Lecture Notes in Mathematics 1514

U. Krengel K. Richter V. Warstat (Eds.)

Ergodic Theory and Related Topics III

Proceedings, Güstrow 1990

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The editors of a volume are strongly advised to inform contributors about these points at an early stage.

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Surveys, if included, should cover a sufficiently broad topic, and should normally not just review the author's own recent research. In the case of surveys, exceptionally, proofs of results may not be necessary.

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§ 6. Proceedings should appear soon after the related meeting. The publisher should therefore receive the complete manuscript (preferably in duplicate) including the Introduction and Table of Contents within nine months of the date of the meeting at the latest.

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Further remarks and relevant addresses at the back of this book.

Lecture Notes in Mathematics 1514

Introduction

In the eighties, Horst Michel organized two conferences "Ergodic theory and Related Topics I and II" held in 1981 at Vitte (Hiddensee), GDR and in 1986 at Georgenthal (Thuringia), GDR. These conferences succeeded in bringing scientists from the East and the West together. Ergodic theorists from Austria, CSSR, France, FRG, GDR, Great Britain, Greece, Japan, the Netherlands, Poland, USA, USSR, and Vietnam discussed their recent results in measure–theoretic and topological dynamical systems as well as connections to other fields. A third conference was in the planning when Horst Michel, his wife Jutta, his younger daughter Kathrin and his mother died in a tragic car accident in December 1987. His colleagues all over the world lost a good friend.

Horst Michel was born in a little town in Thuringia. He studied mathematics at the University of Leipzig. As an assistant at the Technical College Ilmenau and at the University of Halle, he worked on iteration groups of real valued functions using methods of functional analysis. His thesis (1961) dealt with "Continuous and monotone iteration groups of nondifferentiable real valued functions". He then turned to the study of measure theoretical properties and of the classification of special groups of measure preserving transformations. Stimulated by articles of K. Jacobs, H. Furstenberg, and W. Parry, he explored the class of totally ergodic dynamical systems with quasidiscrete spectrum, in particular their embeddability into a flow. After 1970, he became interested in topological dynamics and studied so–called configuration spaces on special lattices. A list of his publications appears in "Kongreß und Tagungsberichte der Martin–Luther–Universität Halle-Wittenberg 1989/54".

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The idea of having a third conference was not given up. Horst Michel's students Karin Richter–Häsler and Volker Warstat organized it, and it was held in October 1990 in Güstrow, GDR, although the political events of 1989–90 caused various difficulties quite different from those of previous years. Fortyfive colleagues from 9 countries participated. This volume contains those results which are not published elsewhere. We thank all the participants of the conference for contributing towards its success, all the authors for their good cooperation with the editors, and the Martin–Luther–University at Halle–Wittenberg for sponsoring the conference.

Our special thanks go to all colleagues who offered their advice in preparing these Proceedings, especially Prof. M. Denker from Göttingen.

Göttingen and Halle
March 1992

Ulrich Krengel
Karin Richter–Häsler
Volker Warstat

Dedicated to the memory of

HORST MICHEL

(1934–1987)

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Symbolic dynamics for angle-doubling on the circle
I. The topology of locally connected Julia sets

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1 Introduction

The study of the dynamics of complex polynomials leads to some problems which belong to topology, combinatorics and number theory rather than complex function theory. Douady and Hubbard used trees to study Julia sets [6, 7], and Thurston [15] introduced invariant laminations of the circle. Our point is to show how symbolic dynamics can be used to strengthen and clarify their results. We restrict ourselves to quadratic polynomials although some of our results extend to polynomials or to invariant factors of shift spaces [1].

The basic concepts are simple. We consider the circle $T = R/Z$ and the angle-doubling map $h : T \to T$, $h(\beta) = 2\beta \bmod 1$. Fix $\alpha \in T$. The diameter between $\frac{\alpha}{2}$ and $\frac{\alpha+1}{2}$ divides T into two open semi-circles T_0^α and T_1^α, where the fixed point $0 = 1$ of h shall belong to T_1^α. The *itinerary* of a point $\beta \in T$ with respect to α is defined as

$$I^\alpha(\beta) = s_1 s_2 s_3 \ldots \quad \text{with} \quad s_i = \begin{cases} 0 & \text{for} \quad h^{i-1}(\beta) \in T_0^\alpha \\ 1 & \text{for} \quad h^{i-1}(\beta) \in T_1^\alpha \\ * & \text{for} \quad h^{i-1}(\beta) \in \{\frac{\alpha}{2}, \frac{\alpha+1}{2}\} \end{cases}$$

The itinerary of α itself, $\hat{\alpha} = I^\alpha(\alpha)$, is called the *kneading sequence* of α. In rough form, our main ideas can be stated as follows.

1. When the Julia set J_c of $p_c(z) = z^2 + c$ is locally connected and c has external angle α then J_c is the quotient space of T obtained by identification of points with equal itineraries.

2. If the boundary of the Mandelbrot set is locally connected, it is the quotient space of T obtained by identification of points with equal kneading sequences.

3. If we confine ourselves to α with $\hat{\alpha} = I^\alpha(1-\alpha)$, we obtain all itineraries and kneading sequences of real unimodal maps [5, 12].

