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- Arabic proverb

INTRODUCTION

This is the third volume of the proceedings of the Caltech-UCLA Logic Seminar, based essentially on material presented and discussed in the period 1979-1981. The last paper "Introduction to Q-theory" includes some very recent work, but it also gives the first exposition in print of some results going back to 1972.

Papers 5-10 form a unit and deal primarily with the question of the extent of definable scales.

Los Angeles
May 1983

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MORE SATURATED IDEALS

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In this paper we prove three theorems relating the consistency strengths of huge cardinals with saturated ideals on regular cardinals and with model theoretic transfer properties.

We prove:

Theorem. Con (ZFC + there is a 2-huge cardinal) $\Rightarrow$ Con (ZFC + for all $m, n \in \omega$ with $m > n$, $(\aleph_{m+1},\aleph_m) \twoheadrightarrow (\aleph_{n+1},\aleph_n)$).

Theorem. Con (ZFC + there is a huge cardinal) $\Rightarrow$ Con (ZFC + for all $n \in \omega$, there is a normal, $\aleph_n$-complete, $\aleph_{n+1}$-saturated ideal on $\aleph_n$ + there is a normal, $\aleph_{\omega+1}$ - complete, $\aleph_{\omega+2}$-saturated ideal on $\aleph_{\omega+1}$).

The theorem above contains all the new ideas necessary to prove the following theorem:

Theorem. Con (ZFC + there is a huge cardinal) $\Rightarrow$ Con (ZFC + every regular cardinal κ carries a κ^+-saturated ideal).

We now make some definitions: Let $\mathcal{L}$ be a countable language with a unary predicate U. A $\mathcal{L}$-structure $\mathfrak{A}$ is said to have type (κ,λ) iff $|\mathfrak{A}| = \kappa$ and $|U^{\mathfrak{A}}| = \lambda$. If $\kappa \geq \kappa'$, $\lambda \geq \lambda'$ we say that $(\kappa,\lambda) \twoheadrightarrow (\kappa',\lambda')$ iff every structure $\mathfrak{A}$ of type (κ,λ) has an elementary substructure $\mathfrak{B} \prec \mathfrak{A}$ of type (κ',λ').

An ideal $\mathcal{J} \subseteq \mathcal{P}$ is said to be α-complete iff whenever $\{X_\gamma : \gamma < \beta\} \subseteq \mathcal{J}$ and $\beta < \alpha$, $\bigcup_{\gamma<\beta} X_\alpha \in \mathcal{J}$. A set $A \subseteq \mathcal{P}(\kappa)$ is said to be positive if $A \notin \mathcal{J}$. $\mathcal{J}$ is normal iff for every positive set $A \subseteq \kappa$ and every regressive function f defined on A there is a $\beta \in \kappa$ such that $\{\alpha : F(\alpha) = \beta\}$ is positive. $\mathcal{J}$ is said to be λ-saturated iff it is normal and $\mathcal{P}(\kappa)/\mathcal{J}$ has the λ-chain condition. (We will never consider "non-normal" saturated ideals.) There is an extensive literature on saturated ideals. (See [6], [7].)

Let $j : V \to M$ be an elementary embedding from V into a transitive class M. Let κ_0 be the critical point of j (i.e. the first ordinal moved by j), Let $\kappa_{i+1} = j(\kappa_i)$. We will call j an n-huge embedding and κ_0 an n-huge cardinal iff M is closed under κ_n-sequences. (This means that if $\langle x_\alpha : \alpha < \kappa_n\rangle \subseteq M$ then

This research was partially supported by NSF Grant No. MCS 78-02989.

$\langle x_\alpha : \alpha < \kappa_n\rangle \in M$.) An almost n-huge embedding is an embedding j as above that is closed under $< \kappa_n$-sequences, i.e. if $\beta < \kappa_n$ and $\langle x_\alpha : \alpha < \beta\rangle \subseteq M$ then $\langle x_\alpha : \alpha < \beta\rangle \in M$. We will use the notation "crit(j)" for the critical point of j.

If $\kappa_0 \leq \lambda < \kappa_1$ and j is at least an almost huge embedding, then j induces a supercompact measure on $\mathcal{P}_\kappa(\lambda)$. In particular j induces a normal measure on κ.

Proposition 1. Let j be an n-huge embedding. Let U be the measure on κ induced by j. Then there is a set A of measure one for U such that for all α and $\beta \in A$, there is an almost n-huge embedding $j_{\alpha,\beta}$ such that the critical point of $j_{\alpha,\beta}$ is α and $j_{\alpha,\beta}(\alpha) = \beta$.

Proof. This is a routine reflection argument.

We will now precisely state the theorems we will prove:

Theorem 1. Con (ZFC + there is a sequence of huge embeddings $\langle j_n : n \in \omega\rangle$ with j_n(critical point of j_n) = critical point of j_{n+1}) $\Rightarrow$ Con (ZFC + for all $m, n \in \omega$, $m > n$ implies $(\aleph_{m+1},\aleph_m) \twoheadrightarrow (\aleph_{n+1},\aleph_n)$).

Theorem 2. Con (ZFC + there is a sequence of almost huge embeddings $\langle j_n : n \in \omega\rangle$ with j_n(critical point of j_n) = critical point of j_{n+1}) $\Rightarrow$ Con (ZFC + for all $n \in \omega$ there is a normal, $\aleph_n$-complete, $\aleph_{n+1}$-saturated ideal on $\aleph_n$).

Theorem 3. Con (ZFC + there is a huge cardinal) $\Rightarrow$ Con (ZFC + $\aleph_{\omega+1}$ carries a normal $\aleph_{\omega+2}$-saturated ideal and for all $n \in \omega$, $\aleph_n$ carries an $\aleph_{n+1}$-saturated ideal.)

