PROCEEDINGS OF THE 1980 GUANGZHOU CONFERENCE ON THEORETICAL PARTICLE PHYSICS

VOLUME TWO

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Generalized WKB Method and Vacuum Wave Functionals*

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ABSTRACT

We study the WKB method in systems with many (finite or infinite) degrees of freedom. Our method provides an intuitive understanding of the Euclidean instanton solutions. Using the concept of "Most Probable Escape Paths" introduced by Banks Bender and Wu, we are able to construct the WKB ground state wave functional for an arbitrary field theory. We illustrate the construction of ground state wave functionals in a simple example. We also describe briefly the possible generalization of our WKB method to finite temperatures.

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1. WKB Method

One dimensional WKB method is well-known, and may be found in any standard quantum mechanics text book. In the following, I shall discuss WKB method applied to higher dimensions.[1] Let us begin with a two dimensional system,

$$H = \frac{1}{2m} (p_x^{\,2} + p_y^{\,2}) + V(x,y) \quad . \tag{1.1}$$

From (1.1), we can write down a Schrödinger equation

$$\left[- \frac{h^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + V(x,y) \right] \psi(x,y) = E\ \psi(x,y) \quad . \tag{1.2}$$

In WKB approximation, we expand $\ell n\ \psi$ (the exponents of ψ) in powers of $\hbar$:

$$\psi = A\ e^{iS/\hbar} \quad , \quad S = S_0 + \hbar S_1 + \ldots \quad . \tag{1.3}$$

Equating coefficients of $\hbar$, we can rewrite (1.2) as

$$\frac{1}{2m} (\nabla S_0)^2 + V(x,y) = E \quad , \tag{1.4}$$

$$-i\nabla^2 S_0 + 2\vec{\nabla} S_0 \cdot \vec{\nabla} S_1 = 0 \quad , \text{ etc.} \tag{1.5}$$

Equation (1.4) is identical to the Hamilton-Jacobi equation in classical mechanics.[2] For every solution to (1.4), we can construct a family of classical trajectories. The construction is analogous to the ray-representation in optics: The classical trajectories are the "rays" which are orthogonal to the "wave fronts" $S_0 = \text{const.}$ (Fig. 1). Along a ray, the change of phase S_0 is stationary,

$$\delta S_0 = \delta \int_1^2 ds \sqrt{2m(E - V)} = 0 \quad . \tag{1.6}$$

It is easy to see that a ray obeys the equation for a classical trajectory.

In the tunneling region,

$$V(x,y) > E \ , \tag{1.7}$$

we can expand ψ as a damping exponential,

$$\psi = A\ e^{-R/\hbar} \ , \quad R = R_0 + R_1\hbar + . . . \quad . \tag{1.8}$$

We can obtain the equations for R's. In particular, R_0 obeys,

$$-\frac{1}{2m}(\nabla R_0)^2 + V(x,y) = E \ . \tag{1.9}$$

In analogy to the classical trajectories, we can construct paths orthogonal to the family of curves R_0 = constant. Along such a path,

$$R_0 = \int_1^2 ds\sqrt{2m(V-E)} \tag{1.10}$$

is stationary and is often a minimum. These minimal-R_0 paths are called "the most probable escape (tunneling) paths" (MPEP).[3] We can show that a most probably escape path obeys the Euclidean equation of motion[4]

$$m\frac{d^2x}{d\tau^2} = \frac{\partial V}{\partial x} \ , \quad m\frac{d^2y}{d\tau^2} = \frac{\partial V}{\partial y} \quad . \tag{1.11}$$

