Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Algebra in a Localic Topos with Applications to Ring Theory

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Berlin Heidelberg New York Tokyo

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AMS Subject Classifications (1980): 18F20, 18C10, 16A64, 16A90

ISBN 3-540-12711-9 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-12711-9 Springer-Verlag New York Heidelberg Berlin Tokyo

Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

Lecture Notes in Mathematics

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0. INTRODUCTION

Sheaves of algebras on a topological space appear in many problems in mathematics and their interest has no longer to be demonstrated. The purpose of this publication is to study the localizations of the category of sheaves of $\mathbb{T}$-algebras, where $\mathbb{T}$ is a finitary algebraic theory, and the extent to which it characterizes the topological base space. The techniques developed to solve these problems, applied to the case of modules on a ring R, provide new results on pure ideals and the representation of rings. As a matter of fact we develop our study in the more general and more natural context of a theory $\mathbb{T}$ internally defined with respect to a topos of sheaves on a frame (i.e. a complete Heyting algebra; for example the algebra of open subsets of a topological space).

We shall normally use the letter $\mathbb{H}$ to denote a frame and, unless stated otherwise, $\mathbb{T}$ will denote a finitary algebraic theory in the topos of sheaves on $\mathbb{H}$. In chapter 1, we recall some basic properties of the categories $Pr(\mathbb{H},\mathbb{T})$ and $Sh(\mathbb{H},\mathbb{T})$ of presheaves and sheaves of $\mathbb{T}$-algebras on $\mathbb{H}$ (limits, colimits, generators, associated sheaf, and so on ...). Reference is made largely to classical texts.

In chapter 2, we first study the Heyting subobjects of a fixed object in $Sh(\mathbb{H},\mathbb{T})$: these are the subobjects wich satisfy properties analogous to the properties of any subobject in a topos. This allows us to describe the formal initial segments of $Sh(\mathbb{H},\mathbb{T})$. If $u\downarrow$ is any initial segment of $\mathbb{H}$ and $\mathbb{T}_{u\downarrow}$ the restriction of $\mathbb{T}$ to $u\downarrow$, $Sh(u\downarrow,\mathbb{T}_{u\downarrow})$ is a subcategory of $Sh(\mathbb{H},\mathbb{T})$ satisfying very special properties. We then define "formal initial segments" to be subcategories of $Sh(\mathbb{H},\mathbb{T})$ satisfying analogous properties. The Heyting subobjects of a fixed algebraic sheaf constitute a frame and the same holds for the formal initial segment of $Sh(\mathbb{H},\mathbb{T})$.

Chapter 3 applies the results developed in chapter 2 to classify the localizations of $Sh(\mathbb{H},\mathbb{T})$ when the theory $\mathbb{T}$ is commutative. We build an object $\Omega_{\mathbb{T}}$ in a topos $\mathcal{E}(\mathbb{H},\mathbb{T})$; the localizations of $Sh(\mathbb{H},\mathbb{T})$ are exactly classified by the Lawvere - Tierney topologies $j : \Omega_{\mathbb{T}} \to \Omega_{\mathbb{T}}$. A characterization in terms of generalized Gabriel - Grothendieck topologies is also given. Examples are produced. A counterexample is given for the case of a non-commutative theory.

When $\mathbb{T}$ is the theory of sets, $\mathbb{H}$ can be easily recovered from the topos $Sh(\mathbb{H},\mathbb{T})$: it is the frame of subobjects of terminal object 1. In chapter 4, we introduce a large class of theories to be called integral : sets, monoids, groups, rings, modules on an integral domain, boolean algebras, ... are examples of them. When $\mathbb{T}$ is integral, the frame $\mathbb{H}$ can be recovered from the category $Sh(\mathbb{H},\mathbb{T})$: it is the frame of formal initial segments defined in chapter 2.

In chapter 5, we expound the results on formal initial segments for a classical finitary algebraic theory $\mathbb{T}$. The category of $\mathbb{T}$-algebras is simply the category of sheaves of $\mathbb{T}$-algebras on the singleton. The frame of formal initial segments turns out to be the Heyting algebra of open subsets of a compact space $Spp(\mathbb{T})$ to be called the spectrum of the theory $\mathbb{T}$. Some results of chapter 2 give rise to a sheaf representation theorem for $\mathbb{T}$-algebras on this spectrum $Spp(\mathbb{T})$.