We will assume that the reader is familiar with iterated forcing. (See [1] for a very good exposition.) All of our partial orderings $\mathbb{P}$ will have a unique greatest element, $1_{\mathbb{P}}$. Our notion of "support" will be the standard one and if p is a condition in an iteration we will write "supp p" for its support. For the inverse limit of a system $\langle P_i : i \in I\rangle$ we will write $\varprojlim \langle P_i : i \in I\rangle$. We will also use the notion of support for products of partial orderings. If $\langle Q_i : i \in I\rangle$ is a collection of partial orderings, we let $\prod\langle Q_i : i \in I\rangle = \{f \mid f$ is a function with domain I and for all $i \in I$ $f(i) \in Q_i\}$. The product $\prod\langle Q_i : i \in I\rangle$ is ordered coordinatewise. If $p \in \prod\langle Q_i : i \in I\rangle$, then supp $p = \{i : p(i) \neq 1_{Q_i}\}$. If $\mathcal{K} \subseteq \mathcal{P}(I)$ is an ideal then $\prod_{\text{supp } p \in \mathcal{K}} \langle Q_i : i \in I\rangle = \{f \in \prod\langle Q_i : i \in I\rangle \mid$ supp $f \in \mathcal{K}\}$.

If $\langle Q_i : i \in \omega\rangle$ is a sequence of terms such that $Q_{i+1} \in V^{Q_0 * Q_1 * \cdots * Q_i}$, we write $\mathop{*}_{i=0}^{n} Q_i$ for the finite iteration $Q_0 * Q_1 * \cdots * Q_n$. If S is a partial ordering with a uniform definition, we will use $S^{\mathbb{P}}$ to denote the partial ordering S defined in $V^{\mathbb{P}}$. To simplify notation, we will write $\mathbb{P} * \overline{S}$ to mean $\mathbb{P} * S^{\mathbb{P}}$.

If $\mathbb{P}$ is a partial ordering we will use $\mathbb{B}(\mathbb{P})$ to denote the canonical complete boolean algebra obtained from $\mathbb{P}$. If $\varphi(\dot{t}_1,\ldots,\dot{t}_n)$ is a formula in the forcing language of $\mathbb{P}$ we will use $\|\varphi(\dot{t}_1,\ldots,\dot{t}_n)\|_{\mathbb{P}}$ to denote the boolean value of $\varphi(\dot{t}_1,\ldots,\dot{t}_n)$ in $\mathbb{B}(\mathbb{P})$. For $p, q \in \mathbb{P}$, we will use the notation $p \wedge q$, and $p \vee q$ to be the meet and join of p and q in $\mathbb{B}(\mathbb{P})$. Similarly $\neg p$ will denote the complement of p in $\mathbb{B}(\mathbb{P})$. If $b \in \mathbb{B}(\mathbb{P})$, we will say that p "decides" b (in symbols $p \| b$) iff $p \leq b$ or $p \leq \neg b$. We will write $p \Vdash b$ if $p \leq b$.

We define $C(\kappa,\gamma) = \langle\{p \mid p : \kappa \to \gamma,\ |p| < \kappa\},\ \supseteq\rangle$. $C(\kappa,\gamma)$ is the partial ordering appropriate for making γ have cardinality κ. We will call this the Levy collapse. Similarly, we will define the Silver collapse $S(\kappa,\lambda)$ by $p \in S(\kappa,\lambda)$ iff

(a) $p : \lambda \times \kappa \to \lambda$

(b) $|p| \leq \kappa$

(c) there is a $\xi < \kappa$, $\operatorname{dom} p \subseteq \lambda \times \xi$

(d) for all $\alpha < \kappa$, $\gamma < \lambda$, $P(\lambda,\alpha) < \gamma$

$S(\kappa,\lambda)$ is ordered by reverse inclusion: Standard arguments show that for inaccessible λ, $S(\kappa,\lambda)$ makes λ into κ^+.

If $\gamma' < \gamma$, and $\kappa < \gamma'$, $C(\kappa,\gamma')$ is a subset of $C(\kappa,\gamma)$. If $p \in C(\kappa,\gamma)$, we define $p \cap C(\kappa,\gamma')$ to be q, where $\operatorname{dom} q = \{\alpha < \kappa : P(\alpha) < \gamma'\}$ and for each $\alpha \in \operatorname{dom} q$, $q(\alpha) = p(\alpha)$. It is easy to see that $p \cap C(\kappa,\gamma') \in C(\kappa,\gamma')$. If $r \in C(\kappa,\gamma')$ and $p \in C(\kappa,\gamma)$ then r is compatible with p in $C(\kappa,\gamma)$ iff r is compatible with $p \cap C(\kappa,\gamma')$. If $p \in C(\kappa,\gamma)$, we define $\sup p$ to be $\sup\{p(\alpha) + 1 : \alpha \in \operatorname{dom} p\}$.

We will write $\vec{\alpha}$ for a finite sequence of ordinals. If $\alpha_0,\alpha_1,\ldots,\alpha_n$ are mentioned in connection with $\vec{\alpha}$ we will assume that $\alpha = (\alpha_0,\alpha_1,\ldots,\alpha_n)$.

If κ and λ are cardinals, with $\kappa < \lambda$ and $x, y \in \mathcal{P}_\kappa(\lambda) = \{z \subseteq \lambda : |z| < \kappa\}$ then we write $x < y$ iff $x \subseteq y$ and the order type of x is less than the order type of $y \cap \kappa$. If $x \in \mathcal{P}_\kappa(\lambda)$, let $\operatorname{crit}(x) =$ order type of $x \cap \kappa$.

If $\mathbb{B}$ and $\mathbb{C}$ are complete Boolean algebras and $\pi : \mathbb{B} \to \mathbb{C}$ is an order preserving function, π is called a projection map if whenever $G \subseteq \mathbb{B}$ is generic, $\pi''G \subseteq \mathbb{C}$ is generic.

Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orderings. If $i : \mathbb{P} \to \mathbb{Q}$ is a one to one, order and incompatibility preserving function with the property that whenever $A \subseteq \mathbb{P}$ is a maximal antichain, $i''A \subseteq \mathbb{Q}$ is a maximal antichain, then we say that i is a <u>neat embedding</u> from $\mathbb{P}$ into $\mathbb{Q}$, and write $i : \mathbb{P} \hookrightarrow \mathbb{Q}$). Standard theory says that if $i : \mathbb{P} \hookrightarrow \mathbb{Q}$ is a neat embedding then there is a projection map $\pi : \mathbb{B}(\mathbb{Q}) \to \mathbb{B}(\mathbb{P})$ such that for all $p \in \mathbb{P}$, $\pi(i(p)) = P$.