Knowing the tunneling path, we can obtain WKB tunneling amplitude between two minima 1 and 2. It is given by

$$\text{Tunneling amplitude} = e^{-R_0/\hbar} \tag{1.12}$$

with

$$R_0 = \left[\int_1^2 ds\sqrt{2m(V-E)}\right]_{MPEP} \quad . \tag{1.13}$$

We can generalize our method easily to a system with n degrees of freedom. Let the system be

$$L = \frac{1}{2} m\dot{\vec{r}}^2 - V(\vec{r}) \tag{1.14}$$

with

$$\vec{r} = (x_1, x_2, \ldots, x_n) \quad . \tag{1.15}$$

We can describe an arbitrary tunneling path between $\vec{r}_1$ and $\vec{r}_2$ by a one-parameter set of points

$$x_i = x_i\ (\tau), \quad \tau_1 < \tau < \tau_2 \tag{1.16}$$

such that

$$\vec{r}(\tau_1) = \vec{r}_1 \ , \quad \vec{r}(\tau_2) = \vec{r}_2 \quad . \tag{1.17}$$

The condition that this path be a MPEP is for the path to obey Euclidean equation of motion[4]

$$m \frac{d^2\vec{r}}{d\tau^2} = \frac{\partial V}{\partial \vec{r}} \quad , \tag{1.18}$$

where τ is an appropriate parametrization.

2. Ground State Wave Functions.

We can use the WKB method to compute the ground state wave function for a system with finite degrees of freedom, or the ground state wave functional for a field theory. Let $\vec{r}_0$ be the location of the classical ground state, and E_0 be the energy of the classical ground state. Then, the ground state WKB wave function is

$$\psi(\vec{r}) = A\, e^{-R_0/\hbar} \tag{2.1}$$

with

$$R_0(\vec{r}) = \left[\int_{\vec{r}_o}^{\vec{r}} ds \sqrt{2m(V - E_0)} \right]_{MPEP} . \tag{2.2}$$

It is easy to see that we are always in the tunneling region when we compute the ground state wave function $\psi(\vec{r})$.

Let us generalize our result to a scalar field theory

$$L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - U(\phi) \tag{2.3}$$

$$H = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + U(\phi) \tag{2.4}$$

where $\Pi \equiv \dot{\phi}$ is the momentum conjugate to $\phi(x)$. A path joining two configurations ϕ_1 and ϕ_2 is a family of field configurations $\phi(\vec{x},\tau)$ such that

$$\phi(\vec{x},\tau_1) = \phi_1(\vec{x}) \quad , \quad \phi(\vec{x},\tau_2) = \phi_2(\vec{x}) \quad . \tag{2.5}$$

In analog to a MPEP in n-dimensional space, we can construct a MPEP in the function space. It is possible to show that a MPEP, $\phi(\vec{x},\tau)$, obeys the Euclidean field equation,

$$\left(\frac{\partial^2}{\partial\tau^2} + \nabla^2 \right) \phi(\vec{x},\tau) - \frac{\partial U}{\partial\phi} = 0 . \tag{2.6}$$

An Euclidean solution to a classical field theory connecting two inequivalent ground states (ϕ_1, ϕ_2) is known as an instanton. In WKB language, instantons are most probably tunneling paths joining inequivalent classical ground states.

Along a MPEP, the problem becomes one-dimensional. The effective Lagrangian is given by

$$L_{eff} = \frac{1}{2}\, m(\tau)\, \dot{\tau}^2 - V(\tau) \tag{2.7}$$

with

$$m(\tau) \equiv \int d^3x \left(\frac{\partial\phi}{\partial\tau}\right)^2 \tag{2.8}$$

$$V(\tau) \equiv \int d^3x \, [\tfrac{1}{2}\, (\nabla\phi)^2 + U(\phi)] \; . \tag{2.9}$$

The WKB exponent is

$$R = \int d\tau \sqrt{2m(\tau)\,(V(\tau) - E)} \quad . \tag{2.10}$$

To obtain WKB wave functional $\psi(\phi)$ around a classical ground state ϕ_0 with classical ground state energy E_0, we construct a MPEP connecting ϕ_0 to ϕ and have

$$\psi(\phi) = \text{const.}\ \exp\left[-\int_{\phi_0}^{\phi} d\tau \sqrt{2m(\tau)\,(V(\tau) - E_0)}\right]_{MPEP} , \tag{2.11}$$

where $m(\tau)$ and $V(\tau)$ are given in (2.8), (2.9). Equation (2.11) provides the probability amplitude for finding the configuration $\phi(\vec{x})$ in the ground state. The method can be extended to non-abelian gauge theories as well.

