BOUNDARY VALUE PROBLEMS AND PARTIAL DIFFERENTIAL EQUATIONS

sixth edition

DAVID L. POWERS

AP

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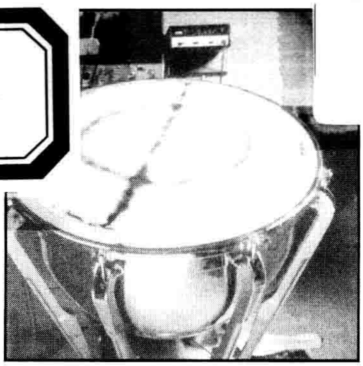

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SIXTH EDITION

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The cover photo shows a kettle drum with a dark pattern on it that is produced in the following way: A powder such as fine sand is sprinkled on the drumhead, and then the surface is vibrated by the transducer that can be seen in the upper left. If the frequency of vibration is a natural frequency of the membrane, there are *nodal curves*—curves and/or lines where the membrane is not moving. The powder collects on and near these curves. Different natural frequencies produce different nodal curves. In the background of the cover is another picture, where the nodal curve is a pair of crossed lines at right angles. The ideas of membrane vibrations and nodal curves are developed in Chapter 5, Section 5.7. See Figure 9 in that section and the associated animations on the website.

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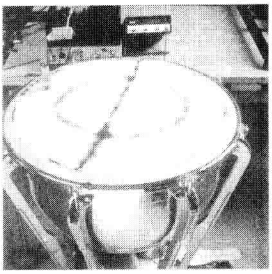
Preface



This text is designed for a one-semester or two-quarter course in partial differential equations given to third- and fourth-year students of engineering and science. It can also be used as the basis for an introductory course for graduate students. Mathematical prerequisites have been kept to a minimum—calculus and differential equations. Vector calculus is used for only one derivation, and necessary linear algebra is limited to determinants of order two. A reader needs enough background in physics to follow the derivations of the heat and wave equations.

The principal objective of the book is solving boundary value problems involving partial differential equations. Separation of variables receives the greatest attention because it is widely used in applications and because it provides a uniform method for solving important cases of the heat, wave, and potential equations. One technique is not enough, of course. D'Alembert's solution of the wave equation is developed in parallel with the series solution, and the distributed-source solution is constructed for the heat equation. In addition, there are chapters on Laplace transform techniques and on numerical methods.

The second objective is to tie together the mathematics developed and the student's physical intuition. This is accomplished by deriving the mathematical model in a number of cases, by using physical reasoning in the mathematical development, by interpreting mathematical results in physical terms, and by studying the heat, wave, and potential equations separately.

In the service of both objectives, there are many fully worked examples and now about 900 exercises, including miscellaneous exercises at the end of each chapter. The level of difficulty ranges from drill and verification of details to development of new material. Answers to odd-numbered exercises are in the back of the book. An Instructor's Manual is available with the answers to the even-numbered problems. A Student Solutions Manual which contains detailed solutions of odd-numbered problems can be found online at:

http://www.elsevierdirect.com/companions/9780123747198.

There are many ways of choosing and arranging topics from the book to provide an interesting and meaningful course. The following sections form the core, requiring at least 14 hours of lecture: Sections 1.1–1.3, 2.1–2.5, 3.1–3.3, 4.1–4.3 and 4.5. These cover the basics of Fourier series and the solutions of heat, wave, and potential equations in finite regions. My choice for the next most important block of material is the Fourier integral and the solution of problems on unbounded regions: Sections 1.9, 2.10–2.12, 3.6 and 4.4. These require at least six more lectures.

The tastes of the instructor and the needs of the audience will govern the choice of further material. A rather theoretical flavor results from including: Sections 1.4–1.7 on convergence of Fourier series; Sections 2.7–2.9 on Sturm-Liouville problems, and the sequel, Section 3.4; and the more difficult parts of Chapter 5, Sections 5.5–5.10 on Bessel functions and Legendre polynomials. On the other hand, inclusion of numerical methods in Sections 1.8, 3.5 and Chapter 7 gives a very applied flavor.

Chapter 0 reviews solution techniques and theory of ordinary differential equations and boundary value problems. Equilibrium forms of the heat and wave equations are derived also. This material belongs in an elementary differential equations course and is strictly optional. However, many students have either forgotten it or never seen it.