If $e : \mathbb{Q} \hookrightarrow \mathbb{P}$ and $S \in V^{\mathbb{Q}}$ is a partial ordering we define $\mathbb{P} *_e S$ to be the iteration with amalgam $e''\mathbb{Q}$. (See [2] for this definition.)

If $\mathbb{P}$ is a partial ordering and $p \in \mathbb{P}$, then $\mathbb{P}/p$ is defined to be $\{q : q \leq p\}$.

§1. In this section we prove Theorems 1 and 2. We begin by mentioning theorems of Magidor and Kunen that we will use extensively in this paper.

Theorem (Kunen [7]) Let $j : V \to M$ be a huge embedding. Suppose $\mathbb{P}' * Q$ is a forcing notion such that

(a) $\mathbb{P}'$ is κ_0 c. c.

(b) Q is κ_0-closed in $V^{\mathbb{P}'}$

(c) $|\mathbb{P}'| = \kappa_0$, $|\mathbb{P}' * Q| = \kappa_1$

(d) there is a projection map $\pi : \mathcal{B}(j(\mathbb{P}')) \to \mathcal{B}(\mathbb{P}' * Q)$ and a $q \in j(\mathbb{P}') * j(Q)$ such that for all $r \in j(\mathbb{P}') * j(Q)$, $r \leq q$ implies $j(\pi(r)) \geq r$ [If $r = (r_0, r_1)$ with $r_0 \in j(\mathbb{P}')$, we take $\pi(r) = \pi(r_0)$]

then in $V^{\mathbb{P}' * Q}$:

(i) κ_0 carries a normal, κ_0-complete, $j(\kappa_0)$-saturated ideal

(ii) $(j(\kappa_0), \kappa_0) \twoheadrightarrow (\kappa_0, < \kappa_0)$

Magidor and others have commented that to get a normal κ_0-complete, $j(\kappa_0)$-saturated ideal on κ_0, it is enough to have j be almost huge and replace (d) by:

(d') there is a neat embedding $i : \mathcal{B}(\mathbb{P}' * Q) \to \mathcal{B}(j(\mathbb{P}'))$.

We shall need the following preservation lemmas.

Lemma 1. Let λ be a regular cardinal. Suppose $\mathbb{P}$ is λ c. c. and Q is λ-closed. (We do not rule out $\mathbb{P} = \{\emptyset\}$.)

(a) If $\kappa < \lambda$ and κ carries a normal λ-saturated ideal in $V^{\mathbb{P}}$, then κ carries a normal λ-saturated ideal in $V^{\mathbb{P}\times Q}$.

(b) If $\lambda' < \lambda$, $\gamma \leq \kappa < \lambda$ and $(\lambda, \lambda') \twoheadrightarrow (\kappa, \gamma)$ in V, then $(\lambda, \lambda') \twoheadrightarrow (\kappa, \gamma)$ in V^Q.

Proof. Since $\mathbb{P}$ is λ c. c., Q is (λ, ∞)-distributive in $V^{\mathbb{P}}$. Thus, if $\mathcal{J}$ is a normal λ-saturated ideal on κ in $V^{\mathbb{P}}$, $\mathcal{J}$ remains a normal ideal in $V^{\mathbb{P}\times Q}$. We must show that $\mathcal{J}$ remains λ-saturated in $(V^{\mathbb{P}})^Q$. Suppose not. Let $\langle \sigma_\alpha : \alpha < \lambda \rangle$ be terms in $V^{\mathbb{P}\times Q}$ for an antichain. We will build a descending sequence $\langle q_\alpha : \alpha < \lambda \rangle \subseteq Q$ in V with the property that there is a term $\tau_\alpha \in V^{\mathbb{P}}$ such that $\| q_\alpha \Vdash_Q \sigma_\alpha = (\tau_\alpha)^{V^{\mathbb{P}}} \|_{\mathbb{P}} = 1$. We do this by induction on α. Let $q_{-1} = 1_Q$.

Assume we have defined $\{q_\beta : \beta < \alpha\}$. Let $q_\alpha^{-1} \leq \{q_\beta : \beta < \alpha\}$. There is such a q_α^{-1} because $\alpha < \lambda$ and Q is λ-closed. We will simultaneously build $\{q_\alpha^\gamma : \gamma < \gamma_\alpha\} \subseteq Q$ and an antichain $\{p_\gamma : \gamma < \gamma_\alpha\} \subseteq \mathbb{P}$. These will have the property that for each γ there is a term $\tau_\alpha^\gamma \in V^{\mathbb{P}}$ such that $(p_\gamma, q_\alpha) \Vdash_{\mathbb{P}\times Q} \sigma_a = (\tau_\alpha^\gamma)^{V^{\mathbb{P}}}$. Assume we have built $\langle q_\alpha^\gamma : \gamma < \xi \rangle$ and $\langle p_\gamma : \gamma < \xi \rangle$ with the above property. Note that $\xi < \lambda$ since $\mathbb{P}$ has the λ-chain condition. If $\langle p_\gamma : \gamma < \xi \rangle$ is not a maximal antichain, pick p_ξ, q_α^ξ such that $q_\alpha^\xi \leq \langle q_\alpha^\gamma : \gamma < \xi \rangle$ and for some $\tau_\alpha^\xi \in V^{\mathbb{P}}$, $(p_\xi, q_\alpha^\xi) \Vdash_{\mathbb{P}\times Q} (\tau_\alpha^\xi)^{V^{\mathbb{P}}} = \sigma_\alpha$ and p_ξ

is incompatible with each p_γ, $\gamma < \xi$.