Chapter 6 is devoted to the case of the theory of modules on an arbitrary ring R with a unit. We establish an isomorphism between the frame of formal initial segments of $\underline{Mod}_R$ and the frame of pure ideals of the ring R. Applying the results of chapter 5, we present R as the ring of global sections of a sheaf of rings on the spectrum of the theory of R-modules; the functorial description of this sheaf is the sheaf of rings of R-linear endomorphisms of the pure ideals of R. An analogous representation theorem holds for any R-module. By interpreting the results of chapter 3, we also obtain the classification of localizations of $\underline{Mod}_R$ as presented by H. Simmons in [22].

All the material of chapter 6 concerning pure ideals and the sheaf representation theorem has been obtained as a corollary of the general theory developed previous chapters. We have found it interesting to rewrite these results using only standard techniques of ring theory. This is the object of chapter 7 which thus provides, in the very special case of modules, an approach of the representation theorem which becomes independant of the general categorical machinery. We include also some additional results and in particular an alternative representation theorem on the same spectrum of the theory of R-modules.

Finally, in chapter 8, we turn to the case of a Gelfand ring R. We first prove some useful properties of pure ideals in Gelfand rings and also some characterizations of Gelfand rings in terms of pure ideals. This allows us to prove that the sheaf representation of chapters 5 - 6 - 7, in the case of Gelfand rings, is the representation studied by Mulvey and Bkouche in [16] and [3]; in particular it coincides with Pierce's representation in the case of a Von Neumann regular ring. As a consequence, we obtain a functorial description of the classical sheaf representations for Gelfand and Von Neumann regular rings : this is simply the sheaf of R-linear endomorphisms of the pure ideals of the ring R. In an appendix, we show that this description in terms of rings of endomorphisms holds in fact for any ring in the case of Pierce's representation.

We are indebted to Harold Simmons for translating in the non commutative case several of our proofs on commutative Gelfand rings. This work has also been improved by fruitful conversations with M. Carral, C. Mulvey and the participants of the category seminar in Louvain-la-Neuve. This is the opportunity for us to thank all of them.

CONTENTS

CHAPTER 1 : CATEGORIES OF ALGEBRAIC SHEAVES

This chapter does not present any new results, except some technical lemmas which will be useful later. We recall some standard facts on sheaves and algebraic theories and take the opportunity to set out the notations and the terminology.

§ 1. ALGEBRAIC THEORIES

A classical or external finitary algebraic theory $\mathbb{T}$ can be presented as a category with a countable set of distinct objects T^0, T^1, T^2, ..., T^n, ... such that T^n is the n-th power of T^1. A (classical) model of $\mathbb{T}$ is a finite product preserving covariant functor from $\mathbb{T}$ to the category Sets of sets; such a model is also called a (classical) $\mathbb{T}$-algebra. A morphism between two $\mathbb{T}$-algebras is simply a natural transformation. We denote by $\underline{\text{Sets}}^{\mathbb{T}}$ the category of $\mathbb{T}$-algebras and their morphisms. There is a forgetful functor $U : \underline{\text{Sets}}^{\mathbb{T}} \to \underline{\text{Sets}}$ which sends a $\mathbb{T}$-algebra A to the underlying set $A(T^1)$. U has a monomorphism preserving left adjoint $F : \underline{\text{Sets}} \to \underline{\text{Sets}}^{\mathbb{T}}$. F is such that for any finite set n, F(n) is isomorphic to $\mathbb{T}(T^n,-)$; so the set underlying F(n) is the set of n-ary operations.