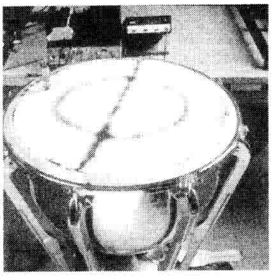
For this sixth edition, I have again made changes in response to students' changing needs and abilities. Many sections have been edited to improve clarity and make solution processes more explicit. Solving potential problems and sketching solutions of the wave equation using d'Alembert's solution get additional attention. In several sections, exercises have been reorganized more logically, old ones revised and new ones added. Many exercises are based on research publications from areas as diverse as chemical, civil, environmental, electrical, mechanical and rehabilitation engineering, physics and finance. I have noted that students' interest is piqued by these real-world problems; they tend to put in a lot of time and effort—and consequently gain greater understanding—in attempting to solve them. To capitalize on this tendency, the book provides over 30 "projects." These are extended problems, most based on research papers with real data, that require going beyond the text. No solutions are provided for projects.

To accompany this edition, the publisher is maintaining a web site with auxiliary materials: animations of convergence of Fourier series; animations of solutions of the heat and wave equations as well as ordinary initial value problems; color graphics of solutions of potential problems; additional exercises in a workbook style; review questions for each chapter. All files are readable with just a browser and Adobe Reader.

I wish to acknowledge these contributors: Cindy Smith, who was the LaTeX compositor and corrected many of my mistakes; Prof. Afshin Ghoreishi of Weber State University and Prof. Scott Fulton of Clarkson University, who provided lists of errata in previous editions; Prof. Joseph Skufca of Clarkson University for classroom experiments that are the basis of two projects; T. Mark Hightower of NASA Ames Research Center for another project; Academic Press editors and consultants; and reviewers for this edition.

David L. Powers

Contents



Ordinary Differential Equations

0.1 Homogeneous Linear Equations

The subject of most of this book is partial differential equations: their physical meaning, problems in which they appear, and their solutions. Our principal solution technique will involve separating a partial differential equation into ordinary differential equations. Therefore, we begin by reviewing some facts about ordinary differential equations and their solutions.

We are interested mainly in linear differential equations of first and second orders, as shown here:

$$\frac{du}{dt} = k(t)u + f(t), \tag{1}$$

$$\frac{d^2u}{dt^2} + k(t)\frac{du}{dt} + p(t)u = f(t). \tag{2}$$

In either equation, if $f(t)$ is 0, the equation is *homogeneous*. (Another test: if the constant function $u(t) \equiv 0$ is a solution, the equation is homogeneous.) In the rest of this section, we review homogeneous linear equations.

A. First-Order Equations

The most general first-order linear homogeneous equation has the form

$$\frac{du}{dt} = k(t)u. \tag{3}$$

doi:10.1016/B978-0-12-374719-8.00001-3.

This equation can be solved by isolating u on one side and then integrating:

$$\begin{aligned}\frac{1}{u}\frac{du}{dt} &= k(t)\\ \ln|u| &= \int k(t)dt + C\\ u(t) &= \pm e^C e^{\int k(t)dt} = ce^{\int k(t)dt}.\end{aligned} \tag{4}$$

It is easy to check directly that the last expression is a solution of the differential equation for any value of c. That is, c is an arbitrary constant and can be used to satisfy an initial condition if one has been specified.

Example.
Solve the homogeneous differential equation

$$\frac{du}{dt} = -tu.$$

The procedure outlined here gives the general solution

$$u(t) = ce^{-t^2/2}$$

for any c. If an initial condition such as $u(0) = 5$ is specified, then c must be chosen to satisfy it ($c = 5$). □

The most common case of this differential equation has $k(t) = k$ constant. The differential equation and its general solution are

$$\frac{du}{dt} = ku, \qquad u(t) = ce^{kt}. \tag{5}$$

If k is negative, then $u(t)$ approaches 0 as t increases. If k is positive, then $u(t)$ increases rapidly in magnitude with t. This kind of exponential growth often signals disaster in physical situations, as it cannot be sustained indefinitely.

