Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1253

Jürgen Fischer

An Approach to the Selberg Trace Formula via the Selberg Zeta-Function

Springer-Verlag

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1253

Jürgen Fischer

An Approach to the Selberg Trace Formula via the Selberg Zeta-Function

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

Author

Jürgen Fischer
Mathematisches Institut
Einsteinstraße 62, 4400 Münster, Federal Republic of Germany

Mathematics Subject Classification (1980): 10D12, 10D40, 10H10, 58G25

ISBN 3-540-15208-3 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-15208-3 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

Lecture Notes in Mathematics

For information about Vols. 1–1034 please contact your bookseller or Springer-Verlag.

Vol. 1035: The Mathematics and Physics of Disordered Media. Proceedings, 1983. Edited by B.D. Hughes and B.W. Ninham. VII, 432 pages. 1983.

Vol. 1036: Combinatorial Mathematics X. Proceedings, 1982. Edited by L.R.A. Casse. XI, 419 pages. 1983.

Vol. 1037: Non-linear Partial Differential Operators and Quantization Procedures. Proceedings, 1981. Edited by S.I. Andersson and H.-D. Doebner. VII, 334 pages. 1983.

Vol. 1038: F. Borceux, G. Van den Bossche, Algebra in a Localic Topos with Applications to Ring Theory. IX, 240 pages. 1983.

Vol. 1039: Analytic Functions, Błażejewko 1982. Proceedings. Edited by J. Ławrynowicz. X, 494 pages. 1983

Vol. 1040: A. Good, Local Analysis of Selberg's Trace Formula. III, 128 pages. 1983.

Vol. 1041: Lie Group Representations II. Proceedings 1982–1983. Edited by R. Herb, S. Kudla, R. Lipsman and J. Rosenberg. IX, 340 pages. 1984.

Vol. 1042: A. Gut, K.D. Schmidt, Amarts and Set Function Processes. III, 258 pages. 1983.

Vol. 1043: Linear and Complex Analysis Problem Book. Edited by V.P. Havin, S.V. Hruščëv and N.K. Nikol'skii. XVIII, 721 pages. 1984.

Vol. 1044: E. Gekeler, Discretization Methods for Stable Initial Value Problems. VIII, 201 pages. 1984.

Vol. 1045: Differential Geometry. Proceedings, 1982. Edited by A.M. Naveira. VIII, 194 pages. 1984.

Vol. 1046: Algebraic K-Theory, Number Theory, Geometry and Analysis. Proceedings, 1982. Edited by A. Bak. IX, 464 pages. 1984.

Vol. 1047: Fluid Dynamics. Seminar, 1982. Edited by H. Beir ao da Veiga. VII, 193 pages. 1984.

Vol. 1048: Kinetic Theories and the Boltzmann Equation. Seminar, 1981. Edited by C. Cercignani. VII, 248 pages. 1984.

Vol. 1049: B. Iochum, Cônes autopolaires et algèbres de ordan. VI, 247 pages. 1984.

Vol. 1050: A. Prestel, P. Roquette, Formally p-adic Fields. V,16 pages. 1984.

Vol. 1051: Algebraic Topology, Aarhus 1982. Proceedings. Edited by I. Madsen and B. Oliver. X, 665 pages. 1984.

Vol. 1052: Number Theory, New York 1982. Seminar. Edited by D.V. Chudnovsky, G.V. Chudnovsky, H. Cohn and M.B. Nathanson. V, 309 pages. 1984.

Vol. 1053: P. Hilton, Nilpotente Gruppen und nilpotente Räume. V, 221 pages. 1984.

Vol. 1054: V. Thomée, Galerkin Finite Element Methods for Parabolic Problems. VII, 237 pages. 1984.

Vol. 1055: Quantum Probability and Applications to the Quantum Theory of Irreversible Processes. Proceedings, 1982. Edited by L. Accardi, A. Frigerio and V. Gorini. VI, 411 pages. 1984.

Vol. 1056: Algebraic Geometry. Bucharest 1982. Proceedings, 1982. Edited by L. Bădescu and D. Popescu. VII, 380 pages. 1984.

Vol. 1057: Bifurcation Theory and Applications. Seminar, 1983. Edited by L. Salvadori. VII, 233 pages. 1984.