In the present paper, we shall work out the first point and its consequences: the $\overline{T_i^\alpha}$ form a Markov partition in the tree-like (non-hyperbolic) case, the branching points of the Julia set can be read from $\hat{\alpha}$ (sec. 6-8) and renormalization can be expressed by substitution of words (sec. 11).

We describe those J_c which are locally connected but the abstract theory is more general, and in some respect more beautiful than the reality of complex polynomials. To *each* angle α on T we construct an abstract Julia set as a quotient space of T. We shall distinguish three cases: the 'tree-like' case that $\hat{\alpha}$ is not periodic, the case where α is periodic under h, and the 'Siegel disk' case where α is not periodic but $\hat{\alpha}$ is.

We prove uniqueness of J in all three cases. Beside the fact that a critical point in J cannot be periodic, we only assume that J is obtained as a quotient of T by a homotopic process in the plane (external rays, cf. sec. 2). This topological condition is crucial for using laminations. In the tree-like case our assumption is a bit weaker (see below). Differentiability is not required in the present paper. Let us note that every continuous, orientation-preserving two-to-one map h' on T with a single fixed point is conjugate to h, so that our topological methods will work for h' as well as for h.

There exist locally disconnected J_c [6, 3, 11] but by the recent remarkable results of Yoccoz (cf. Hubbard [8]) these examples are rare exceptions. It seems that the question whether J_c is locally connected belongs to conformal geometry rather than topology. Roughly speaking, certain topological spaces are too complicated to become realized by the conformal mapping p_c.

To give an impression of the technique, we state a few definitions and results. For fixed α, two inverse branches of h can be defined as $l_i^\alpha : T \setminus \{\alpha\} \to T_i^\alpha$, $i = 0, 1$. A closed equivalence relation $\sim$ on T is said to be an *α-equivalence* if

(a) $\frac{\alpha}{2} \sim \frac{\alpha+1}{2}$

(b) $\beta \sim \gamma$ implies $h(\beta) \sim h(\gamma)$

(c) $\beta \sim \gamma$, $\beta, \gamma \neq \alpha$ implies $l_0^\alpha(\beta) \sim l_0^\alpha(\gamma)$ and $l_1^\alpha(\beta) \sim l_1^\alpha(\gamma)$.

For each α, there is a *minimal* α-equivalence $\sim_\alpha$ which corresponds to Thurston's minimal lamination. The *dynamical* α-equivalence $\approx_\alpha$ is given by the equality of α-itineraries, with $*$ used as a joker for both 0 and 1. We show (sec. 3,4) that for all α in T, the space $T/\approx$ is the invariant factor [1] of the one-sided shift space $\{0,1\}^\infty$, given by the generating relation $0\hat{\alpha} \sim 1\hat{\alpha}$. Let us say that an α-equivalence $\sim$ is *degenerate* if the equivalence class of $\frac{\alpha}{2}$ is periodic under the map $\tilde{h}$ induced by h on $T/\sim$. In sec. 7 we prove

Theorem 1. (Uniqueness of α-equivalences in the tree-like case)
For non-periodic $\hat{\alpha}$, there is only one non-degenerate α-equivalence. Thus minimal and dynamical α-equivalence coincide. If the point c belongs to J_c and has external angle α, then (J_c, p_c) is homeomorphic to $(T/\approx_\alpha, \tilde{h})$.

For the Siegel disk case (sec. 9), the dynamical equivalence will collaps certain Cantor sets and the minimal equivalence will turn them into circles. The latter yields the proper topology. In the periodic case (sec. 10), both equivalences coincide and collapse certain Cantor sets. These will turn into circles when instead of (a), α is identified with a well-defined "conjugate point" β.

To give an idea on how branching points of J_c are connected with $\hat{\alpha}$, consider the fixed points of p_c which correspond to itineraries $\overline{1} = 111\ldots$ and $\overline{0}$. The first one is always an endpoint of J_c while the second is a branching point with $k+1$ branches if $\hat{\alpha}$ starts with 0^k1. In fig. 1, $\hat{\alpha}$ begins with $(001)^3$ but not with $(001)^4$, which implies the existence of branching points with four branches, associated to the sequence $\overline{001}$.

Remark of the first author: in November 1987, six weeks before his death, I met Professor Horst Michel in a curriculum committee and showed him my first rather vague ideas on these questions. In his kind manner, he became interested, gave hints and said he would certainly like to read a written outline. Let me express my gratitude for his encouragement which has contributed to [1, 2] and the present paper.