The category $\underline{\text{Sets}}^{\mathbb{T}}$ is complete and cocomplete. The forgetful functor U preserves and reflects limits and filtered colimits; it is represented by the generator $F(1) \cong \mathbb{T}(T^1,-)$ and thus is faithful. A filtered colimit $L = \varinjlim A_i$ is just the set of all elements in all the A_i divided by the equivalence relation which identifies $x \in A_i$ and $y \in A_j$ if there are morphisms $A_i \to A_k$ and $A_j \to A_k$ which send x and y to the same $z \in A_k$. From this it follows that in $\underline{\text{Sets}}^{\mathbb{T}}$, finite limits commute with filtered colimits. It is also the case that a morphism f in $\underline{\text{Sets}}^{\mathbb{T}}$ is a coequalizer if and only if U(f) is a surjection. Moreover any $\mathbb{T}$-algebra is a quotient of a free $\mathbb{T}$-algebra, i.e. for any $\mathbb{T}$-algebra A there exists a set S and a coequalizer $F(E) \to A$; in fact, E can be chosen to be the underlying set of A.

If $\mathbb{T}$ and $\mathbb{T}'$ are two algebraic theories, a morphism of theories $\mathbb{T} \to \mathbb{T}'$ is a product preserving functor. This induces by composition an algebraic functor $\underline{\text{Sets}}^{\mathbb{T}'} \to \underline{\text{Sets}}^{\mathbb{T}}$; this functor has a left adjoint. It should be noted that a morphism of theories takes any n-ary operation of $\mathbb{T}$ into a n-ary operation of $\mathbb{T}'$.

The results already mentioned can be found in [21], chapter 18. The following facts on commutative theories can be found in [15]. The theory $\mathbb{T}$ is called commutative if for any integers n, m and any operations $\alpha : T^n \to T^1$, $\beta : T^m \to T^1$ the following square commutes :

$$\begin{array}{ccccc} T^{n\times m} & \cong & (T^n)^m & \xrightarrow{\alpha^m} & T^m \\ \wr\Vert & & & & \\ (T^m)^n & & \circlearrowleft & & \\ {\scriptstyle \beta^n}\downarrow & & & & \downarrow{\scriptstyle \beta} \\ T^n & & \xrightarrow[\alpha]{} & & T^1 \end{array}$$

when $\mathbb{T}$ is commutative, $\underline{\text{Sets}}^{\mathbb{T}}$ becomes in a natural way a symmetric monoidal closed category.

§ 2. FRAMES

A lattice H is a partially ordered set in which each pair (u,v) of elements has an infimum $u \wedge v$ and a supremum $u \vee v$. The lattice H is distributive if for any elements u, v, w of H the following equalities hold

$$u \wedge (v \vee w) = (u \wedge v) \vee (u \wedge w)$$
$$u \vee (v \wedge w) = (u \vee v) \wedge (u \vee w);$$

in fact each of these equalities implies the other one. The lattice H is a Heyting algebra if it possesses a smallest element 0, a greatest element 1 and if for any v, w in H there exists some (necessarily unique) $v \Rightarrow w$ in H such that for any u in H

$$u \wedge v \leqslant w \quad \text{iff} \quad u \leqslant v \Rightarrow w;$$

a Heyting algebra is automatically a distributive lattice.

A lattice H is called complete if each subset of H has a supremum or, equivalently, if each subset of H has an infimum. A frame is a complete lattice which satisfies the generalized distributive law

$$u \wedge (\bigvee_{i\in I} v_i) = \bigvee_{i\in I} (u \wedge v_i).$$

A frame is necessarily a distributive lattice but the distributive law

$$u \vee (\bigwedge_{i\in I} v_i) = \bigwedge_{i\in I} (u \vee v_i)$$

holds only for finite I. If H and H' are two frames, a morphism of frames $f : H \to H'$ is a map $f : H \to H'$ preserving finite $\wedge$ and arbitrary $\vee$. The notion of frame is equivalent to that of complete Heyting algebra. A morphism of frames does not preserve the "implication" $v \Rightarrow w$.

If X is a topological space, the lattice of open subsets of X is a frame for the usual laws of intersection and union. If $f : X \to Y$ is a continuous mapping between two spaces, f induces by inverse image a morphism of frames $\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X)$ between the corresponding lattices of open subsets. This gives rise to a contravariant functor from the category of topological spaces to the category of frames; this functor has an adjoint which takes a frame into a sober space (i.e. a space such that any closed subset which is not expressible as the union of two proper closed subsets is the closure of exactly one point). All the material we need concerning lattices and frames can be found in [11].

