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William L. Morris

AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

with difference equations, numerical methods, and applications

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Preface

This text contains an introductory treatment of certain kinds of functional equations, that is, equations in which the "unknown" is a function. The particular kinds of functional equations that are studied are known as *differential equations* and *difference equations*.

The text is intended to provide a reasonable survey of those topics which meet the needs of most students. For the engineering and science students, the book emphasizes various approaches to studying the classes of equations that occur most frequently in their mathematical models. For the mathematics majors, the book can be a source of numerous examples of abstract concepts that are in their immediate future. The choice of topics has been influenced by the effects of the computer revolution, as well as by the increased use of analytical methods in the behavioral sciences.

The presentation in the text presupposes that the reader is familiar with the main ideas treated in a beginning calculus course. In addition, some familiarity with the concepts of elementary linear algebra would be helpful, but this is not essential. Chapter 1 contains a summary of some of the more frequently used topics from algebra and calculus that are needed in the text. It is not expected that the reader have a complete understanding of all of the material in Chapter 1.

The main body of the text is contained in Chapters 2 through 10. The material in these chapters offers students an opportunity to review and reinforce all their prior mathematical training while learning many new and useful concepts. The results that are derived are stated as theorems and are justified by proofs. This is a time-honored manner of mathematical exposition that seems to have more advantages than disadvantages. All the results and methods developed in the text are useful in a wide variety of applications. In this regard, Chapter 11 contains a number of detailed illustrations of applications in engineering, science, and commerce.

The book can be used as the text for a number of one-semester courses, and there is enough material for a two-semester course. Chapter 1 is not designed for classroom presentation. It is intended to be used as a beginning reading assignment in order to introduce the symbolism and terminology within the text. The items in this chapter are referenced throughout the text, so that additional opportunities for discussion of the material in the chapter arise during a course. Chapters 2 and 3 form the core of the text. Each chapter begins with a brief statement that indicates its dependence on other chapters. It is not necessary, or even recommended, that a course consist of consecutively numbered chapters. Some illustrations of one-semester courses are:

(A) Chapters 2, 3, 10 (except Section 47), 7 and 8 can be used for a traditional course for engineering and science majors.
(B) Chapters 2, 3, 4, 5, and 6 can be used for a course that emphasizes computational mathematics.
(C) Chapters 2, 3, 4, 5, and 9 (except Section 42) can be used for a course in which applications of linear algebra are emphasized.

Although the text is separated into chapters by topic, the primary organizational feature is the section. The sections are numbered consecutively with the exception of Chapter 11, which consists of numbered examples. In each section, important items, that is, definitions, theorems, examples, and so on, are numbered consecutively, and these numbers are used for reference throughout the section.

We gratefully acknowledge the help that we have received in editing and class testing the material in this book. In particular, we are indebted to our colleagues Professors Richard Dowell Byrd and Tom Wannamaker. Finally, we are indebted to George J. Telecki and Eleanor Castellano at Harper & Row.

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PRELIMINARIES

Section 1 Introduction

The purpose of this chapter is to present, in summary form, a survey of the material which serves as the mathematical background for most of the main body of the text. This chapter is not necessarily intended for classroom presentation, but rather it, like Chapter 11, is recommended to the reader for frequent reference and review as he or she progresses through the other chapters. However, an initial cursory reading of this chapter is advised, since it provides an introduction to the symbolism and terminology which is used throughout the text.

A study of differential equations presupposes that the reader is familiar with the basic concepts and techniques of calculus. The most frequently used concepts from calculus are reviewed in Sections 2 and 6. The presentation also requires some knowledge of the theory of equations. The essential concepts in this area are covered in the brief review of complex numbers and polynomials that is contained in Section 3.

The major portion of this text deals with linear equations of various kinds. The unifying principles that seem to be most helpful in this regard are the concepts of linear algebra. Therefore, a brief survey of linear algebra and related topics is included in Sections 4 and 5. It is not essential that the reader be familiar with this material, as the approach taken in the beginning chapters of the text is to interpret the properties of differential equations in terms of the concepts of linear algebra. This approach will make more sense after reading Chapters 2 and 3.