B. Second-Order Equations

It is not possible to give a solution method for the general second-order linear homogeneous equation,

$$\frac{d^2u}{dt^2} + k(t)\frac{du}{dt} + p(t)u = 0. \tag{6}$$

Nevertheless, we can solve some important cases that we detail in what follows. The most important point in the general theory is the following.

Principle of Superposition. *If $u_1(t)$ and $u_2(t)$ are solutions of the same linear homogeneous equation (6), then so is any linear combination of them: $u(t) = c_1u_1(t) + c_2u_2(t)$.* □

This theorem, which is very easy to prove, merits the name of *principle* because it applies, with only superficial changes, to many other kinds of linear, homogeneous equations. Later, we will be using the same principle on partial differential equations. To be able to satisfy an unrestricted initial condition, we need two linearly independent solutions of a second-order equation. Two solutions are *linearly independent* on an interval if the only linear combination of them (with constant coefficients) that is identically 0 is the combination with 0 for its coefficients. There is an alternative test: Two solutions of the same linear homogeneous equation (6) are independent on an interval if and only if their *Wronskian*

$$W(u_1, u_2) = \begin{vmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{vmatrix} \tag{7}$$

is nonzero on that interval.

If we have two independent solutions $u_1(t)$, $u_2(t)$ of a linear second-order homogeneous equation, then the linear combination $u(t) = c_1u_1(t) + c_2u_2(t)$ is a general solution of the equation: given any initial conditions, c_1 and c_2 can be chosen so that $u(t)$ satisfies them.

1. Constant coefficients

The most important type of second-order linear differential equation that can be solved in closed form is the one with constant coefficients,

$$\frac{d^2u}{dt^2} + k\frac{du}{dt} + pu = 0 \quad (k, p \text{ are constants}). \tag{8}$$

There is always at least one solution of the form $u(t) = e^{mt}$ for an appropriate constant m. To find m, substitute the proposed solution into the differential equation, obtaining

$$m^2e^{mt} + kme^{mt} + pe^{mt} = 0,$$

or

$$m^2 + km + p = 0 \tag{9}$$

(since e^{mt} is never 0). This is called the *characteristic equation* of the differential equation (8). There are three cases for the roots of the characteristic equation (9), which determine the nature of the general solution of Eq. (8). These are summarized in Table 1.

This method of assuming an exponential form for the solution works for linear homogeneous equations of any order with constant coefficients. In all cases, a pair of complex conjugate roots $m = \alpha \pm i\beta$ leads to a pair of complex solutions

$$e^{\alpha t}e^{i\beta t}, \qquad e^{\alpha t}e^{-i\beta t} \tag{10}$$

Roots of Characteristic Equation	General Solution of Differential Equation
Real, distinct: $m_1 \neq m_2$	$u(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$
Real, double: $m_1 = m_2$	$u(t) = c_1 e^{m_1 t} + c_2 t e^{m_1 t}$
Conjugate complex: $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$	$u(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$

Table 1 Solutions of $\frac{d^2u}{dt^2} + k\frac{du}{dt} + pu = 0$

which can be traded for the pair of real solutions

$$e^{\alpha t} \cos(\beta t), \qquad e^{\alpha t} \sin(\beta t). \tag{11}$$

We include two important examples.

Example.
First, consider the differential equation

$$\frac{d^2u}{dt^2} + \lambda^2 u = 0 \tag{12}$$

where λ is constant. The characteristic equation is $m^2 + \lambda^2 = 0$, with roots $m = \pm i\lambda$. The third case of Table 1 applies if $\lambda \neq 0$; the general solution of the differential equation is

$$u(t) = c_1 \cos(\lambda t) + c_2 \sin(\lambda t). \tag{13}$$

□

Example.
Second, consider the similar differential equation

$$\frac{d^2u}{dt^2} - \lambda^2 u = 0. \tag{14}$$

The characteristic equation now is $m^2 - \lambda^2 = 0$, with roots $m = \pm\lambda$. If $\lambda \neq 0$, the first case of Table 1 applies, and the general solution is

$$u(t) = c_1 e^{\lambda t} + c_2 e^{-\lambda t}. \tag{15}$$

It is sometimes helpful to write the solution in another form. The hyperbolic sine and cosine are defined by

$$\sinh(A) = \frac{1}{2}\left(e^A - e^{-A}\right), \quad \cosh(A) = \frac{1}{2}\left(e^A + e^{-A}\right). \tag{16}$$

Thus, $\sinh(\lambda t)$ and $\cosh(\lambda t)$ are linear combinations of $e^{\lambda t}$ and $e^{-\lambda t}$. By the Principle of Superposition, they too are solutions of Eq. (14). The Wronskian test shows them to be independent. Therefore, we may equally well write

$$u(t) = c_1' \cosh(\lambda t) + c_2' \sinh(\lambda t)$$

as the general solution of Eq. (14), where c_1' and c_2' are arbitrary constants.