Let us work out an explicit example. Consider a free scalar field

$$L = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \quad . \tag{2.12}$$

The classical ground state is at $\phi = 0$ with $E_0 = 0$. A MPEP obeys

$$\left(- \frac{\partial^2}{\partial \tau^2} - \nabla^2 \right) \phi + m^2 \phi = 0 \quad . \tag{2.13}$$

The MPEP joining $\phi = 0$ at $\tau = -\infty$ to an arbitrary $\phi(\vec{x})$ at $\tau = 0$ is

$$\phi(\vec{x},\tau) = \int \frac{d^3k}{(2\pi)^3} \; e^{\sqrt{k^2 + m^2}\,\tau} \; e^{i\vec{k}\cdot\vec{x}} \; \tilde{\phi}(\vec{k}) \tag{2.14}$$

where $\tilde{\phi}(\vec{k})$ is the Fourier transform of $\phi(\vec{x})$. Along the MPEP $\phi(\vec{x},\tau)$, we have

$$m(\tau) = 2\,V(\tau) = \int \frac{d^3k}{(2\pi)^3} \; (k^2 + m^2) \; e^{2\sqrt{k^2 + m^2}\,\tau} \; (\tilde{\phi}(k))^2 \quad . \tag{2.15}$$

Hence, the WKB wave functional is $\psi(\phi) = e^{-R_0}$ with

$$\begin{aligned} R_0 &= \int_{-\infty}^{0} d\tau \; \sqrt{2m(\tau)V(\tau)} \\ &= \int \frac{d^3k}{(2\pi)^3\, 2\,\omega} \; (k^2 + m^2)(\tilde{\phi}(k))^2 \end{aligned} \tag{2.16}$$

where

$$\omega \equiv \sqrt{k^2 + m^2} \; . \tag{2.17}$$

Equation (2.16) shows that field configurations with large ϕ or $\nabla\phi$ are suppressed.

3. MPEP's at Finite Temperatures

We shall now discuss tunnelings at a finite temperature.[5,6,7] In particular, we shall generalize the concept of MPEP to finite temperatures. Consider a system of n degrees of freedom as described in Eq. (1.14). Let E_n and $\psi_n(\vec{r})$ be energy and wave function of a typical eigenstate n. The statistical weight for the system to be at state n is given by the Boltzmann factor $e^{-\beta E_n}$. The probability of finding a particle at $\vec{r}$ is proportional to

$$\rho(\vec{r}) = \sum_n e^{-\beta E_n} \left|\psi_n(\vec{r})\right|^2 \quad . \tag{3.1}$$

Using WKB approximation, we can compute ψ_n for a given n. For $E > V_0$, the spatial region is divided into a classically allowed region $(E > V)$ and a tunneling region $(V > E)$. We should compute the classical trajectory in the classically allowed region and the MPEP in the tunneling region, and match the solutions at the interface. However, this will make the calculation rather difficult. In the following, we shall make further simplifications:

(1) We assume that we can replace $\sum_n$ by an integral over the continuous variable E.

(2) For a given E, the probability distribution function in the classically allowed region is given by the Boltzmann factor $e^{-\beta E}$ times the phase space. The probability distribution function in the tunneling region is given by $e^{-\beta E}$ times the WKB damping factor along the MPEP. The MPEP begins at a point $\vec{r}(E)$ at the boundary of the tunneling region and ends at $\vec{r}$. Hence, we have

$$\rho_E = e^{-\beta E} \exp\left[-2\int_{\vec{r}(E)}^{\vec{r}} ds\sqrt{2m(V-E)}\right]_{MPEP} \quad . \tag{3.2}$$