Because of the usual limitations of space, the brief treatment of prerequisites in this chapter is bound to be inadequate, and so an occasional reference to calculus and linear algebra texts is certainly recommended.

Section 2 Functions and Calculus

Since some assumptions must be made, it is assumed that the reader is very familiar with the concept of a function. By and large our interest in this text will be in the type of functions which are familiar to you from calculus, namely, real-valued functions of a real variable. The first six chapters are devoted almost exclusively to two such types of functions, those whose domain is the set of real numbers on an interval, and those whose domain is the set of nonnegative integers. A function of the latter type is most often called a sequence. In addition to these familiar types of functions, however, we shall also be concerned with functions whose domain and range are vector spaces, especially vector spaces of functions. Such functions are usually called transformations, or operators, and our interest in functions of this type will be indicated in Section 4.

A basic definition connected with the function concept which will be required often in the work which follows is that of equality of functions. This notion is often overlooked in elementary calculus, and so we will state it here. Two functions f and g are *equal*, written $f = g$, if and only if they have the same domain, say X, and $f(x) = g(x)$ for all $x \in X$.

The set of all real numbers will be denoted by $\mathscr{R}$. If x and y are members of $\mathscr{R}$ and $x < y$, then the (open) interval consisting of all real numbers that are greater than x and less than y is denoted by (x, y). In more formal set notation,

$$(x, y) = \{r \in \mathscr{R}: x < r < y\}$$

The other types of (bounded, or finite) intervals are:

$$(x, y] = \{r \in \mathscr{R}: x < r \leq y\}$$

$$[x, y) = \{r \in \mathscr{R}: x \leq r < y\}$$

$$[x, y] = \{r \in \mathscr{R}: x \leq r \leq y\}$$

The last interval is said to be a closed interval. Similarly, the unbounded intervals are specified by using the symbols ∞ and $-\infty$. For example,

$$(-\infty, y) = \{r \in \mathscr{R}: r < y\}$$

is an unbounded open interval.

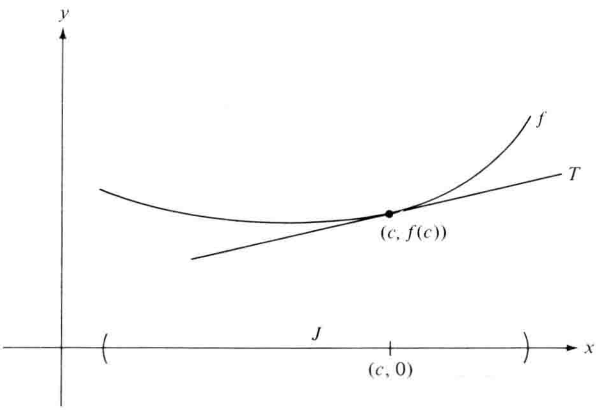
Let J be an open interval and let f be a function whose domain is J. If $c \in J$, then $\lim_{x \to c} f(x) = L$ means that given $\varepsilon > 0$, there is a number $\delta > 0$ such that if $x \in J$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

The function f is *continuous* at the point $c \in J$ if $\lim_{x \to c} f(x) = f(c)$, and f is *continuous* on J if f is continuous at each point of J.

The function f is *differentiable* at the point $c \in J$ if there is a number m, called the *derivative of f at c*, such that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = m$$

The derivative concept also has a useful geometric interpretation. In particular, if f is differentiable at the point $c \in J$, then the derivative of f at c is the slope of the line T which is tangent to the graph of f at the point $(c, f(c))$. See the figure here.



As with continuity, we say that f is *differentiable on J* if f is differentiable at each point of J. Functions which are differentiable on an open interval are studied in some detail in the sequel.

The relationship between continuity and differentiability is given by the following important result.

1 THEOREM

Let J be an open interval and let f be a function whose domain is J. If f is differentiable at the point $c \in J$, then f is continuous at the point c.