§ 3. SHEAVES ON A FRAME

A frame $\mathbf{H}$ can be seen as a category whose objects are the points of $\mathbf{H}$; there is a (single) morphism from u to v if $u \leqslant v$. A presheaf on $\mathbf{H}$ is a contravariant functor $A : \mathbf{H}^{op} \to \underline{\text{Sets}}$; a morphism of presheaves is a natural transformation. If $u \leqslant v$ in $\mathbf{H}$, A is a presheaf on $\mathbf{H}$ and x an element in A(v), we denote by $x|_u$ the image of x in A(u) under the map $A(u \leqslant v)$. The category of presheaves on $\mathbf{H}$ is denoted by $Pr(\mathbf{H})$. A presheaf A is called separated if for any $u = \bigvee_{i\in I} u_i$ in $\mathbf{H}$ and x, y in A(u),

$$x = y \quad \text{iff} \quad \forall\, i \in I \quad x|_{u_i} = y|_{u_i} .$$

A presheaf A is called a sheaf if for any $u = \bigvee_{i\in I} u_i$ in $\mathbf{H}$ and x_i in $A(u_i)$, the condition

$$\forall\, i, j \in I \quad x_i|_{u_i \wedge u_j} = x_j|_{u_i \wedge u_j}$$

implies the existence of a unique x in A(u) such that for any i, $x|_{u_i} = x_i$. A sheaf is necessarily separated. The full subcategory of sheaves is denoted by $Sh(\mathbf{H})$; the canonical inclusion $Sh(\mathbf{H}) \hookrightarrow Pr(\mathbf{H})$ has a left adjoint which preserves finite limits : it is called the associated sheaf functor and denoted by $a : Pr(\mathbf{H}) \to Sh(\mathbf{H})$. (Cf. [1]). If A is a separated presheaf and $u \in \mathbf{H}$, aA(u) has an easy description : consider all the families $(x_i \in A(u_i))_{i\in I}$ for all the coverings $u = \bigvee_{i\in I} u_i$ in $\mathbf{H}$, such that

$$\forall\, i, j \in I \quad x_i|_{u_i \wedge u_j} = x_j|_{u_i \wedge u_j} ;$$

two such families are equivalent if they coincide on all the elements of a common refinement of the coverings; aA(u) is the quotient by the equivalence relation of the set of all such families. In that case the canonical morphism $A \to aA$ is a monomorphism.

§ 4. ALGEBRAIC SHEAVES (EXTERNAL VERSION)

If $\mathbb{T}$ is a (classical) finitary algebraic theory and $\mathbf{H}$ is a frame, a presheaf of $\mathbb{T}$-algebras is a contravariant functor $A : \mathbf{H}^{op} \to \underline{\mathrm{Sets}}^{\mathbb{T}}$; a morphism of presheaves of $\mathbb{T}$-algebras is a natural transformation. The corresponding category is denoted by $\mathrm{Pr}(\mathbf{H}, \mathbb{T})$. There is a forgetful functor $U : \mathrm{Pr}(\mathbf{H}, \mathbb{T}) \to$ $\to \mathrm{Pr}(\mathbf{H})$ obtained by composition with the forgetful functor $U : \underline{\mathrm{Sets}}^{\mathbb{T}} \to \underline{\mathrm{Sets}}$. U has a left adjoint F preserving monomorphisms and such that for any presheaf $A : \mathbf{H} \to \underline{\mathrm{Sets}}$, $FA(u)$ is the free $\mathbb{T}$-algebra on $A(u)$. A sheaf of $\mathbb{T}$-algebras is a presheaf of $\mathbb{T}$-algebras whose underlying presheaf is a sheaf. The corresponding category of sheaves of $\mathbb{T}$-algebras is denoted by $\mathrm{Sh}(\mathbf{H}, \mathbb{T})$. The canonical full inclusion $\mathrm{Sh}(\mathbf{H}, \mathbb{T}) \hookrightarrow \mathrm{Pr}(\mathbf{H}, \mathbb{T})$ has a left adjoint which preserves finite limits; the reflection of a presheaf of $\mathbb{T}$-algebras is the sheaf universally associated to the underlying presheaf. As a consequence there is a forgetful functor $U : \mathrm{Sh}(\mathbf{H}, \mathbb{T}) \to \mathrm{Sh}(\mathbf{H})$ which has a monomorphism preserving left adjoint sending a sheaf A to $aF(A)$. All these results on sheaves can be found in [1].