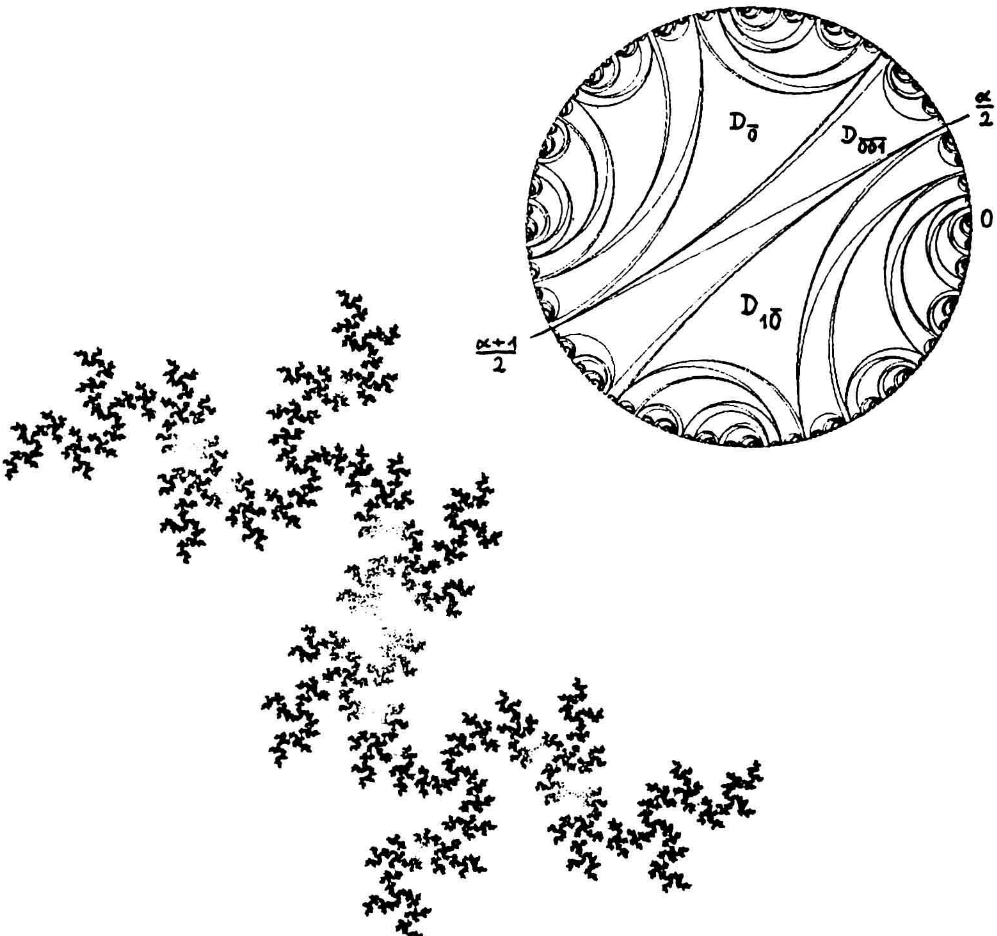

Fig. 1. A Julia set and the corresponding lamination ($\alpha = 0.143$)

2 Parametrization of locally connected Julia sets

We recall some well-known facts [3, 11] concerning Julia sets of $p_c(z) = z^2+c$ in a topological version. For $c, z \in C$, let $O_c(z) = \{z, p_c(z), p_c^2(z), \ldots\}$ denote the forward orbit of z, and call $K_c = \{z \in C |\, O_c(z) \text{ is bounded}\}$ the filled-in Julia set. The boundary J_c of K_c is said to be the Julia set of p_c. $K_0 = D$ is the unit disk.

If J_c is connected, there is a unique conformal isomorphism $\Phi_c : C \setminus K_c \to C \setminus D$ with $\lim_{z\to\infty} \Phi_c(z)/z = 1$ which conjugates p_c and p_0, i.e. $\Phi_c p_c \Phi_c^{-1} = p_0$.

Let us define a potential $u_c(z) = \Phi_c(z)$ on $C \setminus K_c$ with field lines

$$\beta_c = \{z \in C \setminus K_c |\, \arg(\Phi_c(z) = 2\pi\beta\}$$

([6], p.65). Then the mapping $h(\beta) = 2\beta \bmod 1$ fulfils

$$p_c(\beta_c) = (h(\beta))_c \quad \text{and} \quad -\beta_c = ((\beta + \tfrac{1}{2}) \bmod 1)_c \quad \text{for all} \quad \beta \in [0, 1[.$$

According to Caratheodory's theorem, each field line β_c has a continuous extension to a unique point z_β of J_c , and each point of J_c is obtained in this way, if and only if J_c is locally connected. If the field line β_c ends in $z \in J_c$, β is called an external angle of z.

Thus in the locally connected case $\varphi_c(\beta) = z_{\beta \bmod 1}$ and $\varphi_c^-(\beta) = \varphi_c(-\beta)$ are two parametrizations $\varphi : R \to J_c$ of J_c . They are continuous and periodic with minimal period one, and they fulfil the equations

$$\varphi(2\beta) = \varphi(\beta)^2 + c \quad \text{and} \quad -\varphi(\beta) = \varphi(\beta + \tfrac{1}{2}) \ , \ \beta \in R. \tag{1}$$

Proposition 2. Let $c \in C$. Then J_c is locally connected iff the functional equations (1) have a continuous periodic solution. In this case, $J_c = \varphi(R)$.
Moreover, every continuous solution of (1) with minimal period 1 coincides with either φ_c or φ_c^- .

Proof. Since the first equation has no constant solution for $c \neq 0$, we can assume φ is a continuous solution of (1) with minimal period q. We show that $\tilde{\varphi}(\beta) = \varphi(\beta/q)$ agrees with φ_c or φ_c^- . The set of all rationals with odd denominator is dense in R, and all points of $\tilde{\varphi}(A)$ are periodic under p_c , hence contained in J_c , with a finite number of exceptions. Since $\tilde{\varphi}(R)$ is connected and infinite, this shows $\tilde{\varphi}(R) \subseteq J_c$.