The converse of this theorem, namely, if f is continuous at the point $c \in J$, then f is differentiable at the point c, is false. The reader should be able to supply an example.

Of the many symbolic conventions which are used to denote the derivative of f at c, we prefer $Df(c)$. If f is differentiable on J, then we can determine the new function Df whose domain is J by calculating the derivative of f at each point of J. For example, if f is defined by $f(x) = x^2 + 3x - 1$ on the interval $(-\infty, \infty)$, then f is differentiable on this interval and $Df(x) = 2x + 3$ on $(-\infty, \infty)$.

Suppose that f is differentiable on the interval J. Then Df is a function whose domain is J, and it makes sense to inquire as to whether this function is continuous or differentiable. In particular, if Df is continuous, then f is said to be *continuously differentiable on J*. If Df is differentiable on J, then its derivative $D(Df)$ is called the *second derivative of f* and is denoted by D^2f. Thus, in the above example, Df is differentiable on $(-\infty, \infty)$ and D^2f is given by $D^2f(x) = 2$. Of course, we could continue successively in this manner and define the higher derivatives of f. Specifically, the nth derivative of f is denoted by D^nf, with the understanding that $f, Df, D^2f, \ldots, D^{n-1}f$ are each differentiable on the interval J, and that $D^nf = D(D^{n-1}f)$ on J. The function f is said to be *n-times differentiable on J* if D^nf is a function whose domain is J. Similarly, f is *n-times continuously differentiable on J* if D^nf is continuous on J. Finally, for convenience, we define $D^0f = f$ for each function f, and we make no distinction between D^1 and D.

It will become apparent as our work progresses that our interest is not in specific functions defined on some interval J, but rather we will be concerned with sets (collections, families) of functions defined on J. In particular, let J be an open interval. The set of all functions which are continuous on J will be denoted by $\mathscr{C}(J)$, and the set of all n-times continuously differentiable functions on J, n a positive integer, will be denoted by $\mathscr{C}^n(J)$. In addition, if f is n-times differentiable on J for each positive integer n, then f is said to be *infinitely differentiable on J*. Many of the functions which are studied in detail in calculus are infinitely differentiable on some open interval J. For example, the polynomials, the trig functions, $\sin(x)$ and $\cos(x)$, and the exponential function, $\exp(x)$, are each infinitely differentiable on the interval $(-\infty, \infty)$. The set of all infinitely differentiable functions on the interval J is denoted by $\mathscr{C}^\infty(J)$.

There is a relationship between the sets of functions defined above which can be obtained from Theorem 1. In particular, since a differentiable function on an interval J is continuous on J, it follows that $\mathscr{C}^n(J) \subset \mathscr{C}^{n-1}(J)$ for all positive integers $n \geq 2$, $\mathscr{C}^1(J) \subset \mathscr{C}(J)$, and $\mathscr{C}^\infty(J) \subset \mathscr{C}^n(J)$ for all positive integers n.

Members of $\mathscr{C}^n(J)$, n a positive integer, satisfy the hypotheses of Taylor's theorem. A statement of this theorem which is suitable for our purposes is as follows.

2 THEOREM

Taylor's theorem: If J is an open interval, $[a, b] \subset J$, and $f \in \mathscr{C}^n(J)$, then there exists a number $c \in (a, b)$ such that

$$\begin{aligned} f(b) &= f(a) + Df(a)(b - a) \\ &\quad + \frac{D^2f(a)}{2!}(b - a)^2 + \cdots + \frac{D^{n-1}f(a)}{(n-1)!}(b - a)^{n-1} \\ &\quad + \frac{D^nf(c)}{n!}(b - a)^n \end{aligned}$$

The special case $n = 1$ in the theorem yields the mean value theorem, which should be familiar to the reader as one of the most important and useful results in elementary calculus. With this observation, Taylor's theorem could also be called the *extended mean value theorem.*

Every constant function, that is, a function whose range consists of a single number, is a member of $\mathscr{C}^\infty(J)$ for every open interval J, and no symbolic distinction is made between a constant and a constant function. It is an easy consequence of the definition of the derivative and the mean value theorem that $Df(x) = 0$ for all x on an interval J if and only if f is a constant function on J. Equivalently, the functions F and G have the property $DF(x) = DG(x)$ for all $x \in J$ if and only if $G(x) = F(x) + c$ on J, where c is a constant.