Vol. 1058: B. Aulbach, Continuous and Discrete Dynamics near Manifolds of Equilibria. IX, 142 pages. 1984.

Vol. 1059: Séminaire de Probabilités XVIII, 1982/83. Proceedings. Edité par J. Azéma et M. Yor. IV, 518 pages. 1984.

Vol. 1060: Topology. Proceedings, 1982. Edited by L.D. Faddeev and A.A. Mal'cev. VI, 389 pages. 1984.

Vol. 1061: Séminaire de Théorie du Potentiel. Paris, No. 7. Proceedings. Directeurs: M. Brelot, G. Choquet et J. Deny. Rédacteurs: F. Hirsch et G. Mokobodzki. IV, 281 pages. 1984.

Vol. 1062: J. Jost, Harmonic Maps Between Surfaces. X, 133 pages. 1984.

Vol. 1063: Orienting Polymers. Proceedings, 1983. Edited by J.L. Ericksen. VII, 166 pages. 1984.

Vol. 1064: Probability Measures on Groups VII. Proceedings, 1983. Edited by H. Heyer. X, 588 pages. 1984.

Vol. 1065: A. Cuyt, Padé Approximants for Operators: Theory and Applications. IX, 138 pages. 1984.

Vol. 1066: Numerical Analysis. Proceedings, 1983. Edited by D.F. Griffiths. XI, 275 pages. 1984.

Vol. 1067: Yasuo Okuyama, Absolute Summability of Fourier Series and Orthogonal Series. VI, 118 pages. 1984.

Vol. 1068: Number Theory, Noordwijkerhout 1983. Proceedings. Edited by H. Jager. V, 296 pages. 1984.

Vol. 1069: M. Kreck, Bordism of Diffeomorphisms and Related Topics. III, 144 pages. 1984.

Vol. 1070: Interpolation Spaces and Allied Topics in Analysis. Proceedings, 1983. Edited by M. Cwikel and J. Peetre. III, 239 pages. 1984.

Vol. 1071: Padé Approximation and its Applications, Bad Honnef 1983. Prodeedings. Edited by H. Werner and H.J. Bünger. VI, 264 pages. 1984.

Vol. 1072: F. Rothe, Global Solutions of Reaction-Diffusion Systems. V, 216 pages. 1984.

Vol. 1073: Graph Theory, Singapore 1983. Proceedings. Edited by K.M. Koh and H.P. Yap. XIII, 335 pages. 1984.

Vol. 1074: E.W. Stredulinsky, Weighted Inequalities and Degenerate Elliptic Partial Differential Equations. III, 143 pages. 1984.

Vol. 1075: H. Majima, Asymptotic Analysis for Integrable Connections with Irregular Singular Points. IX, 159 pages. 1984.

Vol. 1076: Infinite-Dimensional Systems. Proceedings, 1983. Edited by F. Kappel and W. Schappacher. VII, 278 pages. 1984.

Vol. 1077: Lie Group Representations III. Proceedings, 1982–1983. Edited by R. Herb, R. Johnson, R. Lipsman, J. Rosenberg. XI, 454 pages. 1984.

Vol. 1078: A.J.E.M. Janssen, P. van der Steen, Integration Theory. V, 224 pages. 1984.

Vol. 1079: W. Ruppert. Compact Semitopological Semigroups: An Intrinsic Theory. V, 260 pages. 1984

Vol. 1080: Probability Theory on Vector Spaces III. Proceedings, 1983. Edited by D. Szynal and A. Weron. V, 373 pages. 1984.

Vol. 1081: D. Benson, Modular Representation Theory: New Trends and Methods. XI, 231 pages. 1984.

Vol. 1082: C.-G. Schmidt, Arithmetik Abelscher Varietäten mit komplexer Multiplikation. X, 96 Seiten. 1984.

Vol. 1083: D. Bump, Automorphic Forms on GL (3,IR). XI, 184 pages. 1984.

Vol. 1084: D. Kletzing, Structure and Representations of Q-Groups. VI, 290 pages. 1984.

Vol. 1085: G.K. Immink, Asymptotics of Analytic Difference Equations. V, 134 pages. 1984.