Moreover, if $z \in \tilde{\varphi}(R)$, then by (1) the points of $p_c^{-1}(z)$ – and hence the limit points of the backward orbit of z – also belong to $\tilde{\varphi}(R)$. Thus $\tilde{\varphi}(R) = J_c$, and J_c is locally connected.

Now take $\beta_0 \in R$ such that $\{2^n\beta_0 \bmod 1 | n \in N\}$ is dense in $[0, 1[$. There is an $r \in R$ with $\varphi_c(r\beta_0) = \tilde{\varphi}(\beta_0)$, hence $\varphi_c(r\beta) = \tilde{\varphi}(\beta)$ for $\beta = 2^n\beta_0$ and then for all $\beta \in R$. Since φ_c and $\tilde{\varphi}$ have the same minimal period, $r = \pm 1$. ∎

Thus, from a topological point of view, a locally connected J_c is a factor space of the circle compatible with the angle-doubling function h [6, 7]. Now suppose 0 and hence c belongs to the locally connected Julia set J_c , and α is an external angle of c. Since 0 is the only preimage of c under p_c and $\varphi_c(\alpha) = c$, we obtain $\varphi_c(\frac{\alpha}{2}) = \varphi_c(\frac{\alpha+1}{2}) = 0$. Each other point $\varphi_c(\beta)$ has exactly two preimages under p_c, and it is easy to check that the equivalence relation $\beta \approx \gamma$ if $\varphi_c(\beta) = \varphi_c(\gamma)$ is an α-equivalence.

On the other hand, each α-equivalence $\sim$ defines a factor space J of T and a map $\tilde{h} : J \to J$ such that the projection $\varphi : T \to J$ is a semiconjugacy from h to $\tilde{h}$ (i.e. $\varphi h = \tilde{h}\varphi$). Moreover, $\tilde{h}$ has two inverse branches $\tilde{l}_0^\alpha$ and $\tilde{l}_1^\alpha$ defined on the whole set J: just let $\tilde{l}_i^\alpha(\varphi(\alpha))$ be the equivalence class of $\frac{\alpha}{2}$ for $i = 0, 1$. Clearly, J is Hausdorff iff $\sim$ is closed [9]. We say $\sim$ is non-degenerate if the equivalence class of $\frac{\alpha}{2}$ is not periodic under $\tilde{h}$. An α-equivalence associated with the Julia set of some p_c must be non-degenerate: if 0 has a periodic orbit, this orbit is superstable. For each α, the smallest α-equivalence (the intersection of all α-equivalences in $T \times T$) will be called $\sim_\alpha$.

3 Invariant factors of shift spaces

In [1] we introduced a concept related to α-equivalences: factor spaces A of the one-sided shift space $\{0, 1, \ldots, m\}^\infty$ with mappings semiconjugate to shift maps. Here we shall only need a special case.

Let $\{0,1\}^* = \bigcup_{n=0}^\infty \{0,1\}^n$ be the set of 0-1-words $w = w_1 w_2 \ldots w_n$, and $\{0,1\}^\infty$ the set of one-sided sequences $s = s_1 s_2 \ldots$. Let λ denote the empty word, $|w|$ the length of w, ws and w^k the concatenation, $\overline{w} = www\ldots$ the periodic sequence and $s_{|n} = s_1 \ldots s_n$ the initial subword of s with length n for $w \in \{0,1\}^*$, $s \in \{0,1\}^\infty$. On $\{0,1\}^\infty$ we have the left shift $\sigma(s_1 s_2 \ldots) = s_2 s_3 \ldots$ and right shift maps τ_0, τ_1 defined by $\tau_i(s_1 s_2 \ldots) = i s_1 s_2 \ldots$.

An equivalence relation $\sim$ on $\{0,1\}^\infty$ is said to be *invariant* (strongly invariant in [1]) if for all $s, t \in \{0,1\}^\infty$

(a) $s \sim t$ implies $\sigma(s) \sim \sigma(t)$

(b) $s \sim t$ implies $\tau_0(s) \sim \tau_0(t)$ and $\tau_1(s) \sim \tau_1(t)$.

If $\sim$ is closed, the compact Hausdorff space $F = \{0,1\}^\infty / \sim$ is called an invariant factor. On F there are continuous maps $\tilde{\sigma}([s]) = [\sigma(s)]$ and $\tilde{\tau}_i([s]) = [\tau_i(s)]$, $i = 0, 1$. Conversely, a given compact Hausdorff space A is (homeomorphic to) an invariant factor iff there is a continuous $\tilde{\sigma} : A \to A$ with exactly two inverse branches $\tilde{\tau}_0, \tilde{\tau}_1$ (that is, $\tilde{\sigma} \cdot \tilde{\tau}_0 = \tilde{\sigma} \cdot \tilde{\tau}_1 = id_A$ and $A = \tilde{\tau}_0(A) \cup \tilde{\tau}_1(A)$), such that

$$\bigcap_{n=1}^{\infty} \tilde{\tau}_{s_1} \cdot \tilde{\tau}_{s_2} \cdot \ldots \cdot \tilde{\tau}_{s_n}(A) \quad \text{is a singleton} \quad \psi(s) \quad \text{for each} \quad s \in \{0,1\}^\infty. \tag{2}$$

If (2) is true, then $s \sim t$ iff $\psi(s) = \psi(t)$, and ψ is the projection onto the factor space.