Since $\mathbb{P}$ has the λ c. c., for some $\xi < \lambda$, $\langle p_\gamma : \gamma < \xi \rangle$ is a maximal antichain. Let $q_\gamma \leq \langle q_\alpha^\gamma : \gamma < \xi \rangle$ and $\tau_\alpha \in V^{\mathbb{P}}$ be such that $p_\gamma \Vdash \tau_\alpha^\gamma = \tau_\alpha$. Then $\| q_\alpha \Vdash_{\mathbb{Q}} (\tau_\alpha)^{V^{\mathbb{P}}} = \sigma_\alpha \|_{\mathbb{P}} = 1$.

In $V^{\mathbb{P}}$, $\langle q_\alpha : \alpha < \gamma \rangle$ identifies a sequence $\langle \tau_\alpha : \alpha < \lambda \rangle \subseteq (\mathcal{P}(\kappa))^{V^{\mathbb{P}}}$. If $\alpha < \beta < \lambda$, then $q_\beta \Vdash_{\mathbb{Q}} \tau_\alpha \cap \tau_\beta \in \mathcal{J}$ and $\tau_\alpha, \tau_\beta \notin \mathcal{J}$. But "$\tau_\alpha \cap \tau_\beta \in \mathcal{J}$" and "$\tau_\alpha \notin \mathcal{J}$" is absolute between $V^{\mathbb{P}}$ and $V^{\mathbb{P}\times\mathbb{Q}}$. Hence in $V^{\mathbb{P}}$, $\langle \tau_\alpha : \alpha < \lambda \rangle$ is an antichain of size λ in $\mathcal{P}(\kappa)/\mathcal{J}$, a contradiction.

(b) Let $\mathfrak{A} = \langle \lambda; \lambda', f_i \rangle_{i\in\omega}$ be a fully skolemized structure of type (λ, λ') in $V^{\mathbb{Q}}$. Using the λ-closure of $\mathbb{Q}$, in V we can find a sequence $\langle p_\alpha : \alpha < \lambda \rangle \subseteq \mathbb{Q}$ and a structure $\mathfrak{A}' = \langle \lambda; \lambda', f_i \rangle_{i\in\omega}$ such that f_i' are defined on all of λ and for each $\vec{\beta} \in \lambda^\omega$ and each i there is an $\alpha < \lambda$ $p_\alpha \Vdash f_i(\vec{\beta}) = f_i'(\vec{\beta})$. Let $\mathfrak{B} < \mathfrak{A}'$ be of type (κ, γ). Since $\kappa < \lambda$ and λ is regular there is an α such that for all $\beta \in \mathfrak{B}$, $p_\alpha \Vdash f_i(\vec{\beta}) = f'(\vec{\beta})$. But then $p_\alpha \Vdash \mathfrak{B}$ is closed under all of the f_i's.

Hence $p_\alpha \Vdash \mathfrak{B} < \mathfrak{A}$.

A partial ordering $\mathbb{P}$ will be called κ-centered iff there is a collection $\{C_\alpha : \alpha < \kappa\}$ of disjoint subsets of $\mathbb{P}$ such that

(a) $\mathbb{P} = \bigcup_{\alpha<\kappa} C_\alpha$

(b) If $p_1, \ldots, p_n \in C_\alpha$ then $\bigwedge_{i=1}^{n} p_i \neq 0$.

In particular, if $|\mathbb{P}| \leq \kappa$, $\mathbb{P}$ is κ-centered.

<u>Lemma 2</u>. Let $\mathcal{J}$ be a κ^{++}-saturated ideal on κ^+. Suppose $\mathbb{P}$ is a κ-centered partial ordering. Then $\mathcal{J}$ generates a normal κ^+-complete, κ^{++}-saturated ideal in $V^{\mathbb{P}}$.

(Laver has proven stronger results unknown to the author at the time this work was done. See [8].)

<u>Proof</u>. $\mathbb{P}$ is manifestly κ^+-c. c. Hence $\mathcal{J}$ remains κ^+-complete and normal. We need to see that $\mathcal{J}$ is κ^{++}-saturated. Let $\langle \dot{X}_\alpha : \alpha < \kappa^{++} \rangle$ be a term for an antichain of size κ^{++}. Let $\dot{Y}_{\alpha,\beta} = \dot{X}_\alpha \cap \dot{X}_\beta$. Then $\dot{Y}_{\alpha,\beta} \in V^{\mathbb{P}}$ and $\| \dot{Y}_{\alpha,\beta} \in \bar{\mathcal{J}} \| = 1$. (Here we are using $\bar{\mathcal{J}}$ to stand for the ideal generated by $\mathcal{J}$ in $V^{\mathbb{P}}$.) Hence, for each pair α, β

$$\| \exists Y \in V \ (Y \in \mathcal{J} \text{ and } Y_{\alpha,\beta} \subseteq Y) \| = 1 .$$

Since $\mathbb{P}$ is κ^+-c. c. and $\mathcal{J}$ is κ^+-complete, there is a sequence in V, $\langle Y_{\alpha,\beta} : \alpha < \beta < \kappa^{++} \rangle \subseteq \mathcal{J}$ and $\| X_\alpha \cap X_\beta \subseteq Y_{\alpha,\beta} \| = 1$.

For each X_α and each $\gamma < \kappa$, let $X_{\alpha,\gamma} = \{\xi \mid$ there is a $p \in C_\gamma$, $p \Vdash \xi \in X_\alpha\}$. Since $\|X_\alpha \subseteq \bigcup_{\gamma<\kappa} X_{\alpha,\gamma}\| = 1$ and $\mathcal{J}$ is κ^+-complete, there must be some γ_α with $X_{\alpha,\gamma_\alpha} \notin \mathcal{J}$. By the pigeon-hole principle, there is a set $S \subseteq \kappa^{++}$, $S \in V$ with $|S| = \kappa^{++}$ such that for all $\alpha, \beta \in S$, $\gamma_\alpha = \gamma_\beta$. Fix such a set S and such a γ. We claim that if $\alpha, \beta \in S$ then $X_{\alpha,\gamma} \cap X_{\beta,\gamma} \subseteq Y_{\alpha,\beta}$.