§ 5. ALGEBRAIC SHEAVES (INTERNAL VERSION)

If $\mathbf{H}$ is a frame, $\mathrm{Sh}(\mathbf{H})$ is a topos satisfying the axiom of infinity and it makes sense to speak of a finitary algebraic theory $\mathbb{T}$ internally defined with respect to $\mathrm{Sh}(\mathbf{H})$. This is exactly a sheaf on $\mathbf{H}$ with values in the category of algebraic theories and their morphisms. In other words, $\mathbb{T}$ is a contravariant functor from $\mathbf{H}$ to the category of algebraic theories and their morphisms, such that that for any integer n, the functor $\mathbf{H}^{op} \to \underline{\mathrm{Sets}}$ which sends $u \in \mathbf{H}$ to the set $\mathcal{O}_n(u)$ of n-ary operations of the theory $\mathbb{T}(u)$ is a sheaf in the usual sense.

A $\mathbb{T}$-algebra in $\mathrm{Sh}(\mathbf{H})$ is a sheaf $A : \mathbf{H}^{op} \to \underline{\mathrm{Sets}}$ equipped, for any $u \in \mathbf{H}$, with the structure of a $\mathbb{T}(u)$ algebra on $A(u)$ in such a way that for $u \leqslant v$ in $\mathbf{H}$ and $\alpha \in \mathcal{O}_n(v)$ the following diagram commutes

$$\begin{array}{ccc} A^n(v) & \xrightarrow{\quad \alpha \quad} & A(v) \\ {\scriptstyle A^n(u \leqslant v)} \downarrow & \hookrightarrow & \downarrow {\scriptstyle A(u \leqslant v)} \\ A^n(u) & \xrightarrow[\quad \alpha|_u \quad]{} & A(u). \end{array}$$

A morphism $f : A \to B$ of $\mathbb{T}$-algebras in $Sh(\mathbb{H})$ is a natural transformation such that for any $u \in \mathbb{H}$, f_u is a morphism of $\mathbb{T}(u)$-algebras. The category of $\mathbb{T}$-algebras in $Sh(\mathbb{H})$ is denoted by $Sh(\mathbb{H}, \mathbb{T})$. An analogous definition holds for presheaves and we get a category $Pr(\mathbb{H}, \mathbb{T})$.

$Sh(\mathbb{H}, \mathbb{T})$ is a full subcategory of $Pr(\mathbb{H}, \mathbb{T})$ and the canonical inclusion has a left adjoint a which preserves finite limits and is the associated sheaf functor. Moreover the obvious forgetful functor $U : Pr(\mathbb{H}, \mathbb{T}) \to Pr(\mathbb{H})$ has a monomorphism preserving left adjoint $F : Pr(\mathbb{H}) \to Pr(\mathbb{H}, \mathbb{T})$ such that, for any presheaf A and any element $u \in \mathbb{H}$, $FA(u)$ is the free $\mathbb{T}(u)$-algebra on $A(u)$. This implies that the forgetful functor $U : Sh(\mathbb{H}, \mathbb{T}) \to Sh(\mathbb{H})$ has a monomorphism preserving left adjoint which sends a sheaf A to $aF(A)$. These results on internal algebraic theories can be found in classical texts on topos theory, like [12].

We used the same notation $Sh(\mathbb{H}, \mathbb{T})$ in both cases of a classical theory $\mathbb{T}$ and a theory internally defined with respect to $Sh(\mathbb{H})$. In fact no real confusion arises because the former situation is a special case of the latter as can be seen from the following argument : a classical finitary algebraic theory $\mathbb{T}$ may be identified with a constant presheaf $\Delta\mathbb{T}$ of algebraic theories on $\mathbb{H}$; the corresponding associated sheaf $a\Delta\mathbb{T}$ is a theory internally defined with respect to $Sh(\mathbb{H})$ and the categories $Sh(\mathbb{H}, \mathbb{T})$ and $Sh(\mathbb{H}, a\Delta\mathbb{T})$ coincide. For this reason we shall work in the more general context of a theory $\mathbb{T}$ internally defined with respect to $Sh(\mathbb{H})$.

