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Boris Tsirelson
Wendelin Werner

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Authors

Boris Tsirelson
School of Mathematics
Tel Aviv University
Tel Aviv 69978
Israel

e-mail: tsirel@tau.ac.il

Wendelin Werner
Laboratoire de Mathématiques
Université Paris-Sud
Bât 425, 91405 Orsay Cedex
France

e-mail: wendelin.werner@math.u-psud.fr

Editor

Jean Picard
Laboratoire de Mathématiques Appliquées
UMR CNRS 6620
Université Blaise Pascal Clermont-Ferrand
63177 Aubière Cedex, France

e-mail: Jean.Picard@math.univ-bpclermont.fr

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Preface

Three series of lectures were given at the 32nd Probability Summer School in Saint-Flour (July 7–24, 2002), by Professors Pitman, Tsirelson and Werner. In order to keep the size of the volume not too large, we have decided to split the publication of these courses into two parts. This volume contains the courses of Professors Tsirelson and Werner. The course of Professor Pitman, entitled "Combinatorial stochastic processes", is not yet ready. We thank the authors warmly for their important contribution.

76 participants have attended this school. 33 of them have given a short lecture. The lists of participants and of short lectures are enclosed at the end of the volume.

Finally, we give the numbers of volumes of Springer *Lecture Notes* where previous schools were published.

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Part I

Boris Tsirelson: Scaling Limit, Noise, Stability

Scaling Limit, Noise, Stability

Boris Tsirelson

School of Mathematics, Tel Aviv University, Tel Aviv 69978, Israel
tsirel@tau.ac.il
//www.tau.ac.il/~tsirel/

Summary. Linear functions of many independent random variables lead to classical noises (white, Poisson, and their combinations) in the scaling limit. Some singular stochastic flows and some models of oriented percolation involve very nonlinear functions and lead to nonclassical noises. Two examples are examined, Warren's 'noise made by a Poisson snake' and the author's 'Brownian web as a black noise'. Classical noises are stable, nonclassical are not. A new framework for the scaling limit is proposed. Old and new results are presented about noises, stability, and spectral measures.

Introduction

Functions of n independent random variables and limiting procedures for $n \to \infty$ are a tenor of probability theory.

Classical limit theorems investigate linear functions, such as $f(\xi_1, \dots, \xi_n) = (\xi_1 + \dots + \xi_n)/\sqrt{n}$. The well-known limiting procedure (a classical example of scaling limit) leads to the Brownian motion. Its derivative, the white noise, is not a continuum of independent random variables, but rather an infinitely divisible 'reservoir of independence', a classical example of a continuous product of probability spaces.

Percolation theory investigates some very special nonlinear functions of independent two-valued random variables, either in the limit of an infinite discrete lattice, or in the scaling limit. The latter is now making spectacular progress. The corresponding 'reservoir of independence' is already constructed for oriented percolation (which is much simpler). That is a modern, nonclassical example of a continuous product of probability spaces.

An essential distinction between classical and nonclassical continuous products of probability spaces is revealed by the concept of stability/sensitivity, framed for the discrete case by computer scientists and (in parallel) for the continuous case by probabilists. Everything is stable if and only if the setup is classical.

Some readers prefer discrete models, and treat continuous models as a mean of describing asymptotic behavior. Such readers may skip Sects. 6.2, 6.3, 8.2, 8.3, 8.4. Other readers are interested only in continuous models. They may restrict themselves to Sects. 3.4, 3.5, 4.9, 5.2, 6, 7, 8.

Scaling limit. A new framework for the scaling limit is proposed in Sects. 1.2, 2, 3.1–3.3.

Noise. The idea of a continuous product of probability spaces is formalized by the notions of 'continuous factorization' (Sect. 3.4) and 'noise' (Sect. 3.5). (Some other types of continuous product are considered in [18], [19].) For two nonclassical examples of noise see Sects. 4, 7.