Let $\xi \in X_{\alpha,\gamma} \cap X_{\beta,\gamma}$. Since C_γ is a collection of compatible elements we can find a $q \in C_\gamma$, $q \Vdash \xi \in X_{\alpha,\gamma} \cap X_{\beta,\gamma}$. But then $q \Vdash \xi \in \check{Y}_{\alpha,\beta}$. Hence $\xi \in Y_{\alpha,\beta}$. We now derive our contradiction: $\langle X_{\alpha,\gamma} : \alpha \in S\rangle$ is a set of positive elements of $P(\kappa^+)$ with respect to $\mathcal{J}$ and if $\alpha, \beta \in S$, $\alpha \neq \beta$, $X_{\alpha,\gamma} \cap X_{\beta,\gamma} \subseteq Y_{\alpha,\beta} \in \mathcal{J}$. Hence $\langle X_{\alpha,\gamma} : \alpha \in S\rangle$ are an antichain of size κ^{++} in the ground model, a contradiction.

The following lemma is standard (see [6]).

<u>Lemma 3</u>. Suppose $\lambda' < \kappa' \leq \lambda < \kappa$ are regular cardinals and $\mathbb{P}$ has the κ c. c.

(a) If $(\kappa,\lambda) \twoheadrightarrow (\kappa',\lambda')$ then in $V^{\mathbb{P}}$, $(\kappa,\lambda) \twoheadrightarrow (\kappa',\lambda')$

(b) If κ carries a κ^+-saturated ideal $\mathcal{J}$, then in $V^{\mathbb{P}}$, $\mathcal{J}$ is a κ^+-saturated ideal.

Let $\langle j_n : n \in \omega\rangle$ be a sequence of huge embeddings. Let $\kappa_n = \mathrm{crit}(j_n)$ and suppose $j_n(\kappa_n) = \kappa_{n+1}$. Suppose $\langle \mathbb{R}_n : n \in \omega\rangle$ is a sequence of terms for partial orderings with $\mathbb{R}_{n+1} \in V^{\overset{n}{\underset{i=0}{*}} \mathbb{R}_i}$ such that

(a) $\overset{n}{\underset{i=0}{*}} \mathbb{R}_i$ has the κ_n-chain condition and in $V^{\overset{n}{\underset{i=0}{*}} \mathbb{R}_i}$ $\mathbb{R}_{n+1}$ is κ_n-closed

(b) In $V^{\overset{n}{\underset{i=0}{*}} \mathbb{R}_i}$, for all $i \leq n$ $\kappa_i = \aleph_{i+2}$

(c) In $V^{\overset{n}{\underset{i=0}{*}} \mathbb{R}_i}$, $\mathbb{R}_{n+1}$ has cardinality κ_{n+1}.

(d) For each n, there is a projection map $\pi : j_n(\overset{n}{\underset{i=0}{*}} \mathbb{R}_i) \to \overset{n}{\underset{i=0}{*}} \mathbb{R}_i * \mathbb{R}_{n+1}$ and a condition $q \in j_n(\overset{n}{\underset{i=0}{*}} \mathbb{R}_i) * j_n(\mathbb{R}_{n+1})$ such that for all $r \leq q$, $r \in j_n(\overset{n}{\underset{i=0}{*}} \mathbb{R}_i) * j_n(\mathbb{R}_{n+1})$, $j_n(\pi(r)) \geq r$. (i.e. if $\mathbb{P}' = \overset{n}{\underset{i=0}{*}} \mathbb{R}_i$ and $Q = \mathbb{R}_n$, then $\mathbb{P}'$ and Q satisfy the hypothesis (d) in Kunen's Theorem.)

Let $\mathbb{P} = \lim \langle \overset{n}{\underset{i=0}{*}} \mathbb{R}_i : n \in \omega\rangle$. By Kunen's Theorem, in $V^{\overset{n+1}{\underset{i=0}{*}} \mathbb{R}_i}$ there is κ_{n+1}-saturated ideal on κ_n and $(\kappa_{n+1},\kappa_n) \twoheadrightarrow (\kappa_n,\kappa_{n-1})$. (Let $\kappa_{-1} = \aleph_1$.) But $\mathbb{P}/\overset{n+1}{\underset{i=0}{*}} \mathbb{R}_i$ is κ_{n+1}-closed. Hence by Lemma 1 in $V^{\mathbb{P}}$, for all $m, n \in \omega$, $m > n > 0$ $(\aleph_{m+1},\aleph_m) \twoheadrightarrow (\aleph_{n+1},\aleph_n)$ and $\aleph_m$ carries a normal $\aleph_m$-complete

$\aleph_{m+1}$-saturated ideal.

By Lemmas 2 and 3, in $V^{\mathbb{P}*C(\aleph_0,\aleph_1)}$ for all $m, n \in \omega$, $m > n$, $(\aleph_{m+1},\aleph_m) \twoheadrightarrow (\aleph_{n+1},\aleph_n)$ and $\aleph_m$ carries an $\aleph_m$-complete $\aleph_{m+1}$-saturated ideal.

We will now concentrate our attention on building the $\mathbb{R}$'s. The construction given here is considerably simpler than our original one at the expense of introducing somewhat more technicality. The next few definitions and lemmas explore these technicalities.

Lemma 4. Let $\mathbb{P}$ and Q be partial orderings. Suppose $\pi : \mathbb{P} \to Q$ has the properties:

(1) $p_1 \leq p_2$ implies $\pi(p_1) \leq \pi(p_2)$

(2) for all $p \in \mathbb{P}$ there is a $q \leq \pi(p)$ such that for all $q' \leq q$ there is a $p' \leq p$ such that $\pi(p') \leq q'$.

$$\begin{array}{ccc}
\underline{\mathbb{P}} & & \underline{Q} \\
p & \xrightarrow{\ \pi\ } & \pi(p) \\
 & & \vee\!\mid \\
 & & q \\
\vee\!\mid & & \vee\!\mid \\
 & & q' \\
 & & \vee\!\mid \\
p' & \xrightarrow{\ \pi\ } & \pi(p')
\end{array}$$

Then π is a projection map.