Vol. 1086: Sensitivity of Functionals with Applications to Engineering Sciences. Proceedings, 1983. Edited by V. Komkov. V, 130 pages. 1984

Vol. 1087: W. Narkiewicz, Uniform Distribution of Sequences of Integers in Residue Classes. VIII, 125 pages. 1984.

Vol. 1088: A.V. Kakosyan, L.B. Klebanov, J.A. Melamed, Characterization of Distributions by the Method of Intensively Monotone Operators. X, 175 pages. 1984.

Vol. 1089: Measure Theory, Oberwolfach 1983. Proceedings. Edited by D. Kölzow and D. Maharam-Stone. XIII, 327 pages. 1984.

continued on page 187

TABLE OF CONTENTS

INTRODUCTION

THE MATHEMATICAL BACKGROUND

In 1949 H. Maaß [Ma 1] extended the classical Riemann-Hecke correspondence between Dirichlet series with functional equation and automorphic forms. For that purpose he introduced a new class of automorphic functions which are real-analytic on the upper half-plane $\mathbb{H}$, automorphic with respect to a certain subgroup $\overline{\Gamma} < PSL(2,\mathbb{R})$, and satisfy the wave equation for the Laplacian for the hyperbolic metric on $\mathbb{H}$

$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = \lambda f \tag{1}$$

with some parameter λ. These Maaß wave forms turned out to be of key importance for the subsequent development of the theory of modular forms and its applications to number theory (see e.g. the survey article [I] by H. Iwaniec, [He 1], [He 2], [He 3], [Ve 1]). In addition, the Maaß wave forms come up in a natural way jointly with the classical holomorphic automorphic forms in the representation theory of $SL(2,\mathbb{R})$ (see [GGPS], [La]).

The eigenvalue problem (1) was considered by W. Roelcke [Ro 3] and A. Selberg [Se 1], [Se 2] as an eigenvalue problem of a self-adjoint linear operator, that is, the Laplacian defined on a suitable domain in the Hilbert space $L^2(\overline{\Gamma} \setminus \mathbb{H})$. These researches finally led to the celebrated Selberg Trace Formula which is a relation between the eigenvalues of Δ and some data determined by $\overline{\Gamma}$. For a more detailed explanation of this background we introduce some notations.

A fundamental domain $\mathcal{F}$ of $\overline{\Gamma}$ is a set of representatives of the orbits of $\overline{\Gamma}$ in $\mathbb{H}$, measurable with respect to the hyperbolic area measure ω. The ω-measurable functions $f: \mathbb{H} \longrightarrow \mathbb{C}$ such that $f \circ M = f$ $(M \in \overline{\Gamma})$ and $\int_{\mathcal{F}} |f|^2 d\omega < \infty$ constitute a Hilbert space $\mathcal{H} \simeq L^2(\overline{\Gamma} \setminus \mathbb{H})$, equipped with the scalar product $(f,g) = \int_{\mathcal{F}} \overline{f}\, g\, d\omega$. Since Δf is invariant under $\overline{\Gamma}$ whenever f has this property, $\Delta : \mathcal{D} \longrightarrow \mathcal{H}$ defines a linear operator on an appropriate domain $\mathcal{D} \subset \mathcal{H}$. Roelcke has shown in [Ro 1] that Δ is essentially self-adjoint on $\mathcal{D}$. The key problem now is to determine the spectral decomposition of Δ. This is called the eigenvalue problem of automorphic forms. Up to now this problem was solved only for a certain class of Fuchsian groups of the second kind ([E1], [E2], [E3], [Pa 1], [Pa 2], [Pa 3]). These groups are not of arithmetic interest, however. The really interesting class of groups are the finitely generated Fuchsian groups of the first kind, i.e. the groups $\overline{\Gamma}$ with a fundamental domain of finite hyperbolic area. These groups are briefly called cofinite groups. For cofinite groups the continuous spectrum of Δ was completely determined by Selberg [Se 1], [Se 2] and Roelcke [Ro 1], [Ro 2], [Ro 3] in terms of the analytically continued Eisenstein series. Hence, the main problems left are concerned with the eigenvalues of Δ, and these problems turn out to be of upmost complexity. Not a single example of a cofinite group is known for which the sequence of eigenvalues can be explicitly determined. (It is not even clear beyond doubt what one should mean by an explicit determination of the eigenvalues.) For example, only the eigenvalue 0 is really explicitly known for the rational modular group $PSL(2,\mathbb{Z})$, and although much effort was spent in determining the first eigenvalues numerically on a computer ([He 2], [He 4]), none of the other eigenvalues for $PSL(2,\mathbb{Z})$ is "explicitly known". For arbitrary cofinite groups it is not even known whether a single eigenvalue different from 0 exists at all. Recent research has even led to the conjecture that the generic cofinite group has very few Maaß wave forms

