Problem Book in High-School Mathematics

Edited by A.I. Prilepko, D.Sc.

Mir Publishers

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СБОРНИК ЗАДАЧ ПО МАТЕМАТИКЕ

для поступающих в вузы

Под редакцией профессора
А. И. ПРИЛЕПКО

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Preface

The present problem book is meant for high-school students who intend to enter technical colleges. It contains more than two thousand problems and examples covering all divisions of high-school mathematics.

The main aim of the book is to help students to revise their school knowledge of mathematics and develop a technique in solving a variety of problems.

The book consists of nine chapters divided into sections, each of which deals with a certain theme. The problems on a definite theme are arranged in the order of increasing difficulty, which makes it possible for a student to gradually acquire the necessary techniques and experience in problem solving. Thus, the problems are classified as far as possible. Most of the problems were given at the entrance examinations in various colleges to the USSR in recent years. All the problems are supplied with answers, and some of them with solutions or instructions. The words "Solution" and "Hint" are replaced by the signs ▲ and ● respectively. The list of designations makes the use of the book more convenient.

All the contributors to the book have a long experience as lecturers at preparatory courses of colleges, as teachers at high schools specializing in physics and mathematics and as examiners in mathematics.

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The authors

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На английском языке

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List of Designations Accepted in the Book

$\mathbb{N}$, the set of all natural numbers
$\mathbb{Z}_0$, the set of all nonnegative integers
$\mathbb{Z}$, the set of all integers
$\mathbb{Q}$, the set of all rational numbers
$\mathbb{R}$, the set of all real numbers
$[a; b]$, a closed interval from a to b, $a < b$
$(a; b)$, an open interval from a to b, $a < b$
$(a; b]$, $[a; b)$, semi-open intervals from a to b, $a < b$
$(a; \infty)$, $[a; \infty)$, $(-\infty; b]$, $(-\infty; b)$, infinite intervals, rays of a number line
$(-\infty; \infty)$, an infinite interval, a number line
$\Rightarrow$, sign of implication
$\Leftrightarrow$, sign of equivalence
$\in$, sign of membership relation
$n \in \mathbb{N}$, the number n belongs to the set of natural numbers
$\subset$, sign of inclusion
$C \subset D$, the set C is included into the set D, or C is a subset of D
$\cup$, sign of union
$C \cup D$, union of the sets C and D
$(a - \varepsilon; a + \varepsilon) - \varepsilon$, the neighbourhood of the point a
$\{a; b; \ldots\}$, a set consisting of the elements $a, b, \ldots$
$(a; b)$, an ordered pair
$(a; b; c)$, an ordered triple
$n!$, an n-factorial, the product of the first n natural numbers $(1! = 1)$
$[x]$, the integral part of the number x
$\{x\}$, the fractional part of the number x
$|x|$, the modulus (absolute value) of the number x
(x_n), (a_n), an infinite number sequence

$\lim_{n\to\infty} x_n = a$, the number a is the limit of the sequence (x_n)
$f(x)$, the value of the function f at the point x
$D(f)$, the domain of definition of the function f
$E(f)$, the range of the function f
Δx, an increment of the variable x
$\Delta f(x_0)$, Δf, an increment of the function f at the point x_0
$\lim_{x\to a} f(x) = b$, the number b is a limit of the function f as x tends to a
$f'(x_0)$, the derivative of the function f at the point x_0
log, decimal logarithm
ln, natural logarithm (logarithm to the base e)
$\max_{[a;\,b]} f$, the greatest value of the function f on the interval $[a;\,b]$
$\min_{[a;\,b]} f$, the least value of the function f on the interval $[a;\,b]$
$\int f(x)\,dx$, the general form of the antiderivatives of the function $f(x)$
$\int_a^b f(x)\,dx$, the integral of the function f in the limits from a to b
$A \in \Phi$, the point A belongs to the figure Φ
$A \notin \Phi$, the point A does not belong to the figure Φ
$\Phi_1 \cap \Phi_2$, the intersection of the figures Φ_1 and Φ_2
$\varnothing$, an empty set
$\Phi_1 \cong \Phi_2$, the figures Φ_1 and Φ_2 are congruent (equal)
$\Phi_1 \sim \Phi_2$, the figures Φ_1 and Φ_2 are similar
$\uparrow\uparrow$ $(\downarrow\uparrow)$, similarly (oppositely) directed
$\parallel$, parallel
$\perp$, perpendicular
$\angle$, an angle, a dihedral angle, a trihedral angle
$\wedge$, the magnitude (degree, measure) of the angle
$\widehat{(a,\,b)}$, the magnitude of the angle between straight lines
$\widehat{(a,\,\alpha)}$, the magnitude of the angle between a line and a plane
$\widehat{(\alpha,\,\beta)}$, the magnitude of the angle between planes
$(\ B)$, a straight line AB
$[AB]$, a segment AB