Proof. This is standard.

Definition (Laver). Let $\mathbb{P}$ be a partial ordering and $Q \in V^{\mathbb{P}}$ be a term for a partial ordering. We define the termspace partial ordering Q^* to be the following partial order:

$$\text{dom } Q^* = \{\tau : \tau \in V^{\mathbb{P}} \text{ and } \|\tau \in Q\|_{\mathbb{P}} = 1 \text{ and for all } \tau' \in V^{\mathbb{P}}, \text{ if } \operatorname{rank}(\tau') < \operatorname{rank}(\tau) \text{ then } \|\tau' = \tau\|_{\mathbb{P}} \neq 1\} .$$

For $\tau, \tau' \in \text{dom } Q^*$,

$$\tau \leq_{Q^*} \tau' \quad \text{iff} \quad \|\tau \leq_Q \tau'\|_{\mathbb{P}} = 1 .$$

(Technically we want to take the universe of Q^* to be equivalence classes of terms modulo the relation $\tau_1 \sim \tau_2$ iff $\|\tau_1 = \tau_2\|_{\mathbb{P}} = 1$. In practice we ignore this distinction.)

The main use of the termspace partial ordering is summed up by:

Lemma 5. (a) Let $Q \in V^{\mathbb{P}}$ be a term for a partial ordering. There is a projection map $\pi : \mathbb{P} \times Q^* \to \mathbb{P} * Q$

(b) Let $\langle Q_i : i \in \omega \rangle \subseteq V^{\mathbb{P}}$ be terms for partial orderings. Then there is a projection map

$$\pi : \mathbb{P} \times \prod_{i\in\omega} (Q_i)^* \longrightarrow \mathbb{P} * (\prod_{i\in\omega} Q_i)^{V^{\mathbb{P}}}$$

(c) If $\langle Q_i : i \in \omega \rangle$, $\langle \mathbb{P}_i : i \in \omega \rangle$ are two sequences of partial orderings and for each $i \in \omega$ $\varphi_i : Q_i \to \mathbb{P}_i$ is a neat embedding, then there is a $\varphi : \prod_{i\in\omega} Q_i \to \prod_{i\in\omega} \mathbb{P}_i$ extending each φ_i.

Proof. (a) Define π as follows: If $(p,\tau) \in \mathbb{P} \times Q^*$, let $\pi(p,\tau) = (p,\tau) \in \mathbb{P} * Q$. We want to apply Lemma 4. It is enough to see that if $(p,\tau) \in \mathbb{P} \times Q^*$ and $(q,\sigma) \in \mathbb{P} * Q$ with $(q,\sigma) \leq_{\mathbb{P}*Q} (p,\tau)$ there is a $\sigma' \in Q^*$ such that $(q,\sigma') \leq_{\mathbb{P}\times Q^*} (p,\tau)$ and $(q,\sigma') \leq_{\mathbb{P}*Q} (q,\sigma)$. Fix $(p,\tau) \in \mathbb{P} \times Q^*$ and $(q,\sigma) \in \mathbb{P} * Q$ with $(q,\sigma) \leq_{\mathbb{P}*Q} (p,\tau)$. Let σ' be a term of minimal rank such that $\|\sigma' = \sigma\|_{\mathbb{P}} \geq q$ and $\|\sigma' = \tau\|_{\mathbb{P}} \geq \neg q$. Then $q \Vdash \sigma' \leq \sigma$, hence $(q,\sigma') \leq_{\mathbb{P}*Q} (q,\sigma)$. Also $\neg q \leq \|\sigma' = \tau\|$, hence $q \vee (\neg q) \leq \|\sigma' \leq \tau\|$ so $\|\sigma' \leq \tau\| = 1$ and $(q,\sigma') \leq_{\mathbb{P}\times Q^*} (p,\tau)$ as desired.

The proof of (b) is similar and (c) is standard.

If Q is in $V^{\mathbb{P}}$ we will use the notation $A(\mathbb{P};Q)$ for the termspace partial ordering.

We now make a definition we will use to get an easy hold on the chain condition of the termspace partial ordering.

Definition. (a) Let $\mathbb{P}$ and Q be partial orderings. Q is said to be representible in $\mathbb{P}$ iff there is an incompatibility preserving map $i : Q \to \mathbb{P}$. (i.e. if q and q' are incompatible in Q, $i(q)$ and $i(q')$ are incompatible in $\mathbb{P}$.)

(b) Let $\mathfrak{F}(\mathbb{P},S)$ be the partial ordering with domain

$$\{f \mid f : \mathbb{P} \to S \text{ and } f \text{ is order preserving}\}$$

The ordering of $\mathfrak{F}(\mathbb{P},S)$ is $f \leq g$ iff for all $p \in \mathbb{P}$ $f(p) \leq g(p)$.

If Q is representible in $\mathbb{P}$ then the chain condition of $\mathbb{P}$ gives an upper bound on the chain condition for Q.

Example. (a) Let $\mathbb{P}$ and S be partial orderings and $\prod_{p\in\mathbb{P}} S$ be the product of $|\mathbb{P}|$ copies of S, one for each element of $\mathbb{P}$. Then $\mathfrak{F}(\mathbb{P},S)$ is representible in $\prod_{p \in \mathbb{P}} S$, namely map $f \in \mathfrak{F}(\mathbb{P},S)$ to $g \in \prod_{p\in\mathbb{P}} S$ where g has the value $f(p)$ on the p^{th} copy of S.