([DIPS], [PS 1], [PS 2]), although large classes of arithmetically or geometrically "nice" groups are known which have infinitely many eigenvalues. The latter class includes of course all cocompact groups. Needless to say that our knowledge with respect to the eigenfunctions is still more defective despite some interesting numerical attempts for $PSL(2,\mathbb{Z})$ (see [St]).

Since precise results on the individual eigenvalues are out of scope one has to have recourse to asymptotic methods. Asymptotic results on the eigenvalues can be obtained from the Selberg Trace Formula. This formula takes its simplest form for cocompact groups. Let us assume for simplicity for the moment that $\overline{\Gamma} < PSL(2,\mathbb{R})$ is a cocompact discrete group without elliptic elements, and let

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots, \quad \lambda_n = \tfrac{1}{4} + r_n^2$$

be the eigenvalues of $-\Delta$. Suppose that the function

(2) $$h: \{r \in \mathbb{C}: |\mathrm{Im}\, r| < \tfrac{1}{2}+\delta\} \longrightarrow \mathbb{C}$$

$(\delta > 0)$ is holomorphic and even and satisfies the growth condition

(3) $$h(r) = O\left((1+|r|^2)^{-1-\delta}\right)$$

for $|r| \longrightarrow \infty$ uniformly in the strip. Let

(4) $$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) e^{-iru}\, dr$$

be the Fourier transform of h. Then the Selberg Trace Formula states:

(5) $$\sum_{n=0}^{\infty} h(r_n) = \frac{\omega(\mathcal{F})}{4\pi} \int_{-\infty}^{\infty} r h(r) \tanh \pi r\, dr + \sum_{\{P\}_{\overline{\Gamma}}} \frac{\log N(P_0)}{N(P)^{\frac{1}{2}} - N(P)^{-\frac{1}{2}}}\, g(\log N(P)) .$$

The sum on the right-hand side extends over all $\overline{\Gamma}$-congugacy classes $\{P\}_{\overline{\Gamma}}$ of the hyperbolic elements $P \in \overline{\Gamma} \setminus \{I\}$. $N(P)$ denotes the norm of P, that is, $N(P)$ is equal to the square of the eigenvalue of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\left(Pz = \frac{az+b}{cz+d}\text{ , det }\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1\right)$ with larger absolute value. Finally, P_o is the primitive hyperbolic element associated with P, that is, $P = P_o^m$ with $P_o \in \overline{\Gamma}$ and $m \geq 1$ maximal. All the sums and integrals in the above trace formula (5) are absolutely convergent.

D. A. Hejhal [He 1] gives a detailed proof of the trace formula in the cocompact case and some of its applications; see also the survey articles by Elstrodt [E4], Hejhal [He 3], Venkov [Ve 1] and Wallach [Wa]. The proof of the trace formula rests on the basic fact that the eigenfunctions of the differential operator Δ are simultaneously eigenfunctions of all integral operators associated with point-pair invariants in the following manner: For every compactly supported continuous function $\Psi\colon [0,\infty[\longrightarrow \mathbb{C}$, the function

$$k(z,z') := \Psi\left(\frac{|z-z'|^2}{\text{Im } z \quad \text{Im } z'}\right)$$

is a point-pair invariant, that is,

$$k(Mz,Mz') = k(z,z') \qquad (M \in PSL(2,\mathbb{R})).$$

The series

$$K(z,z') := \sum_{M\in\overline{\Gamma}} k(z,Mz')$$

is locally finite and $\overline{\Gamma}$-invariant in both variables and hence defines a linear operator