Example. Let $A = [0,1]$ and $\tilde{\sigma}$ the tent map, $\tilde{\sigma}(x) = 2x$ for $0 \le x \le \frac{1}{2}$ and $\tilde{\sigma}(x) = 2(1-x)$ for $\frac{1}{2} \le x \le 1$. Condition (2) is fulfilled for $\tilde{\tau}_0(x) = \frac{x}{2}$, $\tilde{\tau}_1(x) = 1 - \frac{x}{2}$. The fixed point of $\tilde{\tau}_0$ is $\psi(\overline{0}) = 0$. So $01\overline{0}$ and $11\overline{0}$ are the two sequences assigned to the critical point $\frac{1}{2}$ since $\tilde{\sigma}^2(\frac{1}{2}) = 0$. In fact $\sim$ is the smallest invariant equivalence relation which identifies $01\overline{0}$ and $11\overline{0}$. Since $k(x) = 2\cos \pi x$ is a conjugacy from $\tilde{\sigma}$ to p_{-2}, this invariant factor can also be considered as Julia set $J_{-2} = [-2, 2]$.

If $A = \tilde{\tau}_0(A) \cup \tilde{\tau}_1(A)$ is an invariant factor, the points $x \in \tilde{\tau}_0(A) \cap \tilde{\tau}_1(A)$ will be called critical points since $\tilde{\sigma}(x)$ has no other preimage than x. We are interested in factors with a single critical point. These spaces are dendrites (simply connected Peano continua) and hence embeddable into the plane [9]. Note that all locally connected and connected J_c with $J_c = K_c$ belong to this class.

We give a construction for such factors. For $s \in \{0,1\}^\infty$, let $\sim_s$ be the smallest closed invariant equivalence relation containing the identification $0s \sim 1s$ and $F(s)$ the corresponding factor. If s is not periodic, $\sim_s$ is algebraically generated by the invariance condition (b):

$$r \sim_s t \text{ iff } r = t \text{ or there is a word } w \text{ with } r = w0s, t = w1s \text{ or } r = w1s, t = w0s.$$

This relation is closed since the sets $U_n = \{(r,t) |\ s_i = t_i \text{ for } i = 1,\dots,n\}$ form a neighbourhood base of the diagonal in $\{0,1\}^\infty \times \{0,1\}^\infty$, and only finitely many non-trivial equivalence classes intersect the complement of U_n.

A closed invariant $\sim$ generated by a single equation $t \sim wt$ looks more complicated. Condition (a) implies $t \sim w^n t$ for $n = 1,2,\dots$ and $t \sim \overline{w}$. Moreover, if $t = \overline{v}$ is periodic, all ut with $u \in \{v,w\}^*$ and hence $\{v,w\}^\infty$ belong to the class of t.

Thus for $s = \overline{u0}$ as well as for $s = \overline{u1}$, the equivalence class of s with respect to $\sim_s$ contains $\bullet_s = \{0u, 1u\}^\infty$. If $|0u| = |1u|$ is the minimal period of s, then $\sigma^k(\bullet_s) \cap \bullet_s = \emptyset$ unless k is a multiple of $|0u|$, and $\bullet_s$ is a full equivalence class. If s has smaller period, the class is larger. For example, $u = \lambda, u = 000$ and $u = 1$ all yield the trivial relation $\bullet_s = \{0,1\}^\infty$.

Theorem 3. (Classification of topologically self-similar dendrites with two pieces)

(a) A compact space A is an invariant factor with a single critical point iff $A = F(s)$ for some $s \in \{0,1\}^\infty$.

(b) The non-trivial equivalence classes of $\sim_s$ are the sets $w\bullet_s, w \in \{0,1\}^*$, where $\bullet_s = \{0s, 1s\}$ for non-periodic s and $\bullet_s = \{0u, 1u\}^\infty$ for periodic s with minimal period $u0$ or $u1$.

(c) $\sim_s$ has finite equivalence classes iff s is not periodic.

(d) For all $s,t \in \{0,1\}^* \cup \{0,1\}^\infty$ we have $\sim_s \subset \sim_t$ iff $\bullet_s \subset \bullet_t$.

Proof. For the remaining part of (a), suppose A is an invariant factor with one critical point $\{x\} = \tilde{r}_0(A) \cap \tilde{r}_1(A)$, and $x = \varphi(0wr) = \varphi(1wt)$ with $w = w_1 \dots w_n$ and $r_1 \neq t_1$. The σ-invariance implies $r \sim t$, so $\varphi(r) = \varphi(t) = x$. As above, $0wr \sim r \sim t \sim 1wt$ yields an equivalence class $\bullet$ containing $\{0w, 1w\}^\infty$.