(b) Let $\mathbb{P}$ be a κ c. c. partial ordering. Let $S^{\mathbb{P}}(\kappa,\lambda)$ be the Silver collapse of λ to κ^+ defined in $V^{\mathbb{P}}$. Let $S(\kappa,\lambda)$ be the Silver collapse of λ to κ^+ defined in V. Then $(S^{\mathbb{P}}(\kappa,\lambda))^*$ is representible in $\mathfrak{F}(\mathbb{P};S(\kappa,\lambda))$: Define a map $i : (S^{\mathbb{P}}(\kappa,\lambda))^* \to \mathfrak{F}(\mathbb{P};S(\kappa,\lambda))$ by $i(\tau) = f_\tau$ where

$$f_\tau(p) = \{(\alpha,\beta,\gamma) : p \Vdash \tau(\alpha,\beta) = \gamma\} .$$

Using the κ c. c. of $\mathbb{P}$ we see that $|f_\tau(p)| \le \kappa$. Further, $p \Vdash f_\tau(p) \in S^{\mathbb{P}}(\kappa,\lambda)$, so there is a $\xi < \kappa$ such that $\operatorname{dom} f_\tau(p) \subseteq \lambda \times \xi$. Hence $f_\tau(p) \in S(\kappa,\lambda)$. If $p \le_{\mathbb{P}} p'$ and $p' \Vdash \tau(\alpha,\beta) = \gamma$ then $p \Vdash \tau(\alpha,\beta) = \gamma$. Thus f_τ is order preserving and $f_\tau \in \mathfrak{F}(\mathbb{P},S)$.

We must see that i preserves incompatibility. Let $\tau, \tau' \in (S^{\mathbb{P}}(\kappa,\lambda))^*$, τ and τ' incompatible. Then there is a $p \in \mathbb{P}$,

$$p \Vdash_{\mathbb{P}} \tau \text{ and } \tau' \text{ are incompatible in } S^{\mathbb{P}}(\kappa,\lambda) .$$

Pick $p' \le p$, $\gamma \ne \gamma'$, $p' \Vdash \tau(\alpha,\beta) = \gamma$ and $p' \Vdash \tau'(\alpha,\beta) = \gamma'$. Then $f_\tau(p')$ is incompatible with $f_{\tau'}(p')$, and hence f_τ is incompatible with $f_{\tau'}$.

(In fact if $|\mathbb{P}| = \kappa$ it is possible to see that $(S^{\mathbb{P}}(\kappa,\lambda))^* \cong \{f \mid f : \mathbb{P} \to S(\kappa,\lambda),\ f$ is order preserving and there is a $\xi < \kappa$ $\bigcup_{p\in\mathbb{P}} \operatorname{dom} f(p) \subseteq \lambda \times \xi\}$. We leave this to the reader.)

If $Q \in V^{\mathbb{P}}$ and $\|Q$ is λ-closed$\|_{\mathbb{P}} = 1$ and $\langle \tau_\alpha : \alpha < \beta\rangle$ $(\beta < \lambda)$ is a descending sequence in Q^* then $\|$there is a $\tau \in Q$, $\tau \le \tau_\alpha$ for all $\alpha\|_{\mathbb{P}} = 1$. Let τ be a term for an element of Q such that for each α $\|\tau \le \tau_\alpha\|_{\mathbb{P}} = 1$. Then $\tau \le_{Q^*} \inf_{\alpha<\beta} \tau_\alpha$. Hence Q^* is λ-closed.

<u>Lemma 6</u>. Let $\mathbb{R} \in V^{\mathbb{P}*Q}$ be a term for a partial ordering. Then $A(\mathbb{P};A(Q;\mathbb{R})) \approx A(\mathbb{P} * Q;\mathbb{R})$.

<u>Proof</u>. If $\tau \in A(\mathbb{P};A(Q,\mathbb{R}))$ let $\pi(\tau) \in A(\mathbb{P} * Q;\mathbb{R})$ be a term τ^* such that $\|\tau^* = ((\tau)^{V^{\mathbb{P}}})^{V^Q}\|_{\mathbb{P}*Q} = 1$. We verify that π is an isomorphism. π is clearly order preserving. Suppose $\tau, \sigma \in A(\mathbb{P};S(Q;\mathbb{R}))$, $\|\tau = \sigma\|_{\mathbb{P}} \ne 1$. Let $p \in \mathbb{P}$, $p \Vdash \tau \ne \sigma$ in $A(Q;\mathbb{R})$. Pick $q \in V^{\mathbb{P}}$, $\|q \in Q\| \ge p$, $p \Vdash (q \Vdash ((\tau)^{V^{\mathbb{P}}})^{V^Q} \ne ((\sigma)^{V^{\mathbb{P}}})^{V^Q})$. Then $(p,q) \in \mathbb{P} * Q$ and $(p,q) \Vdash \tau^* \ne \sigma^*$. Let $\tau^* \in V^{\mathbb{P}*Q}$, $\|\tau^* \in \mathbb{R}\|_{\mathbb{P}*Q} = 1$. In $V^{\mathbb{P}}$, there is a term τ for an element of $V^{\mathbb{P}*Q}$ such that $\| \|\tau = \tau^*\|_Q = 1\|_{\mathbb{P}} = 1$. Then $\pi(\tau) = \tau^*$.

The following lemma is standard.

Lemma 7. Let λ be a measurable cardinal. Let $\alpha < \lambda$ and $\langle \mathbb{P}_\gamma : \gamma < \lambda \rangle$ be a sequence of λ-c. c. partial orderings. Then $\prod_{\alpha \text{ supports}} \{\mathbb{P}_\gamma : \gamma < \lambda\}$ has the λ c. c. ("α supports" denotes the ideal of subsets of λ with cardinality $\leq \alpha$.)

We will be interested in constructing partial orderings with nice embedding properties. We make the following definition, which is similar to one that will appear in [3].

Our partial orderings will be in the form $\mathbb{R}(\kappa,\lambda)$ for all regular $\kappa > \omega$ and all measurable $\lambda > \kappa$. $\mathbb{R}$ is the uniform definition given below.

If λ is greater than the first measurable above κ, the partial ordering $\mathbb{R}(\kappa,\lambda)$ will be the product $\prod_{n\in\omega} S^n(\kappa,\lambda)$ for partial orderings $S^n(\kappa,\lambda)$ defined inductively.

Assume that we have defined $\mathbb{R}(\beta,\alpha)$ for all regular β, $\omega < \beta < \alpha$ and all measurable $\alpha < \lambda$. We now define $\mathbb{R}(\beta,\lambda)$.