$[AB)$, a ray AB
$|AB|$, the length of the segment AB
$\mathbf{a}$, $\overrightarrow{AB}$, a vector
$\mathbf{0}$, $\overrightarrow{AA}$, a zero vector
$\widehat{(\mathbf{a}, \mathbf{b})}$, the magnitude of the angle between two vectors
$|\mathbf{a}| = a$, $|\overrightarrow{AB}| = |AB|$, the length of a vector
$(\mathbf{i}, \mathbf{j}, \mathbf{k})$ an orthogonal basis
$\mathbf{a} = (x, y, z)$, a vector with coordinates x, y, z
$\mathbf{a} \cdot \mathbf{b}$, $\overrightarrow{AB} \cdot \overrightarrow{CD}$, a scalar product of vectors

Chapter 1

RATIONAL EQUATIONS, INEQUALITIES AND FUNCTIONS IN ONE VARIABLE

1.1. Linear Equations and Inequalities in One Variable. A Linear Function

Solve the following equations:

1. $(3x + 7) - (2x + 5) = 3$.

2. $\frac{x}{2} + \frac{x}{6} + \frac{x}{12} + \frac{x}{20} + \frac{x}{30} + \frac{x}{42} = -6$.

3. (a) $3x + 1 = (4x - 3) - (x - 4)$;
(b) $3x + 1 = (4x - 3) - (x - 5)$.

4. $ax = a^2$. **5.** $(a - 2)\,x = a^2 - 4$.

6. $(a^2 - 9)\,x = a^3 + 27$.

Solve the following inequalities:

7. (a) $7x > 3$; (b) $-4x > 5$; (c) $5x + 6 \leqslant 3x - 8$;
(d) $x/2 + 1 \leqslant x/\sqrt{3} + 1/2$.

8. (a) $ax \leqslant 1$; (b) $ax > 1$.

Solve the following systems of inequalities:

9. (a) $\begin{cases} 3x > 1, \\ -x < 3; \end{cases}$ (b) $\begin{cases} 2x < \pi, \\ -x > -1.6; \end{cases}$

(c) $\begin{cases} x > -1, \\ 2x + 1 \leqslant 5; \end{cases}$ (b) $\begin{cases} 3x + 2 \geqslant 0, \\ x + \sqrt{5} < 0. \end{cases}$

10. Find all the values of $x \in N$ satisfying the inequality $5x - 7 < 2x + 8$.

Solve the following equations:

11. (a) $|x - 1| = 3$; (b) $|x + 2| = -1$.

12. (a) $|3 - x| = a$; (b) $|x - a| = 2$.

13. (a) $|x - 3| = x - 3$; (b) $|x - 3| = 3 - x$;
(c) $|x - 3| = x$.

14. $|2x - 1| = |x + 3|$.

15. $|x - a| = |x - 4|$.

16. (a) $|x-4|+|x+4|=9$; (b) $|x-4|+|x+4|=8$; (c) $|x-4|-|x+4|=8$; (d) $|x+4|-|x-4|=8$.

17. $|x-3|+|x+2|-|x-4|=3$.

18. At what values of a does the equation $|2x+3|+|2x-3|=ax+6$ possess more than two roots?

Solve the following equations:

19. (a) $|x|>a$; (b) $|x-1|>-1$; (c) $|x-1|>1$.