Now we claim that $\bullet \neq \{0w, 1w\}^\infty$ implies the existence of some $k < n$ with $\bullet \supseteq \{0w_1 \dots w_k, 1w_1 \dots w_k\}^\infty$. Indeed, a sequence in $\bullet \setminus \{0w, 1w\}^\infty$ can be written as $ujw_1 \dots w_k v$, where $u \in \{0w, 1w\}^*, j \in \{0,1\}, k < n$ and $v \in \{0,1\}^\infty, v_1 \neq w_{k+1}$. Since $ujw\overline{0w}$ also belongs to $\bullet$, we conclude $v \sim w_{k+1} \dots w_n \overline{0w}$ by σ-invariance. Since we have only one critical point, these two sequences belong to $\bullet$. Now $jw_1 \dots w_k \bullet \subseteq \bullet$ since v and $jw_1 \dots w_k v$ are in $\bullet$, and $(1-j)w_1 \dots w_k \bullet \subseteq \bullet$ because $w_{k+1} \dots w_n \overline{0w}$ and $(1-j)w\overline{0w}$ are in $\bullet$. The claim is proved.

When we apply the above conclusion finitely often, we either end with $\bullet = \{0,1\}^\infty$, or with $\bullet = \{0w', 1w'\}^\infty$, where w' is a subword of w. This proves (a), and the other assertions follow easily. ∎

4 Itineraries and kneading sequences

Let X be a topological space, $f: X \to X$ a continuous map and $\mathcal{P} = \{P_0, P_1, \ldots\}$ a partition of X. The *symbolic dynamics* of a point x in X with respect to f and $\mathcal{P}$ is the sequence $I(x) = s_1 s_2 \ldots$ with $s_i = k$ iff $f^{i-1}(x) \in P_k$. This is an old idea. The binary representation $b(\beta) = b_1 b_2 \ldots$ of $\beta \in T$, for instance (with $\ldots\overline{1}$ excluded), is the dynamics of β with respect to h and $\mathcal{P} = \{[0, \frac{1}{2}[, [\frac{1}{2}, 1[\}$. Our $I^0(\beta)$ (sec. 1) is obtained from $b(\beta)$ by writing $\overline{*}$ for $\overline{0}$ and $w\overline{*}$ for $w1\overline{0}$.

For the topologist it is somewhat disgusting that I is not a continuous map unless $\mathcal{P}$ consists of open-and-closed sets. However, in our case $\mathcal{P}$ contains the open semi-circles T_0^α and T_1^α which both become closed when we add the rest, $P_*^\alpha = \{\frac{\alpha}{2}, \frac{\alpha+1}{2}\}$. Using $*$ as a joker for both 0 and 1, and replacing the shift space by an invariant factor, we shall succeed in making I^α continuous.

We start with some simple remarks. The n-th coordinates of $I^\alpha(\beta)$ and $I^0(\beta)$ are different iff $h^{n-1}(\beta)$ lies in $[0, \frac{\alpha}{2}]$ or $[\frac{1}{2}, \frac{\alpha+1}{2}]$. Thus we can calculate itineraries directly from the binary representation:

$$I^\alpha(\beta) = s_1 s_2 \ldots \quad \text{with} \quad s_i = \begin{cases} b_i(\beta) & for \quad \sigma^i(b(\alpha)) > b(\beta) \\ 1 - b_i(\beta) & for \quad \sigma^i(b(\alpha)) < b(\beta) \\ * & for \quad \sigma^i(b(\alpha)) = b(\beta) \end{cases} \tag{3}$$

The *kneading sequence* $\hat{\alpha} = I^\alpha(\alpha)$ always starts with 0. Since $I^\alpha(\beta) = I^{1-\alpha}(1-\beta)$ and in particular $\hat{\alpha} = \widehat{1-\alpha}$ for all α, β, we *assume throughout that* $\alpha \le \frac{1}{2}$.

A point $\beta \in T$ is periodic under h iff it is $0 = 1$ or rational with odd denominator: β has period p if we can write $\beta = m/(2^p - 1)$. Periodic points have periodic itineraries. The converse is true, except for one case (see proposition 6.2). $\hat{\alpha}$ is periodic iff it contains $*$.

β is called *preperiodic* under h if some $h^n(\beta) = h^{n+p}(\beta)$ for some minimal n, p but β is not periodic. These are the rationals with even denominator, $\beta = m/2^n(2^p - 1)$. Preperiodic points have preperiodic itineraries, and a preperiodic α has preperiodic $\hat{\alpha}$. (To see that $\hat{\alpha}$ is not periodic, write $b(\alpha) = b_1 \ldots b_k \overline{b_{k+1} \ldots b_{k+p}}$ with $b_k \ne b_{k+p}$. By (3), $\hat{\alpha}(k) \ne \hat{\alpha}(k+p)$.)