Case 1. λ is the first inaccessible above β. Let $\mathbb{R}(\beta,\lambda) = S(\beta,\lambda) =$ the Silver collapse of λ to β^+.

Case 2. Otherwise.

By induction on n, we define $S^n(\alpha,\lambda)$ for all regular α, $\beta \leq \alpha < \lambda$. We will have a uniform definition of $S^n(\alpha,\lambda)$ no matter which model we define it in, hence we define it only in the ground model. $(S^n(\alpha,\lambda))^{V^{\mathbb{R}(\beta,\alpha)}}$ is the partial ordering constructed in $V^{\mathbb{R}(\beta,\alpha)}$ using this definition.

If $n = 0$:

$$\text{Let } S^0(\alpha,\lambda) = S(\alpha,\lambda) = \text{the Silver collapse of } \lambda \text{ to } \alpha^+ .$$

Assume that we have defined $S^n(\alpha,\lambda)$ for all regular α.

For $n + 1$:

$$\text{Let } S^{n+1}(\alpha,\lambda) = \prod_{\alpha \text{ supports}} \{A(\mathbb{R}(\alpha,\gamma);S^n(\gamma,\lambda)) : \alpha < \gamma < \lambda \text{ and } \gamma \text{ is measurable}\} .$$

Lemma 8. For all regular $\beta > \omega$ and all measurable λ, $\mathbb{R}(\beta,\lambda)$ is λ-c. c. and has cardinality λ. (In fact λ Mahlo works but we leave this to the reader.)

Proof. By Lemma 7, it is enough to see that each $S^n(\beta,\lambda)$ has the λ-c. c.

We show by induction on λ and n, that for all measurable $\alpha_1,\ldots,\alpha_m$ $A(\overset{m-1}{\underset{i=1}{*}} \mathbb{R}(\alpha_i,\alpha_{i+1});S^n(\alpha_m,\lambda))$ has cardinality λ and the λ c. c.

Assume that this is true for all $\lambda' < \lambda$. By our example we know that $A(\overset{m-1}{\underset{i=1}{*}} \mathbb{R}(\alpha_i,\alpha_{i+1});S^0(\alpha_m,\lambda)$ is representible in $\mathfrak{F}(\overset{m-1}{\underset{i=1}{*}} \mathbb{R}(\alpha_i,\alpha_{i+1});S^0(\alpha_m,\lambda))$. This in turn is representible in the product of α_m copies of $S^0(\alpha_m,\lambda)$. By Lemma 7, this has the λ c. c.

Assume that $A(\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1});S^n(\alpha_m,\lambda))$ is λ-c. c. for all $\alpha_1,\dots,\alpha_m$ measurable. We want that for each $\alpha_1,\dots,\alpha_m$ measurable, $A(\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1});S^{n+1}(\alpha_m,\lambda))$ is λ-c. c.

Fix such a sequence $\alpha_1,\dots,\alpha_m$. In $V^{\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1})}$

$$S^{n+1}(\alpha_m,\lambda) = \prod_{\alpha_{m+1}\text{ support}} \{A(\mathbb{R}(\alpha_m,\beta);S^n(\beta,\lambda)) \quad \alpha_{n+1} < \beta < \lambda \text{ and } \beta \text{ is measurable}\}\ .$$

Since $\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1})$ has the α_m-c. c., if $t \in A(\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1});S^{n+1}(\alpha_m,\lambda))$ there is a set $D \subseteq \lambda$ such that $|D| = \alpha_m$ and $\|\text{supp } t \subseteq D\|_{\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1})} = 1$.

For each $\beta \in D$, we get a term τ_β such that $\|\tau(\beta) = \tau_\beta\|_{\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1})} = 1$ and $\tau_\beta \in A(\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1});\ (\mathbb{R}(\alpha_m,\beta);S^n(\beta,\lambda)))$. It is easy to verify that the map $\tau \longmapsto \langle\tau_\beta : \beta \in D\rangle$ is an isomorphism between $A(\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1});S^{n+1}(\alpha_m,\lambda))$ and $\prod_{\alpha_m\text{ supports}} \{A(\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1});A(\mathbb{R}(\alpha_m,\beta);S^n(\beta,\lambda))) \quad \alpha_m < \beta < \lambda$ and β is measurable$\}$. By Lemma 6,

$$A(\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1});A(\mathbb{R}(\alpha_m,\beta);S^n(\beta,\lambda))) \approx A(\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1}) \ast \mathbb{R}(\alpha_m,\beta);S^n(\beta,\lambda))\ .$$

By the induction hypothesis for n, $A(\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1}) \ast \mathbb{R}(\alpha_m,\beta);S^n(\beta,\lambda))$ has the λ c. c. and thus by Lemma 7 $A(\ast_{i=1}^{m-1}\ \mathbb{R}(\alpha_i,\alpha_{i+1});S^{n+1}(\alpha_m,\lambda))$ has the λ-c. c.

We now establish the properties we need to satisfy the conditions of Kunen's Theorem:

Lemma 9. For each $\kappa > \omega$, κ regular and each λ measurable:

(a) $\prod_{\substack{n\in\omega\\ n\geq 1}} S^n(\kappa,\lambda)$ is κ^+-closed

(b) $\prod_{n\in\omega} S^n(\kappa,\lambda)$ is κ-closed

(c) If α is measurable and $id : \mathbb{R}(\kappa,\alpha) \hookrightarrow \mathbb{R}(\kappa,\lambda)$ then there is a map φ extending id such that

$$\varphi : \mathbb{R}(\kappa,\alpha) \times \prod_{n\in\omega} A(\mathbb{R}(\kappa,\alpha);S^n(\alpha,\lambda)) \hookrightarrow \mathbb{R}(\kappa,\lambda)$$

(d) If $id : \mathbb{R}(\kappa,\alpha) \to \mathbb{R}(\kappa,\lambda)$ then there is a map ψ extending id,

$$\psi : \mathcal{B}(\mathbb{R}(\kappa,\alpha) \ast \mathbb{R}(\alpha,\lambda)) \hookrightarrow \mathcal{B}(\mathbb{R}(\kappa,\lambda))$$