The familiar rules concerning the derivative of the sum of two functions and the derivative of a constant times a function are:

$$D(f + g) = Df + Dg$$

and

$$D(cf) = c\,Df$$

where c is a constant. Each of the equations above expresses the equality of two functions and thus, by the definition, the equations mean

$$D(f + g)(x) = Df(x) + Dg(x)$$

and

$$D(cf)(x) = c\,Df(x)$$

for all x in J, where J is the domain of the functions f and g. The general versions of the rules above are:

(3) $$D^n(f + g) = D^n f + D^n g$$

(4) $$D^n(cf) = c\,D^n f$$

where it is assumed that each of f and g is n-times differentiable on some interval J, and where c is a constant.

Since a complete and motivated definition of the definite integral of a function is a lengthy process, we assume that the reader has some familiarity with the meaning of

$$\int_a^b f(t)\,dt$$

Recall that the variable t in this expression has no special significance, and that $\int_a^b f(t)\,dt$, $\int_a^b f(u)\,du$, $\int_a^b f(s)\,ds$, and so forth, all denote the same number.

The important relationship between D and $\int$ is expressed in the following theorem.

5 THEOREM

Let J be an open interval. If $f \in \mathscr{C}(J)$, $a \in J$, and

$$F(x) = \int_a^x f(t)\,dt, \qquad x \in J$$

then

$$F \in \mathscr{C}^1(J) \qquad \text{and} \qquad DF = f$$

This theorem provides a means of obtaining a function F such that $DF = f$. As noted above, if G is any other function whose derivative is f, then G differs from F by a constant; that is, $G(x) = F(x) + c$ for some constant c. Equivalently, the collection of all functions of the form $F(x) + c$, where c is any constant, represents all functions whose derivative is f. The collection of functions $F(x) + c$ is often called a *one-parameter family*. A consequence of Theorem 5 and these observations is the fundamental theorem of calculus.

6 THEOREM

Let J be an open interval and let $f \in \mathscr{C}(J)$. If $a, b \in J$, $a < b$, and if G is any function such that $DG = f$ on J, then

$$\int_a^b f(t)\,dt = G(b) - G(a)$$

For the most part, the function F of Theorem 5 has no better representation than that given. In special cases, of course, there might be more convenient representations of F. For example, if $f(x) = \cos x$ on $(-\infty, \infty)$, then

$$F_1(x) = \int_0^x \cos(t)\,dt = \sin x$$

is one example of a function whose derivative is f. However, the function F_2 given by

$$F_2(x) = \int_1^x \cos(t)\,dt = \sin x - \sin 1$$

is also a function whose derivative is f. Of course, $F_2 - F_1$ is a constant. In contrast to the function $f(x) = \cos x$, consider the function $g(x) = \sin x^2$. The function G given by

$$G(x) = \int_0^x \sin(t^2)\,dt$$

has no other convenient representation. The reader is urged to try some "methods of integration" to find another representation of G. Thus, in addition to generating functions in $\mathscr{C}^1(J)$, Theorem 5 indicates that there are many more members in this set than those few functions studied in calculus.

The following properties of $\int$ are derived from corresponding properties of D:

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

An additional property of $\int$, which follows either from Theorem 6 or from the geometric interpretation of the integral of f as being the area bounded by the graph of f and the x-axis, is:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

for any $a, b, c \in J$.

