is very useful to verify the continuity or to compute the coefficients. For more details, consult [2] or [3].

The simple poles of an arc are unique, and define the arc independently of the parameter (generalized poles are dependent on the parameter sequence). They are also closer to the curve. For any algebraic arc AB, the unique and non-ambiguous references are the simple poles of the arc. It is recognized that Riesenfeld points provide a more condensed form when the degree and the number of points are large.

Algebraic calculus using the poles

The algebraic calculations are based on parameter insertion. For any sequence of parameters $t_0, \ldots, t_i, t_j, \ldots, t_f$, in order to get the simple poles of the arc $[t_i t_j]$ we insert the parameter t_i $n - p$ times and the parameter t_j $n - q$ times, where n is the order of the curve and p and q are the original repetitions of the indices of i and j, respectively.

At the beginning of the arc (defined by t_i with initial repetition p), because we have inserted the parameter t_i $n - p$ times, this will generate $n - p$ algebraic common conditions in the difference table, and therefore set the continuity c to $n - p$. We can see that, not including the point at the beginning, we have, on both consecutive arcs, $n - p$ simple poles which are derived from the same sequence of poles.

Example

Given degree 5, and the parameter sequence 012223345566, what is the continuity c at the point P_3 of parameter $t = 3$? To define arcs $[t_2, t_3]$ and $[t_3, t_4]$, we need at least five indices before 3 and after 5. It is necessary to insert index 3 three times ($n = 5$, $p = 2$), and then inserting index 4

P_{22233}			
	P_{22333}		
P_{22334}		P_{23333}	
	P_{23334}		P_{33333}
P_{23345}		P_{33334}	
	P_{33345}		
P_{33455}			

then: (17)

$$\mathbf{P}_{33333}$$

$$\mathbf{P}_{33334}$$

$$\mathbf{P}_{33344}$$

$$P_{33345} \qquad \mathbf{P}_{33444}$$

$$P_{33445}$$

$$P_{33455}$$

We see a continuity $c = n - p = 3$. The reason is that the common algebraic elements are up to P_{22233} or P_{33444} on the simple poles (set in bold). The other simple poles will use different elements containing other indices, and are of a different algebraic nature.

If we want to obtain the endpoints of a curve, it is necessary to repeat the first and the last indices n times each. On the other hand, it is possible to close a curve by taking the parameter sequence as a circular list: $t_0 \leqslant t_1 \leqslant \ldots \leqslant t_f = t'_0 \leqslant t'_1 \ldots \leqslant t'_f$, etc.

Construction of a point on the curve

One has to insert the index t q times where, as above, q is the original repetition of the index. As an example, for $n = 3$ and $q = 2$ we have (see Figure 1.2):

Riesenfeld points: *Simple poles:*

$$\mathbf{P}_{012} \qquad\qquad\qquad \mathbf{P}_{222}$$

$$\mathbf{P}_{12t} \qquad\qquad\qquad \mathbf{P}_{22t}$$

$$P_{123} \quad \mathbf{P}_{2tt} \qquad P_{223} \quad \mathbf{P}_{2tt}$$

$$P_{2t3} \quad P_{ttt} \qquad P_{2t3} \quad P_{ttt} \tag{18}$$

$$P_{234} \quad \mathbf{P}_{tt3} \qquad P_{233} \quad \mathbf{P}_{tt3}$$

$$\mathbf{P}_{t34} \qquad\qquad\qquad \mathbf{P}_{t33}$$

$$\mathbf{P}_{345} \qquad\qquad\qquad \mathbf{P}_{333}$$

We have also solved the subdivision of the arc, also called *sub-poles* (bold).

Subdivision of an arc

This is derived from the polar definition. The $q = 2$ insertions of the parameter t (see above) generate a new sequence where the poles (bold) replace the old ones. This problem has also been solved by Boehm for B-splines. The advantage of the poles resides in their simplicity. All the problems have their solution in the multilinear definition. Furthermore, no distinction is made between 'knots' and 'nodes', because the interpolation is completely disconnected from the definition of the arc.

Geometric construction

Geometric construction reflects exactly what has already been said so far. On any line, the poles are different by only one index, and the corresponding points divide the legs in the same ratio as do the parameters t_i the parameter line (see Figure 1.2). Geometry is independent of the origin of the indices and their absolute length, provided that the proportions are presented. Therefore, the application of poles is a wonderful example of affine geometry. In Figure 1.2, the geometric construction is illustrated using generalized as well as simple poles (which are indicated by bold lines).

Homogeneous form

The homogeneous form $u + t = 1$ is well adapted to compute the simple poles and to preserve the proportion u/t (see Figure 1.2):

$$P_{222}$$

$$uP_{222} + tP_{223} \rightarrow P_{22t}$$

$$P_{223} \qquad uP_{22t} + tP_{2t3} \rightarrow P_{2tt}$$

$$uP_{223} + tP_{233} \rightarrow P_{2t3} \qquad uP_{2tt} + tP_{tt3} \rightarrow P_{ttt}$$

$$P_{233} \qquad uP_{2t3} + tP_{t33} \rightarrow P_{tt3}$$

$$uP_{233} + tP_{333} \rightarrow P_{t33}$$

$$P_{333} \tag{19}$$

and

$$P(t) = P(ttt) = u^3P_{222} + 3u^2tP_{223} + 3ut^2P_{233} + t^3P_{333}$$

The terms of the expansion of $(u + t)^3$ are the coefficients of the poles. The sub-poles are P_{222}, P_{22t}, P_{2tt}, P_{ttt} and P_{ttt}, P_{tt3}, P_{t33}, P_{333}.