For $\beta \ne \alpha$ let $l_i^\alpha(\beta)$ be the point in $T_i^\alpha \cap \{\beta/2, (\beta+1)/2\}$, $i = 0, 1$. Now take a word $w \in \{0,1\}^n$. The mapping $l_w^\alpha = l_{w_1}^\alpha \cdot \ldots \cdot l_{w_n}^\alpha$ is defined and continuous on the set of $\beta \in T$ with $h^i(\alpha) \ne \beta, i = 0, \ldots, n-1$. It is easy to see that $T_w^\alpha = l_w^\alpha(T)$ is the set of all β such that the itinerary $I^\alpha(\beta)$ starts with w. This is a finite union of open intervals, with total length 2^{-n}. The itineraries of the endpoints of the intervals are obtained from w by replacing one or more w_i by $*$. If we define

$$C_w^\alpha = \{\beta \in T \mid I^\alpha(\beta)(i) \in \{w_i, *\} \text{ for } i = 1, \ldots, n\},$$

then $C_w^\alpha \supseteq \overline{T_w^\alpha}$. Moreover, equality holds unless α is periodic. (If a point $\beta \in C_w^\alpha$ is not in T_w^α and neither a right nor a left endpoint of some interval of T_w^α, there must exist two different integers $i, j \le n$ with $h^i(\beta) = h^j(\beta) = \alpha$.)

Now let $t \in \{0,1\}^\infty$, and write $t_{|n}$ for $t_1 \ldots t_n$. By compactness,

$$C_t^\alpha := \{\beta \mid I^\alpha(\beta)(i) \in \{t_i, *\} \text{ for } i = 1, 2, \ldots\} = \bigcap_{n=1}^\infty C_{t_{|n}}^\alpha \supseteq \bigcap_{n=1}^\infty \overline{T_{t_{|n}}^\alpha} \ne \emptyset.$$

There are points with itinerary t, maybe with some t_i replaced by $*$. However, the $*$ in an itinerary has to be followed by $\hat{\alpha}$, so there are points with proper itinerary t unless t has the form $w\hat{\alpha}$.

Let us define what we call the dynamical α-equivalence on T. If β,γ are points such that for each i, either $I^\alpha(\beta)(i) = I^\alpha(\gamma)(i)$ or $I^\alpha(\beta)(i) = *$ or $I^\alpha(\gamma)(i) = *$, then β and γ should be equivalent. Let $\approx_\alpha$ denote the smallest closed equivalence relation with this property.

If $\hat{\alpha}$ is non-periodic, then $\beta \approx_\alpha \gamma$ iff either $I^\alpha(\beta) = I^\alpha(\gamma)$ or $I^\alpha(\beta) = wu\hat{\alpha}$ and $I^\alpha(\gamma) = wv\hat{\alpha}$ for some $w \in \{0,1\}^*$ and $u,v \in \{0,1,*\}$. If $\hat{\alpha} = \overline{u*}$, $\overline{u0}$ or $\overline{u1}$ for some word u, all points with itineraries in $w\{0u, 1u, *u\}^\infty$ are identified for each w.

Theorem 4. (Invariant factors of the circle are invariant factors of shift space)

(a) For each α in T , the spaces $T/\approx_\alpha$ and $F(\hat{\alpha})$ are homeomorphic, and the homeomorphism is a semiconjugacy from $\tilde{h}$ and $\tilde{l}_i^\alpha$ to $\tilde{\sigma}$ and $\tilde{\tau}_i$.

(b) $\approx_\alpha$ is degenerate iff $\hat{\alpha}$ is periodic.

Proof. We saw that I^α can be considered as a map from T onto $F(\hat{\alpha})$, with $\beta \approx_\alpha \gamma$ iff the itineraries represent the same point of $F(\hat{\alpha})$. We show that I^α is continuous, hence a quotient map. Basic neighbourhoods of $I^\alpha(\gamma)$ are given by fixing a finite number of coordinates of $I^\alpha(\gamma)$ which are $\neq *$. For all β sufficiently near to γ, the itineraries of β and γ will agree in these coordinates. Observing that $I^\alpha(h(\beta)) = \sigma(I^\alpha(\beta))$ and $I^\alpha(l_i^\alpha(\beta)) = \tau_i(I^\alpha(\beta))$ we finish the proof of (a). If $\hat{\alpha} = \overline{u*}$ then $\bullet$ is invariant under $\tilde{\sigma}^{|u*|}$ while for non-periodic $\hat{\alpha}$, the image of $\bullet = \{0\hat{\alpha}, 1\hat{\alpha}, *\hat{\alpha}\}$ under σ^n does not contain a point of $\bullet$ for any n. ∎

Remark. Since we proved that all 0-1-sequences are itineraries with respect to every α (provided $*$ is used as a joker) we should also mention that only few sequences are kneading sequences. Some 0-1-words, as 010011, cannot be initial subwords of any $\hat{\alpha}$. In fact, $\hat{\alpha}$ starts with 01 iff $2\alpha > \frac{\alpha+1}{2}$. Now since $2 > 4\alpha > \alpha + 1$, the next digits 001 imply $4\alpha < \frac{\alpha+3}{2}$, $8\alpha > 3 + \frac{\alpha}{2}$ and $16\alpha - 6 \in]\frac{\alpha+1}{2}, 2\alpha[$. This means 32α mod 1 is in $]\alpha, \frac{\alpha+1}{2}[$ so that the sixth letter of $\hat{\alpha}$ must be zero.