20. (a) $|x|<a$; (b) $|x+2|\leqslant -2$; (c) $|x+2|<2$.

21. (a) $|2x+1|>x$; (b) $|2x+3|\leqslant 4x$.

22. $|1-3x|-|x+2|\leqslant 2$.

23. $|x+2|+|x-3|>5$.

Solve the following inequalities:

24. $\begin{cases} |x|\geqslant x, \\ 2x-1>3. \end{cases}$ 25. $\begin{cases} |x|\leqslant -x, \\ |x+2|>1. \end{cases}$

26. At what value of a is the function $f(x)=(a-2)x+3a-4$, $x\in(-\infty;+\infty)$ (a) even; (b) odd?

27. At what value of a is the function $f(x)=(a+3)x+5a$, $x\in(-\infty;\infty)$ periodic?

28. Find the values of k for which the function $f(x)=(k-1)x+k^2-3$, $x\in(-\infty;\infty)$ (a) increases monotonically; (b) decreases monotonically.

29. Determine the values of m for which the function $y=(m^2-4)x+|m|$, $x\in(-\infty;\infty)$ has an inverse. Find the inverse function.

30. Given the linear function $f(x)$. Prove that the function $F(x)=f(f(x))$ is also linear.

Construct the graphs of the following functions:

31. (a) $y=2x$, (b) $y=-\frac{1}{3}x$.

32. (a) $y=x-2$; (b) $y=3-x$.

33. (a) $y=2x-1$; (b) $y=1-3x$.

34. (a) $y=|x-1|$; (b) $y=-|x+2|$.

35. (a) $y=|1+2x|$; (b) $y=-|-4x+2|$.

36. $y=||x-1|-2|$. 37. $y=|x+2|+|x-3|$.

38. $y = |2x+1| - |2x-2|$. 39. $y = x + x/|x|$.

40. $y = x + |x-1| + \frac{|x-2|}{x-2}$.

Construct the graphs of the following functions:

41. $\frac{y}{x+1} = -1$. 42. $|y| + x = -1$.

43. $|x| + |y| = 2$. 44. $|y-3| = |x-1|$.

Indicate the points on the plane xOy which satisfy the following inequalities:

45. $y > x$. 46.. $y < -x$.

47. $y \geqslant |x|$. 48. $x > |y|$.

Indicate the points on the plane xOy which satisfy the following equations:

49. $y + |y| - x - |x| = 0$.

50. $|x+y| + |x-y| = 4$.

51. $|y|x = x$. 52. $|x-y| + y = 0$.

53. Find the value of a for which the function y is continuous at the point $x = 0$, if

$$y = \begin{cases} 2x+1 & \text{for} \quad x \leqslant 0, \\ -x+a & \text{for} \quad x > 0. \end{cases}$$

54. Find the critical points of the function (a) $y = |3x+1|$; (b) $y = |x+1| + x + 1$; (c) $y = |x+1| + |x-1|$; (d) $y = |x-3| - |x+3|$.

Find the intervals of the monotonic increase and decrease of the following functions:

55. (a) $y = 3 - x$; (b) $y = \frac{1}{4}x + 1$.

56. (a) $y = 2 + |x-4|$; (b) $y = 3 - |x|$.

57. (a) $y = -(|x+10| + |x-10|)$; (b) $y = |x-4| - |x+5|$; (c) $y = |x+4| - |x+3| + |x+2| - |x+1| + |x|$.

Find the points of extremum of the following functions:

58. (a) $y = |2x-1|$; (b) $y = 2 - |3-4x|$.

59. (a) $y = |3x+2| + |2x-3|$; (b) $y = |x+7| - 2|x-2|$.

60. $y = 2\,|\,x-1\,| - 3\,|\,x+2\,| + x$.

61. $y = |\,x-2\,| + |\,x-a\,|$.