$$K\colon \mathcal{H} \longrightarrow \mathcal{H}$$

$$(Kf)(z) := \int_{\mathcal{F}} K(z,z')f(z')d\omega(z') .$$

Now, if $\lambda_n = \frac{1}{4}+r_n^2$ is an eigenvalue of $-\Delta$ with eigenfunction f_n, then $Kf_n = h(r_n)f_n$ where the even entire function h, independent of λ_n, is constructed by the following chain of integral transformations:

$$\Psi \longmapsto Q: [0,\infty[\longrightarrow \mathbb{C}, \quad Q(x) := \int_x^\infty \frac{\Psi(t)}{\sqrt{t-x}}\,dt$$

(Abel's integral transform),

$$Q \longmapsto g: \mathbb{R} \longrightarrow \mathbb{C}, \quad g(u) := Q(e^u + e^{-u} - 2),$$

$$g \longmapsto h: \mathbb{C} \longrightarrow \mathbb{C}, \quad h(r) := \int_{-\infty}^{\infty} e^{iru} g(u)\,du .$$

The kernel K has the eigenfunction expansion

$$K(z,z') = \sum_{n=0}^{\infty} h(r_n) f_n(z) f_n(z') .$$

Under certain mild additional assumptions on Ψ, this series converges uniformly on $\mathbb{H}\times\mathbb{H}$, hence $\int_{\mathcal{F}} K(z,z)d\omega(z)$ can be computed by termwise integration. This yields the preliminary trace formula

$$(6) \qquad \sum_{n=0}^{\infty} h(r_n) = \int_{\mathcal{F}} K(z,z)d\omega(z) .$$

Representing $K(z,z)$ by the series $\sum_{M\in\overline{\Gamma}} k(z,Mz)$ and interchanging integration and summation one can transform the last integral into the right-hand side of the trace formula by some calculations. Then suitable approximation arguments complete the proof of the trace formula for arbitrary pairs of functions (h,g) with the properties (2), (3) and (4). The proof is considerably more difficult if $\overline{\Gamma}$ is not cocompact but still cofinite. Then $\overline{\Gamma}$ also contains parabolic elements and Δ has a continuous spectrum in addition to the discrete one. In this case a term derived from certain eigenpackets associated with the Eisenstein series has to be subtracted from $K(z,z)$ on the right-hand side of the preliminary trace formula (6).

Selberg also discussed trace formulae of the type (6) in a more general geometrical setting. He proved the trace formula for cofinite groups where Δ is applied to vector-valued functions on $\mathbb{H}$ satisfying $f\circ M = \chi(M)f \quad (M \in \overline{\Gamma})$ with a unitary character χ, but up to now he has refrained from publishing a proof. In [He 2] Hejhal proved the trace formula for cofinite groups in the following more general framework. Instead of Δ he considered the differential operator of real weight $2k$:

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - i\,2ky\,\frac{\partial}{\partial x} .$$

The character χ has now to be replaced by a so-called unitary multiplier system χ of weight $2k$ (cf. section 1.3). It is convenient to define χ on the subgroup Γ of $SL(2,\mathbb{R})$ corresponding to $\overline{\Gamma}$ and containing the element $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. It is known that Δ_k is an essentially self-adjoint linear operator on a dense subspace $\mathfrak{D}_k$ of a Hilbert space $\mathfrak{H}_k$. The elements of $\mathfrak{H}_k$ are functions defined on $\mathbb{H}$ with values in a finite-dimensional vector-space V and with the transformation behaviour

$$f\left(\frac{az+b}{cz+d}\right) = \exp\Big(i\,2k\arg(cz+d)\Big)\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f(z)$$

$$\left(z \in \mathbb{H},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ \arg\colon \mathbb{C} \setminus\ ]-\infty,0] \longrightarrow\ ]-\pi,\pi]\right) .$$

There exists a close connection with the so-called classical entire automorphic forms. If g is a classical entire automorphic form, then the function f defined by $f(x+iy) = y^k g(x+iy)$ belongs to $\mathfrak{D}_k$ and is an eigenfunction of $-\Delta_k$ with eigenvalue $k(1-k)$. The proof of the trace formula is similar to the case $k = 0$, the technical expenditure is higher at some points. An eigenpacket part arises if and only if the underlying multiplier system χ is singular (cf. section 1.5).