5 Invariant laminations

Thurston [15] uses a more geometric approach. He considers T as the boundary of the unit disc D. A *lamination of the disc* is a set $\mathcal{S}$ of chords of T such that $\cup\mathcal{S}$ is closed in D, and that any two of these chords do not intersect except at their endpoints (cf. fig. 1). Points of T are considered as degenerate chords. A *gap* of $\mathcal{S}$ is the closure of a component of $D \setminus \cup\mathcal{S}$. For any chord or gap $S \in \mathcal{S}$ let $h(S) = \text{conv}\ h(S \cap T)$, where conv means convex hull. The lamination $\mathcal{S}$ is called *invariant* with respect to h if [14]

- For each chord S in $\mathcal{S}$, the image chord $h(S)$ and the opposite chord $-S$ belong to $\mathcal{S}$, and there is a chord S' with $h(S') = S$.

Obviously, every S will have two preimage chords so that the conditions are almost the same as in the definition of α-equivalence. Let us fix a *non-periodic* α and construct an invariant lamination containing the chord $S_* = \text{conv}\ \{\frac{\alpha}{2}, \frac{\alpha+1}{2}\}$. Let w be a 0-1-word.

Besides $C_w = C_w^\alpha = \overline{T_w^\alpha}$ we now consider $D_w = \text{conv } C_w$, and we define S_{w*} to be the chord in D connecting $l_w(\frac{\alpha}{2})$ and $l_w(\frac{\alpha+1}{2})$. Then $S_{w*} = D_{w0} \cap D_{w1}$ and $D_w = D_{w0} \cup D_{w1}$, which can be proved by induction on $|w|$.

Let $\mathcal{S}^\alpha$ consist of the S_{w*}, $w \in \{0,1\}^*$, and of their limit chords. It is not difficult to see that $\mathcal{S}^\alpha$ is the smallest invariant lamination containing S_* [15], prop. II.4.5. Moreover, $\mathcal{S}^\alpha$ defines the minimal α-equivalence: $\beta \sim_\alpha \gamma$ iff β and γ can be joined through a finite number of chords from $\mathcal{S}^\alpha$. (An elementary argument shows that this defines a closed relation.)

This lamination is tightly connected to our itineraries. By construction, each gap is obtained as $D_s^\alpha := \text{conv } C_s = \bigcap_{n=1}^\infty D_{s_{|n}}$ for some $s \in \{0,1\}^\infty$. Moreover, each chord in $\mathcal{S}^\alpha$ which does not bound a gap also coincides with some D_s. Thus the family of all gaps of $\mathcal{S}^\alpha$, and of all chords not contained in gaps, coincides with the family conv C_s, $s \in \{0,1\}^\infty$.

Let S be a chord, a the length of the subtended arc. Note that $a \leq \frac{1}{2}$ since T has perimeter 1. The length a' of the arc subtended by $h(S)$ is given by the tent map: $a' = 2a$ for $a \leq \frac{1}{4}$ and $a' = 1 - 2a$ for $a \geq \frac{1}{4}$. This implies a simple but important fact ([15], II.5.1):

Lemma 5.1 Among all chords $h^i(S)$, $i = 1, 2, \ldots$ the first chord longer than S coincides with the first chord which lies between S and $-S$. ■

The following result of Thurston [15] is a combinatorial analogue of Sullivan's celebrated theorem on the non-existence of wandering domains [14]. We give a new proof.

Theorem 5.2 (Thurston's structure theorem for quadratic laminations)

(a) On (T, h) there are no wandering triangles: If $\alpha, \beta, \gamma \in T$, then some of the sets $\Delta_k = \text{conv } \{h^k(\alpha), h^k(\beta), h^k(\gamma)\}$, $k = 0, 1, 2, \ldots$ will intersect each other, or there exist n such that Δ_k collapses to a chord for $k > n$.

(b) If G is a gap in an invariant lamination, there exists an integer $n \geq 0$ such that either

- $h^n(G)$ is periodic: $h^n(G) = h^{n+p}(G)$, or
- $h^n(G)$ is a triangle with a diameter of T as side, or a rectangle with a diameter of T as diagonal.

Proof. First we note that (b) follows from (a). Note that h maps each gap onto a gap or a chord. Given a gap G, take $\alpha, \beta, \gamma \in G \cap T$. If Δ_n and Δ_{n+p} intersect then $h^n(G) = h^{n+p}(G)$. The same is true if $h^n(G)$ and $h^{n+p}(G)$ contain a diameter in their interior. However, if some Δ_k collapses, then $h^n(G)$ contains a diameter for some $n < k$, and if $h^n(G)$ is not a triangle or rectangle, $h^{n+1}(G)$ is still a gap.

To prove (a), let $c \geq b \geq a$ denote the arc lengths of the sides of a triangle on T. Either $c = b + a$ or $a + b + c = 1$, but any two triangles from the latter case will intersect. Thus we can assume $a + b = c < \frac{1}{2}$. Moreover, there are at most three disjoint triangles with $c \geq \frac{1}{3} > b$. So we find n_0 such that for $k > n_0$, either $a \leq b \leq c \leq \frac{1}{3}$ or $a < \frac{1}{3} \leq b \leq c$.

Let $c' \geq b' \geq a'$ denote the lengths of the sides of the image triangle. In the first case, $a' = 2a$, $b' = 2b$, $c' = 2c$. In the second case $1 - 2c \leq 1 - 2b$ and $a = c - b < \frac{1}{2} - b$