62. Find the values of a for which the function y possesses a maximum at the point $x = 2$, if

$$y = \begin{cases} x+1 & \text{for} \quad x < 2, \\ a & \text{for} \quad x = 2, \\ 5-x & \text{for} \quad x > 2. \end{cases}$$

63. Find the least and the greatest value of the function $y = |\,x-a\,|$ on the interval $[1;\ 2]$ $(a \neq 1;\ a \neq 2)$.

1.2. Quadratic Equations and Inequalities. A Quadratic Function

Solve the following equations and inequalities:

1. (a) $x^2 - 7x + 12 = 0$; (b) $-x^2 + 4x + 5 = 0$; (c) $6x^2 - 5x + 1 = 0$; (d) $3x^2 + 10x + 3 = 0$; (e) $x^2 - 2x - 5 = 0$; (f) $2x^2 + x - 8 = 0$.

2. (a) $x^2 - 3x - 4 > 0$; (b) $x^2 - 3x - 4 \leqslant 0$; (c) $x^2 + 4x + 4 > 0$; (d) $4x^2 + 4x + 1 \leqslant 0$; (e) $2x^2 - x + 5 > 0$; (f) $x^2 - x + 1 < 0$.

3. Find solutions to the following systems:

(a) $\begin{cases} 2x^2 - 5x + 2 = 0, \\ x - 2 < 0; \end{cases}$ (b) $\begin{cases} x^2 - 2x - 3 = 0, \\ x + 4 \geqslant 0; \end{cases}$

(c) $\begin{cases} x^2 - 9 \geqslant 0, \\ x - 4 < 0; \end{cases}$ (d) $\begin{cases} x^2 - 6x + 5 \geqslant 0, \\ x^2 - 25 \leqslant 0; \end{cases}$

(e) $\begin{cases} x^2 + 6x + 9 \leqslant 0, \\ 2x - 5 > 0; \end{cases}$ (f) $\begin{cases} x^2 + x + 8 < 0, \\ x^2 + 6x + 5 = 0; \end{cases}$

(g) $\begin{cases} |\,x-2\,| + |\,x-3\,| = 1, \\ 813x - 974 \leqslant 163x^2. \end{cases}$

4. Suppose x_1 and x_2 are roots of the equation $x^2 + x - 7 = 0$. Find (a) $x_1^2 + x_2^2$; (b) $x_1^3 + x_2^3$; (c) $x_1^4 + x_2^4$ without solving the equation.

5. Given the equation $ax^2 + bx + c = 0$. Prove that if x_1, x_2 and x_3 are pairwise distinct real roots of this equation, then $a = b = c = 0$.

6. At what values of a does the equation

$$(a^2 - 3a + 2)x^2 - (a^2 - 5a + 4)x + a - a^2 = 0$$

possess more than two roots?

7. The equation $x^2 + px + q = 0$, where $p \in \mathbb{Z}$, $q \in \mathbb{Z}$, has rational roots. Prove that those roots are integers.

8. Prove that the equation $x^2 + (2m + 1)x + 2n + 1 = 0$ does not possess any rational roots if $m \in \mathbb{Z}$, $n \in \mathbb{Z}$.

9. At what values of a does the equation $2x^2 - (a^3 + 8a - 1)x + a^2 - 4a = 0$ possess roots of opposite signs?

10. Find all the values of a for which the equation $x^2 - ax + 1 = 0$ does not possess any real roots.

11. At what values of k does the equation $x^2 + 2(k - 1)x + k + 5 = 0$ possess at least one positive root?

12. Find all the values of m for which both roots of the equation $2x^2 + mx + m^2 - 5 = 0$ (a) are less than 1; (b) exceed -1.

13. Find all the values of k for which one root of the equation $x^2 - (k + 1)x + k^2 + k - 8 = 0$ exceeds 2 and the other root is smaller than 2.

14. Suppose x_1 and x_2 are roots of the equation $x^2 + 2(k - 3)x + 9 = 0$ $(x_1 \neq x_2)$. At what values of k do the inequalities $-6 < x_1 < 1$ and $-6 < x_2 < 1$ hold true?

15. Find all the values of k for which one root of the equation $(k - 5)x^2 - 2kx + k - 4 = 0$ is smaller than 1 and the other root exceeds 2.