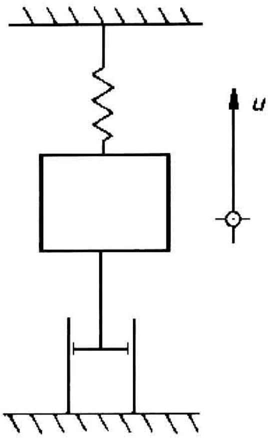


Figure 1 Mass-spring-damper system.

Example: Mass-spring-damper system.
The displacement of a mass in a mass-spring-damper system (Fig. 1) is described by the initial value problem

$$\frac{d^2u}{dt^2} + b\frac{du}{dt} + \omega^2 u = 0$$

$$u(0) = u_0, \quad \frac{du}{dt}(0) = v_0.$$

The equation is derived from Newton's second law. Coefficients b and ω^2 are proportional to characteristic constants of the damper and the spring, respectively. The characteristic equation of the differential equation is

$$m^2 + bm + \omega^2 = 0$$

with roots

$$\frac{-b \pm \sqrt{b^2 - 4\omega^2}}{2} = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - \omega^2}$$

The nature of the solution, and therefore the motion of the mass, is determined by the relation between $b/2$ and ω.

$b = 0$: *undamped.* The roots are $\pm i\omega$ and the general solution of the differential equation is

$$u(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

The mass oscillates forever.

$0 < b/2 < \omega$: *under-damped.* The roots are complex conjugates $\alpha \pm i\beta$ with $\alpha = -b/2$, $\beta = \sqrt{\omega^2 - (b/2)^2}$. The general solution of the differential equation is

$$u(t) = e^{-bt/2}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right).$$

The mass oscillates, but approaches equilibrium as t increases.

$b/2 = \omega$: *critically damped.* The roots are both equal to $b/2$. The general solution of the differential equation is

$$u(t) = e^{-bt/2}(c_1 + c_2 t).$$

The mass approaches equilibrium as t increases and may pass through equilibrium ($u(t)$ may change sign) at most once.

$b/2 > \omega$: *over-damped.* Both roots of the characteristic equation are real and negative, say m_1 and m_2. The general solution of the differential equation is

$$u(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}.$$

The mass approaches equilibrium as t increases and $u(t)$ may change sign at most once. These cases are illustrated on the website. □

2. Cauchy-Euler equation

One of the few equations with variable coefficients that can be solved in complete generality is the Cauchy-Euler equation:

$$t^2 \frac{d^2u}{dt^2} + kt\frac{du}{dt} + pu = 0. \tag{17}$$

The distinguishing feature of this equation is that the coefficient of the nth derivative is the nth power of t, multiplied by a constant. The style of solution for this equation is quite similar to the preceding: assume that a solution has the form $u(t) = t^m$, then find m. Substituting u in this form into Eq. (17) leads to

$$t^2 m(m-1)t^{m-2} + ktmt^{m-1} + pt^m = 0, \quad \text{or}$$

$$m(m-1) + km + p = 0 \quad (k,\, p \text{ are constants}). \tag{18}$$

This is the characteristic equation for Eq. (17), and the nature of its roots determines the solution as summarized in Table 2.

Example.

One important example of the Cauchy-Euler equation is

$$t^2 \frac{d^2u}{dt^2} + t\frac{du}{dt} - \lambda^2 u = 0 \tag{19}$$

where $\lambda > 0$. The characteristic equation is $m(m-1) + m - \lambda^2 = m^2 - \lambda^2 = 0$.