Stability. Stability (and sensitivity) is studied in Sects. 5, 6.1, 6.4. For an interplay between discrete and continuous forms of stability/sensitivity, see especially Sects. 5.3, 6.4.

The spectral theory of noises, presented in Sects. 3.3, 3.4 and used in Sects. 5, 6, generalizes both the Fourier transform on the discrete group $\mathbb{Z}_2^n$ (the Fourier-Walsh transform) and the Itô decomposition into multiple stochastic integrals. For the scaling limit of spectral measures, see Sect. 3.3.

Throughout, either by assumption or by construction, all probability spaces will be Lebesgue-Rokhlin spaces; that is, isomorphic mod 0 to an interval with Lebesgue measure, or a discrete (finite or countable) measure space, or a combination of both.

1 A First Look

1.1 Two Toy Models

The most interesting thing is a scaling limit as a transition from a lattice model to a continuous model. A transition from a finite sequence to an infinite sequence is much simpler, but still nontrivial, as we'll see on simple toy models.

Classical theorems about independent increments are exhaustive, but a small twist may surprise us. I demonstrate the twist on two models, 'discrete' and 'continuous'. The 'continuous' model is a Brownian motion on the circle. The 'discrete' model takes on two values ± 1 only, and increments are treated multiplicatively: $X(t)/X(s)$ instead of the usual $X(t) - X(s)$. Or equivalently, the 'discrete' process takes on its values in the two-element group $\mathbb{Z}_2$; using additive notation we have $\mathbb{Z}_2 = \{0, 1\}$, $1+1 = 0$, increments being $X(t) - X(s)$. In any case, the twist stipulates values in a compact group (the circle, $\mathbb{Z}_2$, etc.), in contrast to the classical theory, where values are in $\mathbb{R}$ (or another linear space). Also, the classical theory assumes continuity (in probability), while our twist does not. The 'continuous' process (in spite of its name) is discontinuous at a single instant $t = 0$. The 'discrete' process is discontinuous at $t = \frac{1}{n}$, $n = 1, 2, \dots$, and also at $t = 0$; it is constant on $[\frac{1}{n+1}, \frac{1}{n})$ for every n.

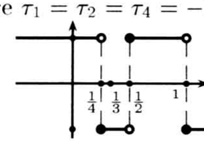
Example 1.1. Introduce an infinite sequence of random signs $\tau_1, \tau_2, \dots$; that is,

$$\mathbb{P}\left(\tau_k = -1\right) = \mathbb{P}\left(\tau_k = +1\right) = \frac{1}{2} \quad \text{for each } k,$$
$$\tau_1, \tau_2, \dots \quad \text{are independent.}$$

For each n we define a stochastic process $X_n(\cdot)$, driven by $\tau_1, \dots, \tau_n$, as follows:

$$X_n(t) = \prod_{k: 1/n \le 1/k \le t} \tau_k \,.$$

a sample path of X_4
(here $\tau_1 = \tau_2 = \tau_4 = -1, \tau_3 = +1$)



For $n \to \infty$, finite-dimensional distributions of X_n converge to those of a process $X(\cdot)$. Namely, X consists of countably many random signs, situated on intervals $[\frac{1}{k+1}, \frac{1}{k})$. Almost surely, X has no limit at $0+$. We have

$$\frac{X(t)}{X(s)} = \prod_{k: s < 1/k \le t} \tau_k \tag{1.1}$$

whenever $0 < s < t < \infty$. However, (1.1) does not hold when $s < 0 < t$. Here, the product contains infinitely many factors and diverges almost surely;