Selberg noted a striking analogy of his trace formula with certain "explicit formulae" in analytic number theory. On the one hand of these "explicit formulae" the non-trivial zeros of the Riemann zeta-function are inserted into a holomorphic function h. On the other hand the Fourier transform of h is applied to the logarithms of the powers of the primes. Proceeding from this analogy Selberg introduced a zeta-function associated with Γ and χ which has properties similar to those of the Riemann zeta-function. The Selberg zeta-function arises as follows. Consider the trace formula with

$$h(r) = \frac{1}{(s-\frac{1}{2})^2+r^2} - \frac{1}{(a-\frac{1}{2})^2+r^2} ,$$

where the parameters s,a satisfy $\mathrm{Re}\, s > 1$, $\mathrm{Re}\, a > 1$. Then on the right-hand side of the trace formula there appears a contribution of the hyperbolic elements of Γ which as a function of s can be written as the logarithmic derivative of the product

$$Z(s) = \prod_{\substack{\{P_o\}_\Gamma \\ \text{hyp., prim.}}} \prod_{m=0}^{\infty} \det\left(id_V - \chi(P_o)N(P_o)^{-s-m}\right) .$$

This product converges absolutely and uniformly on compact sets in $\{s \in \mathbb{C}\colon \mathrm{Re}\, s > 1\}$. The trace formula immediately yields that the function $Z(s)$ has a meromorphic continuation to the whole s-plane and satisfies a functional equation. There exists a series of trivial zeros of Z . The non-trivial zeros are precisely the numbers $\frac{1}{2}+ir_n$, $\frac{1}{2}-ir_n$, where $\lambda_n = \frac{1}{4}+r_n^2$ is an eigenvalue of $-\Delta_k$. All but finitely many of the non-trivial zeros of Z are on the critical line $\{s \in \mathbb{C}\colon \mathrm{Re}\, s = \frac{1}{2}\}$, that is, the analogue of the Riemann Hypothesis is true for Z . Moreover, the definition of Z closely resembles the Euler product expansion of the Riemann zeta-function. All these properties are in striking analogy with the standard properties of the zeta-functions and L-series arising in number theory.

THE CONTENTS OF THIS VOLUME

A more direct approach to the Selberg zeta-function was suggested by J. Elstrodt in [E4] for the case of a fixed point free cocompact group with trivial multiplier system of weight 0 . For the analogous situation in three-dimensional hyperbolic space Elstrodt, Grunewald and Mennicke [EGM] explained a corresponding procedure. The elaboration of this approach for an arbitrary cofinite Fuchsian group with a unitary multiplier system of real weight $2k$ is an essential part of the contents of this volume. The papers [Ro 1], [Ro 2] by W. Roelcke and [E1], [E2] by J. Elstrodt form the basis of our considerations. Both these authors show that a complex number under appropriate assumptions belongs to the resolvent set $\rho(-\tilde{\Delta}_k)$ of the self-adjoint extension $-\tilde{\Delta}_k: \tilde{\mathfrak{D}}_k \longrightarrow \mathfrak{H}_k$ of $-\Delta_k$ and that the resolvent operator has an integral representation of the form

$$(-\tilde{\Delta}_k-\lambda)^{-1}f = \int_{\mathcal{F}} G_{k\lambda}(\ ,z)f(z)d\omega(z) \qquad (f \in \mathfrak{H}_k) .$$

For $z \not\equiv z'$ mod Γ the kernel $G_{k\lambda}(z,z')$ is determined by a normally convergent matrix series, the summation being extended over all $M \in \Gamma$ (cf. p. 26 , (1.4.7)). The integral representation of the resolvent operator is stated in Theorem 1.4.10 (p. 27). Apart from some definitions and simple considerations needed later on, the second important result reported on in the first chapter is a theorem by Roelcke on orthogonal expansions of the functions $f \in \tilde{\mathfrak{D}}_k$ with respect to a complete system of orthonormal eigenfunctions f_n $(n \geq 0)$ and eigenpackets of $-\tilde{\Delta}_k$; see Expansion Theorem 1.6.4 (p. 37). In section 2.1 we transform the integral