Roots of Characteristic Polynomial	General Solution of Differential Equation
Real, distinct roots: $m_1 \neq m_2$	$u(t) = c_1 t^{m_1} + c_2 t^{m_2}$
Real, double root: $m_1 = m_2$	$u(t) = c_1 t^{m_1} + c_2 (\ln t) t^{m_1}$
Conjugate complex roots: $m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta$	$u(t) = c_1 t^{\alpha} \cos(\beta \ln t) + c_2 t^{\alpha} \sin(\beta \ln t)$

Table 2 Solutions of $t^2 \frac{d^2u}{dt^2} + kt \frac{du}{dt} + pu = 0$

The roots are $m = \pm\lambda$, so the first case of Table 2 applies, and

$$u(t) = c_1 t^{\lambda} + c_2 t^{-\lambda} \tag{20}$$

is the general solution of Eq. (19). □

For the general linear equation

$$\frac{d^2u}{dt^2} + k(t)\frac{du}{dt} + p(t)u = 0,$$

any point where $k(t)$ or $p(t)$ fails to be continuous is a *singular point* of the differential equation. At such a point, solutions may break down in various ways. However, if t_0 is a singular point where both of the functions

$$(t - t_0)k(t) \text{ and } (t - t_0)^2 p(t) \tag{21}$$

have Taylor series expansions, then t_0 is called a *regular singular point.* The Cauchy-Euler equation is an example of an important differential equation having a regular singular point (at $t_0 = 0$). The behavior of its solution near that point provides a model for more general equations.

3. *Other equations*

Other second-order equations may be solved by power series, by change of variable to a kind already solved, or by sheer luck. For example, the equation

$$t^4 \frac{d^2u}{dt^2} + \lambda^2 u = 0, \tag{22}$$

which occurs in the theory of beams, can be solved by the change of variables

$$t = \frac{1}{z}, \qquad u(t) = \frac{1}{z}v(z).$$

Here are the details. The second derivative of u has to be replaced by its expression in terms of v, using the chain rule. Start by finding

$$\frac{du}{dt} = \frac{d}{dz}\left(\frac{v}{z}\right) \cdot \frac{dz}{dt}.$$

Since $t = 1/z$, also $z = 1/t$, and $dz/dt = -1/t^2 = -z^2$. Thus

$$\frac{du}{dt} = -z^2\left(\frac{zv' - v}{z^2}\right) = -zv' + v.$$

Similarly we find the second derivative

$$\frac{d^2u}{dt^2} = \frac{d}{dz}\left(\frac{du}{dt}\right)\frac{dz}{dt} = \frac{d}{dz}(-zv' + v)(-z^2)$$
$$= -z^2(-zv'' - v' + v') = z^3v''.$$

Finally, replace both terms of the differential equation:

$$\left(\frac{1}{z}\right)^4 z^3v'' + \lambda^2\frac{v}{z} = 0$$

or

$$v'' + \lambda^2 v = 0.$$

This equation is easily solved, and the solution of the original is then found by reversing the change of variables:

$$u(t) = t(c_1\cos(\lambda/t) + c_2\sin(\lambda/t)). \tag{23}$$

C. Second Independent Solution

Although it is not generally possible to solve a second-order linear homogeneous equation with variable coefficients, we can always find a second independent solution if one solution is known. This method is called *reduction of order.*

Suppose $u_1(t)$ is a solution of the general equation

$$\frac{d^2u}{dt^2} + k(t)\frac{du}{dt} + p(t)u = 0. \tag{24}$$

Assume that $u_2(t) = v(t)u_1(t)$ is a solution. We wish to find $v(t)$ so that u_2 is indeed a solution. However, $v(t)$ must not be constant, as that would not supply an independent solution. A straightforward substitution of $u_2 = vu_1$ into the differential equation leads to

$$v''u_1 + 2v'u_1' + vu_1'' + k(t)(v'u_1 + vu_1') + p(t)vu_1 = 0.$$

Now collect terms in the derivatives of v. The preceding equation becomes

$$u_1v'' + (2u_1' + k(t)u_1)v' + (u_1'' + k(t)u_1' + p(t)u_1)v = 0.$$

However, u_1 is a solution of Eq. (24), so the coefficient of v is 0. This leaves

$$u_1v'' + (2u_1' + k(t)u_1)v' = 0, \tag{25}$$

which is a first-order linear equation for v'. Thus, a nonconstant v can be found, at least in terms of some integrals.