nevertheless, the increment $X(t)/X(s)$ is well-defined. Each X_n satisfies (1.1) for all s,t (including $s < 0 < t$; of course, $k \le n$), but X does not. Still, X is an independent increment process (multiplicatively); that is, $X(t_2)/X(t_1), \dots, X(t_n)/X(t_{n-1})$ are independent whenever $-\infty < t_1 < \dots < t_n < \infty$. However, we cannot describe the whole X by a countable collection of its independent increments. The infinite sequence of $\tau_k = X(\frac{1}{k}+)/X(\frac{1}{k}-)$ does not suffice since, say, $X(1)$ is independent of $(\tau_1, \tau_2, \dots)$. Indeed, the global sign change $x(\cdot) \mapsto -x(\cdot)$ is a measure-preserving transformation that leaves all τ_k invariant. The conditional distribution of $X(\cdot)$ given $\tau_1, \tau_2, \dots$ is concentrated at two functions of opposite global sign. It may seem that we should add to $(\tau_1, \tau_2, \dots)$ one more random sign τ_∞ independent of $(\tau_1, \tau_2, \dots)$ such that $X(\frac{1}{k})$ is a measurable function of $\tau_k, \tau_{k+1}, \dots$ and τ_∞. However, it is impossible. Indeed, $X(1) = \tau_1 \dots \tau_k X(\frac{1}{k})$. Assuming $X(\frac{1}{k}) = f_k(\tau_k, \tau_{k+1}, \dots; \tau_\infty)$ we get $f_1(\tau_1, \tau_2, \dots; \tau_\infty) = \tau_1 \dots \tau_{k-1} f_k(\tau_k, \tau_{k+1}, \dots; \tau_\infty)$ for all k. It follows that $f_1(\tau_1, \tau_2, \dots; \tau_\infty)$ is orthogonal to all functions of the form $g(\tau_1, \dots, \tau_n) h(\tau_\infty)$ for all n, and thus, to a dense (in L_2) set of functions of $\tau_1, \tau_2, \dots; \tau_\infty$; a contradiction.

So, for each n the process X_n is driven by (τ_k), but the limiting process X is not.

Example 1.2. (See also [3].) We turn to the other, the 'continuous' model. For any $\varepsilon \in (0,1)$ we introduce a (complex-valued) stochastic process

$$Y_\varepsilon(t) = \begin{cases} \exp\bigl(\mathrm{i}B(\ln t) - \mathrm{i}B(\ln \varepsilon)\bigr) & \text{for } t \ge \varepsilon, \\ 1 & \text{otherwise,} \end{cases}$$

where $B(\cdot)$ is the usual Brownian motion; or rather, $\bigl(B(t)\bigr)_{t\in[0,\infty)}$ and $\bigl(B(-t)\bigr)_{t\in[0,\infty)}$ are two independent copies of the usual Brownian motion. Multiplicative increments $Y_\varepsilon(t_2)/Y_\varepsilon(t_1), \dots, Y_\varepsilon(t_n)/Y_\varepsilon(t_{n-1})$ are independent whenever $-\infty < t_1 < \dots < t_n < \infty$, and the distribution of $Y_\varepsilon(t)/Y_\varepsilon(s)$ does not depend on ε as far as $\varepsilon < s < t$ (in fact, the distribution depends on t/s only). The distribution of $Y_\varepsilon(1)$ converges for $\varepsilon \to 0$ to the uniform distribution on the circle $|z| = 1$. The same for each $Y_\varepsilon(t)$. It follows easily that, when $\varepsilon \to 0$, finite dimensional distributions of Y_ε converge to those of some process Y. For every $t > 0$, $Y(t)$ is distributed uniformly on the circle; Y is an independent increment process (multiplicatively), and $Y(t) = 1$ for $t \le 0$. Almost surely, $Y(\cdot)$ is continuous on $(0,\infty)$, but has no limit at $0+$. We may define $B(\cdot)$ by

$$\begin{gathered} Y(t) = Y(1)\exp\bigl(\mathrm{i}B(\ln t)\bigr) \quad \text{for } t \in \mathbb{R}, \\ B(\cdot) \quad \text{is continuous on } \mathbb{R}. \end{gathered}$$

Then B is the usual Brownian motion, and

$$\frac{Y(t)}{Y(s)} = \frac{\exp(\mathrm{i}B(\ln t))}{\exp(\mathrm{i}B(\ln s))} \quad \text{for } 0 < s < t < \infty.$$