From now on and through this chapter $\mathbb{H}$ is a frame and $\mathbb{T}$ is a finitary algebraic theory internally defined with respect to $Sh(\mathbb{H})$. We recall and establish some basic facts about $Sh(\mathbb{H}, \mathbb{T})$.

§ 6. LIMITS AND COLIMITS

Proposition 1.

The categories $Pr(\mathbb{H}, \mathbb{T})$ *and* $Sh(\mathbb{H}, \mathbb{T})$ *are complete, cocomplete and regular.*

Any algebraic category is complete, cocomplete and regular. Now in $Pr(\mathbb{H}, \mathbb{T})$ limits, colimits and images are computed pointwise : this implies that $Pr(\mathbb{H}, \mathbb{T})$ is complete, cocomplete and regular. $Sh(\mathbb{H}, \mathbb{T})$ is complete and cocomplete as a full reflective subcategory of $Pr(\mathbb{H}, \mathbb{T})$; it is regular because the reflection is exact. (Cfr. [2]). ■

Proposition 2.

The forgetful functors $U : \mathrm{Pr}(\mathbb{H}, \mathbb{T}) \to \mathrm{Pr}(\mathbb{H})$ *and* $U : \mathrm{Sh}(\mathbb{H}, \mathbb{T}) \to \mathrm{Sh}(\mathbb{H})$ *preserve and reflect filtered colimits.*

In any algebraic category the filtered colimits are computed as in the category of sets. In $\mathrm{Pr}(\mathbb{H}, \mathbb{T})$ and $\mathrm{Pr}(\mathbb{H})$ all colimits are computed pointwise. Therefore the filtered colimits in $\mathrm{Pr}(\mathbb{H}, \mathbb{T})$ are computed as in $\mathrm{Pr}(\mathbb{H})$.

To compute an arbitrary colimit in $\mathrm{Sh}(\mathbb{H}, \mathbb{T})$ or in $\mathrm{Pr}(\mathbb{H}, \mathbb{T})$, we need to compute it in $\mathrm{Pr}(\mathbb{H}, \mathbb{T})$ or $\mathrm{Pr}(\mathbb{H})$ and apply the associated sheaf functor. But filtered colimits are computed in the same way in $\mathrm{Pr}(\mathbb{H}, \mathbb{T})$ and $\mathrm{Pr}(\mathbb{H})$ and the associated sheaf functor preserves them. So the result holds in the case of sheaves. ■

Proposition 3.

In $\mathrm{Pr}(\mathbb{H}, \mathbb{T})$ *and* $\mathrm{Sh}(\mathbb{H}, \mathbb{T})$, *finite limits commute with filtered colimits.*

This is true in any algebraic category and hence it is in $\mathrm{Pr}(\mathbb{H}, \mathbb{T})$ where limits and colimits are computed pointwise. In $\mathrm{Sh}(\mathbb{H}, \mathbb{T})$, a limit or a colimit is the reflection of the corresponding limit or colimit in $\mathrm{Pr}(\mathbb{H}, \mathbb{T})$; as the reflection preserves colimits and finite limits, the commutation property transfers to $\mathrm{Sh}(\mathbb{H}, \mathbb{T})$. ■

If u is some element in $\mathbb{H}$, we denote by $h_u : \mathbb{H}^{op} \to \underline{\mathrm{Sets}}$ the presheaf represented by u; the continuity of a representable functor implies immediately that h_u is in fact a sheaf.

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Proposition 4.

Consider $u \in \mathbb{H}$ *and* $A \in \mathrm{Pr}(\mathbb{H}, \mathbb{T})$. *The following natural isomorphism holds*

$$A(u) \cong (F\, h_u, A).$$

$A(u) \cong UA(u)$	
$\cong (h_u, UA)$	Yoneda lemma
$\cong (F\, h_u, A)$	adjunction $F \dashv U$. ■