We assume that the reader is familiar with the fact that the concepts of continuity, differentiation, and integration, as discussed briefly here, can be extended to functions of more than one variable. Since this text requires only a limited amount of background information from the multivariable calculus, we will not go through a corresponding review of the basic concepts for functions of several variables. One concept which will arise, however, is that of a partial derivative of a function of several variables, and so we give an appropriate definition and the notation. Let f be a function of two variables, say x and y, whose domain is some region A in the x-y plane. Let $(x_0, y_0) \in A$. Then the *partial derivative of f with respect to x at the point* (x_0, y_0) is given by

$$\lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

provided this limit exists. Similarly, *the partial derivative of f with respect to y at the point* (x_0, y_0) is given by

$$\lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

provided this limit exists. The function f is *differentiable on A* if each of the partial derivatives of f exists at each point $(x, y) \in A$. If f is differentiable on A, then the notation

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

is used to denote the partial derivatives of f with respect to x and y, respectively. For additional information concerning multivariable calculus, the reader is urged to consult a calculus text.

The preceding discussion is a brief summary of some of the basic concepts and principal results of calculus. We shall assume throughout the remainder of this text that the reader is familiar with this basic material. We conclude this section by indicating some additional concepts and facts from calculus which might not be as familiar as those above, but which will be useful in work which follows.

Let J be an interval and let f be a function whose domain is J. Then f is *bounded on J* if there exists a positive number M such that

$$|f(x)| \leq M$$

for all $x \in J$. The following theorem relates the concepts of continuity and boundedness.

7 THEOREM

If f is continuous on the finite, closed interval $J = [a, b]$, then f is bounded on J.

The familiar rule for differentiating the product of two functions is:

$$D(fg) = f\,Dg + g\,Df$$

If f and g are elements of $\mathscr{C}^2(J)$, then by differentiating this equation and using (3), we have

$$\begin{aligned} D^2(fg) = D[D(fg)] &= D[f\,Dg] + D[g\,Df] \\ &= f\,D^2g + 2(Df)(Dg) + g\,D^2f \end{aligned}$$

The reader should recognize that this result is analogous to the expansion

$$(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$$

Proceeding by induction, we can obtain a formula for the nth derivative of the product $f \cdot g$. This formula is the analog of the binomial theorem, that is, the expansion of $(\alpha + \beta)^n$, and it is known as *Leibnitz's rule.*

8 THEOREM

Leibnitz's rule: Let J be an open interval. If $f, g \in \mathscr{C}^n(J)$, then

$$D^n(fg) = \sum_{i=0}^{n} \binom{n}{i} (D^i f)(D^{n-i} g)$$

The symbol

$$\binom{n}{i}$$

is a *binomial number*, and it is defined by

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

Also, by our convention indicated previously, $D^0 f = f$.

It is often necessary to evaluate a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}, \qquad a \in \mathscr{R}, \qquad \text{or} \qquad a = \pm\infty$$

where either

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$$

or

$$\lim_{x \to a} f(x) = \infty = \lim_{x \to a} g(x)$$

Such limits are called *indeterminate forms* of type $0/0$ or ∞/∞.

The following theorem, known as *L'Hôpital's rule*, provides a method for treating limits of quotients having an indeterminate form.

9 THEOREM

L'Hôpital's rule: Let f and g be differentiable functions. If

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$$

and if

$$\lim_{x \to a} \frac{Df(x)}{Dg(x)} = L$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L$$

This conclusion also holds for the case

$$\lim_{x \to a} f(x) = \infty = \lim_{x \to a} g(x)$$

---

10 Examples

a. The limit

$$\lim_{x \to 0} \frac{\sin x}{x}$$

has the indeterminate form 0/0. Since $D[\sin x] = \cos x$, $D[x] = 1$, and

$$\lim_{x \to 0} \frac{D[\sin x]}{D[x]} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

we have, by L'Hôpital's rule,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

b. Consider

$$\lim_{x \to \infty} \frac{x^2}{e^x}$$

This limit has the indeterminate form ∞/∞. The derivatives of x^2 and e^x are $2x$ and e^x, respectively, and

$$\lim_{x \to \infty} \frac{2x}{e^x}$$