16. At what values of m is the inequality $mx^2 - 9mx + 5m + 1 > 0$ satisfied for any $x \in \mathbb{R}$?

17. Find all the values of m for which every solution of the inequality $1 \leqslant x \leqslant 2$ is a solution of the inequality $x^2 - mx + 1 < 0$.

Solve the following equations:

18. (a) $x^2 - |x| - 2 = 0$; (b) $x^2 + 5|x| + 4 = 0$.

19. (a) $2x^2 - |5x - 2| = 0$; (b) $x^2 - |x - 1| = 0$.

20. (a) $|x^2 + x - 6| = x^2 + x - 6$; (b) $|6x^2 - 5x + 1| = 5x - 6x^2 - 1$; (c) $|x^2 + x| = x^2 + x$; (d) $|x^2 - x + 5| = x - x^2 - 5$.

21. (a) $|x^2 - 1| = x + 3$; (b) $|x^2 - 1| = |x + 3|$.

22. (a) $|2x^2 - 1| = x^2 - 2x - 3$;
(b) $|2x^2 - 1| = |x^2 - 2x - 3|$.

23. $|x^2 - 3|x| + 2| = x^2 - 2x$.

Solve the following inequalities:

24. (a) $x^2 - |x| - 12 < 0$; (b) $x^2 + 2|x| - 15 \geqslant 0$;
(c) $x^2 - 7|x| + 10 \leqslant 0$; (d) $8x^2 + |-x| + 1 > 0$;
(d) $4x^2 + 2|x| + 0.25 < 0$.

25. (a) $3x^2 - |10x - 3| > 0$; (b) $x^2 \leqslant |x - 2|$.

26. (a) $|x^2 + x - 20| \leqslant x^2 + x - 20$; (b) $|x - 2x^2| > 2x^2 - x$; (c) $|x^2 + 6x + 8| \leqslant -x^2 - 6x - 8$.

27. (a) $|x^2 - 6| > 4x + 1$; (b) $|x - 3| > |x^2 - 3|$.

28. (a) $|2x^2 - x - 10| > |x^2 - 8x - 22|$;
(b) $|x^2 - 5|x| + 4| \geqslant |2x^2 - 3|x| + 1|$.

29. Find the domain of definition of the function (a) $y = \sqrt{-x^2}$; (b) $y = x - \sqrt{1 - x^2}$; (c) $y = \sqrt{x^2 - 4x + 3}$; (d) $y = 1/\sqrt{x^2 + 4x}$; (e) $y = \sqrt{16 - x^2} + \sqrt{x^2 + x}$; (f) $y = \sqrt{x^2 - |x|} + 1/\sqrt{9 - x^2}$.

30. Prove that the function $f(x) = ax^2 + bx + c$, $a \neq 0$, is not periodic.

31. At what values of a does the function $f(x) = -x^2 + (a - 1)x + 2$ increase monotonically on the interval $(1; 2)$?

32. Find the inverse of the function $f(x) = x^2 - 6x + 1$ (a) on the interval $(-\infty; 3)$; (b) on the interval $(5; 7)$.

Construct the graphs of the following functions:

33. (a) $y = x^2$; (b) $y = (x - 2)^2$;
(c) $y = 2(x - 2)^2$; (d) $y = 2(x - 2)^2 - 1$.

34. (a) $y = -x^2$; (b) $y = -(x + 1)^2$;
(c) $y = -0.5(x + 1)^2$; (d) $y = 2 - 0.5(x + 1)^2$.

35. (a) $y = x^2 + 5x + 6$; (b) $y = 4x^2 + 4x + 1$;
(c) $y = x^2 + x + 1$.

36. (a) $y = 3x - x^2 - 2$; (b) $y = 2x - x^2 - 1$;
(c) $y = x - x^2 - 1$.

37. (a) $y = x^2 - 4|x| + 3$; (b) $y = x^2 + 4|x| + 3$;
(c) $y = 2 - |x| - x^2$.

38. (a) $y = |x^2 + x|$; (b) $y = -|x^2 - 2x|$.