$$(\lambda-\mu) \int_{\mathcal{F}} \mathrm{tr}\Big(G_{k\lambda}(z,z')G_{k\mu}(z',z)\Big)d\omega(z')$$

into the sum of the series

$$(7) \qquad \sum_{n\geq 0}\left(\frac{1}{\lambda_n-\lambda} - \frac{1}{\lambda_n-\mu}\right)|f_n(z)|^2$$

and a contribution of the Eisenstein series. On the other hand Hilbert's resolvent equation yields

$$(\lambda-\mu) \int_{\mathcal{F}} \operatorname{tr}\Big(G_{k\lambda}(z,z')G_{k\mu}(z',z)\Big)d\omega(z')$$

$$= \lim_{z'\longrightarrow z} \operatorname{tr}\Big(G_{k\lambda}(z,z')-G_{k\mu}(z,z')\Big) \quad .$$

Integrating (7) over $\mathcal{F}$ we obtain

$$(8) \qquad \sum_{n\geq 0} \left(\frac{1}{\lambda_n-\lambda} - \frac{1}{\lambda_n-\mu}\right)$$

as the trace of the iterated resolvent kernel. The preliminary version of the resolvent trace formula (p. 46, Theorem 2.1.2) states that the series (8) is equal to the integral of the difference of $\lim_{z\longrightarrow z'} \operatorname{tr}\Big(G_{k\lambda}(z,z')-G_{k\mu}(z,z')\Big)$ and the eigenpacket part, the resolvent kernels being represented by the series (1.4.7).

This integral is computed in sections 2.1, 2.2, 2.3 and 2.4 after the integrand has been split into four sums corresponding to the identity, the hyperbolic the elliptic and the parabolic elements of Γ, respectively. The latter of these sums must be integrated jointly with the eigenpacket part since the single integrals do not exist. After the substitution $\lambda = s(1-s)$, $\mu = a(1-a)$ (Re s, Re a > 1) and some calculations where certain formulae on the hypergeometric function are quite useful there appear terms which may be looked at as logarithmic derivatives of holomorphic functions in s resp. a. The contribution of the hyperbolic elements has the form

$$\frac{1}{2s-1}\frac{Z'}{Z}(s) - \frac{1}{2a-1}\frac{Z'}{Z}(a)$$

where Z denotes the Selberg zeta-function (p. 50, Proposition 2.2.5, p. 56, Corollary 2.2.6). The contributions of the other elements turn out to be logarithmic derivatives of elementary functions involving the gamma function resp. the Barnes G-function in the case of the identity. The computation of the elliptic and especially the parabolic terms is considerably more complicated than the hyperbolic part. We state the results of our computations in Proposition 2.3.4 (p. 61), Corollary 2.3.5

(p. 68) for the elliptic and in Proposition 2.4.21 (p. 102), Corollary 2.4.22 (p. 104) for the parabolic case. In section 2.5 we obtain the Resolvent Trace Formula (p. 106, Theorem 2.5.1, p. 108, Theorem 2.5.2), an important special case of the Selberg Trace Formula, by combining the results achieved before. From the Resolvent Trace Formula we conclude well-known formulae for the dimensions of the spaces of classical entire automorphic forms of weight $2k > 2$ and necessary conditions for the existence of unitary multiplier systems χ of weight $2k \in \mathbb{R}$ in the cocompact case.

This approach to the Selberg zeta-function clarifies the origin of the apparently arbitrarily chosen function

$$h(r) = \frac{1}{(s-\frac{1}{2})^2+r^2} - \frac{1}{(a-\frac{1}{2})^2+r^2}$$

which makes the zeta-function arise from the general Selberg Trace Formula. The direct approach bears another advantage, as some otherwise necessary very technical approximation arguments are avoided. Moreover, our computation of the non-hyperbolic contributions automatically yields the appropriate elementary factors by which the Selberg zeta-function Z must be multiplied in order to obtain a function Ξ that enjoys very simple properties. To be more explicit, we mention that Ξ is an entire function the zeros of which are exactly the numbers $\frac{1}{2} \pm ir_n$ and which satisfies the simple functional equation

$$\Xi(s) = \Xi(1-s) \ .$$

The notations Z, Ξ are analogous to the usual notations ζ, ξ for the Riemann zeta-function ζ and its associated entire function