However, $Y(1)$ is independent of $B(\cdot)$. Indeed, the global phase change $y(\cdot) \mapsto e^{i\alpha}y(\cdot)$ is a measure preserving transformation that leaves $B(\cdot)$ invariant. The conditional distribution of $Y(\cdot)$ given $B(\cdot)$ is concentrated on a continuum of functions that differ by a global phase (distributed uniformly on the circle). Similarly to the 'discrete' example, we cannot introduce a random variable $B(-\infty)$ independent of $B(\cdot)$, such that $Y(t)$ is a function of $B(-\infty)$ and increments of $B(r)$ for $-\infty < r < \ln t$.

So, for each ε, the process Y_ε is driven by the Brownian motion, but the limiting process Y is not.

Both toy models are singular at a given instant $t = 0$. Interestingly, continuous stationary processes can demonstrate such strange behavior, distributed in time! (See Sects. 4, 7).

1.2 Our Limiting Procedures

Imagine a sequence of elementary probabilistic models such that the n-th model is driven by a finite sequence $(\tau_1, \dots, \tau_n)$ of random signs (independent, as before). A limiting procedure may lead to a model driven by an infinite sequence $(\tau_1, \tau_2, \dots)$ of random signs. However, it may also lead to something else, as shown in Sect. 1.1. This is an opportunity to ask ourselves: what do we mean by a limiting procedure?

The n-th model is naturally described by the finite probability space $\Omega_n = \{-1, +1\}^n$ with the uniform measure. A prerequisite to any limiting procedure is some structure able to join these Ω_n somehow. It may be a sequence of 'observables', that is, functions on the disjoint union,

$$f_k : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to \mathbb{R} .$$

Example 1.3. Let $f_k(\tau_1, \dots, \tau_n) = \tau_k$ for $n \ge k$. Though f_k is defined only on $\Omega_k \uplus \Omega_{k+1} \uplus \dots$, it is enough. For every k, the joint distribution of $f_1, \dots, f_k$ on Ω_n has a limit for $n \to \infty$ (moreover, the distribution does not depend on n, as far as $n \ge k$). The limiting procedure should extend each f_k to a new probability space Ω such that the joint distribution of $f_1, \dots, f_k$ on Ω_n converges for $n \to \infty$ to their joint distribution on Ω. Clearly, we may take the space of infinite sequences $\Omega = \{-1, +1\}^\infty$ with the product measure, and let f_k be the k-th coordinate function.

Example 1.4. Still $f_k(\tau_1, \dots, \tau_n) = \tau_k$ (for $n \ge k \ge 1$), but in addition, the product $f_0(\tau_1, \dots, \tau_n) = \tau_1 \dots \tau_n$ is included. For every k, the joint distribution of $f_0, f_1, \dots, f_k$ on Ω_n has a limit for $n \to \infty$; in fact, the distribution does not depend on n, as far as $n > k$ (this time, not just $n \ge k$). Thus, in the limit, $f_0, f_1, f_2, \dots$ become independent random signs. The functional dependence $f_0 = f_1 f_2 \dots$ holds for each n, but disappears in the limit. We still may take $\Omega = \{-1, +1\}^\infty$, however, f_0 becomes a new coordinate.

This is instructive; the limiting model depends on the class of 'observables'.

Example 1.5. Let $f_k(\tau_1, \dots, \tau_n) = \tau_k \dots \tau_n$ for $n \ge k \ge 1$. In the limit, f_k become independent random signs. We may define τ_k in the limiting model by $\tau_k = f_k / f_{k+1}$; however, we cannot express f_k in terms of τ_k. Clearly, it is the same as the 'discrete' toy model of Sect. 1.1.

The second and third examples are isomorphic. Indeed, renaming f_k of the third example as g_k (and retaining f_k of the second example) we have

$$g_k = \frac{f_0}{f_1 \dots f_{k-1}} \, ; \qquad f_k = \frac{g_k}{g_{k+1}} \text{ for } k > 0 \, , \quad \text{and} \quad f_0 = g_1 \, ;$$

these relations hold for every n (provided that the same $\Omega_n = \{-1, +1\}^n$ is used for both examples) and naturally, give us an isomorphism between the two limiting models.