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) \ .$$

The parabolic terms account for a product $P(s)$ introduced in section 3.2, and the investigation of ΞP in sections 3.2, 3.3 and 3.4 will yield interesting results. For example, ΞP is an entire function of order precisely 2, whereas Ξ and P are entire functions of order at most 2 (p. 125, Theorem 3.2.11, p. 126, Corollary 3.2.13). The series

$$\sum_{\substack{n\geq 0 \\ \lambda_n \neq 0}} \lambda_n^{-s}$$

converges for $\operatorname{Re} s > 1$ (p. 125, Corollary 3.2.12). It remains an open question whether or not 1 is the exact abscissa of convergence of this series for all cofinite groups. An estimate on the argument function $\arg \Xi(\frac{1}{2}+iT)$ for $T \longrightarrow +\infty$ renders to the Weyl-Selberg asymptotic formula for the distribution of the eigenvalues: Let $N(T)$ denote the number of zeros $\frac{1}{2}+ir_n$ of Ξ such that $r_n \in]0,T[$, and let $\varphi(s,\chi)$ denote the determinant of the so-called scattering matrix which is defined by means of the Eisenstein series. Then the Weyl-Selberg asymptotic formula states:

$$N(T) - \frac{1}{4\pi}\int_{-T}^{T} \frac{\varphi'}{\varphi}(\tfrac{1}{2}+it,\chi)\,dt = d\frac{\omega(\mathcal{F})}{4\pi}T^2 + R(T) \tag{9}$$

where d denotes the dimension of the range space V of the automorphic forms under consideration and where $R(T)$ is an error term that will be discussed later (see p. 138, Theorem 3.3.13). Commenting upon the growth of the terms on the left-hand side of (9), Selberg remarks in his Göttingen lectures: "Unfortunately however, we have in the general case no means of separately estimating the two terms on the left-hand side of [(9)] so that the asymptotic formula for the distribution of the eigenvalues r_i cannot be given. Only in some special cases when the function $\varphi(s,\chi)$ can be expressed in terms of functions that are known from analytic number theory can we do this, and in all these special cases the second term on the left-hand side of [(9)] is $O(R \log R)$ as $R \longrightarrow \infty$." On the basis of this result for congruence subgroups of the modular group, Selberg conjectured that $N(T)$ is always dominant. A strong form of this so-called Selberg conjecture was recently disproved ([DIPS], [PS1] , [PS2])under certain assumptions, such as extended Riemann hypotheses, but there still remain difficult open problems. We shall comment further on this topic at the end of section 3.3.

Since the function Ξ is at most of order 2 , it admits a factorization in the form of a Weierstraß product multiplied by e^{Q} with a polynomial

$$Q(s) = a_2(s-\tfrac{1}{2})^2 + a_1(s-\tfrac{1}{2}) + a_o$$

of degree at most 2 . We develop certain formulae for the coefficients of Q in section 3.4. There is an amazing analogy of the highest coefficient a_2 with the Euler-Mascheroni constant which governs the highest coefficient of the polynomial in the analogous canonical factorization of the Riemann zeta-function. For example, if Γ is cocompact or if the multiplier system χ is regular, we have

$$a_2 = \lim_{T\to\infty}\Big(\sum_{\substack{n\geq 0\\ r_n\neq 0\\ \mathrm{Re}\, r_n<T}} \frac{1}{r_n^2} - d\frac{\omega(\mathcal{F})}{2\pi}\log T\Big) .$$

The full results are summarized in Theorem 3.4.8 (p. 157).

The resolvent method yields no less than the usual approach to the Selberg Trace Formula and to the zeta-function. This is explained in the fourth and last chapter. In particular, the general Selberg Trace Formula for pairs of functions (h,g) satisfying the above conditions (2), (3), (4) is deduced from the Resolvent Trace Formula by simple use of the calculus of residues. Thus the resolvent method does not lead to a "loss of information", the Resolvent Trace Formula resp. the Selberg zeta-function turn out to inherit just as much information as the general Selberg Trace Formula.