That is also instructive; some changes of the class of 'observables' are essential, some are not.

It means that the sequence (f_k) is not really the structure responsible for the limiting procedure. Rather, f_k are generators of the relevant structure. The second and third examples differ only by the choice of generators for the same structure. In contrast, the first example uses a different structure. So, what is the mysterious structure?

I can describe the structure in two equivalent ways. Here is the first description. In the commutative Banach algebra $l_\infty(\Omega_1 \uplus \Omega_2 \uplus \dots)$ of all bounded functions on the disjoint union, we select a subset C (its elements will be called observables) such that

$$C \text{ is a separable closed subalgebra of } l_\infty(\Omega_1 \uplus \Omega_2 \uplus \dots) \text{ containing the unit.} \tag{1.2}$$

In other words,

$$\begin{gathered} C \text{ contains a sequence dense in the uniform topology;} \\ f_n \in C,\ f_n \to f \text{ uniformly} \quad \Longrightarrow \quad f \in C \, ; \\ f, g \in C,\ a, b \in \mathbb{R} \quad \Longrightarrow \quad af + bg \in C \, ; \\ \mathbf{1} \in C \, ; \\ f, g \in C \quad \Longrightarrow \quad fg \in C \end{gathered} \tag{1.3}$$

(here $\mathbf{1}$ stands for the unity, $\mathbf{1}(\omega) = 1$ for all ω). Or equivalently,

$$\begin{gathered} C \text{ contains a sequence dense in the uniform topology;} \\ f_n \in C,\ f_n \to f \text{ uniformly} \quad \Longrightarrow \quad f \in C \, ; \\ f, g \in C,\ \varphi : \mathbb{R}^2 \to \mathbb{R} \text{ continuous} \quad \Longrightarrow \quad \varphi(f, g) \in C \, . \end{gathered} \tag{1.4}$$

Indeed, on one hand, both $af + bg$ and fg (and $\mathbf{1}$) are special cases of $\varphi(f, g)$. On the other hand, every continuous function on a bounded subset of $\mathbb{R}^2$ can be uniformly approximated by polynomials. The same holds for $\varphi(f_1, \dots, f_n)$

where $f_1, \dots, f_n \in C$, and $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Another equivalent set of conditions is also well-known:

$$\begin{gathered} C \text{ contains a sequence dense in the uniform topology;} \\ f_n \in C,\ f_n \to f \text{ uniformly} \quad \Longrightarrow \quad f \in C\,; \\ f, g \in C,\ a, b \in \mathbb{R} \quad \Longrightarrow \quad af + bg \in C\,; \\ \mathbf{1} \in C\,; \\ f \in C \quad \Longrightarrow \quad |f| \in C\,; \end{gathered} \tag{1.5}$$

here $|f|$ is the pointwise absolute value, $|f|(\omega) = |f(\omega)|$.

The smallest set C satisfying these (equivalent) conditions (1.2)–(1.5) and containing all given functions f_k is, by definition, generated by these f_k.

Recall that C consists of functions defined on the disjoint union of finite probability spaces Ω_n; a probability measure P_n is given on each Ω_n. The following condition is relevant:

$$\lim_{n \to \infty} \int_{\Omega_n} f \, dP_n \text{ exists for every } f \in C\,. \tag{1.6}$$

Assume that C is generated by given functions f_k. Then the property (1.6) of C is equivalent to such a property of functions f_k:

For each k, the joint distribution of $f_1, \dots, f_k$ on Ω_n weakly converges, when $n \to \infty$. (1.7)

Proof: (1.7) means convergence of $\int \varphi(f_1, \dots, f_k)\, dP_n$ for every continuous function $\varphi : \mathbb{R}^k \to \mathbb{R}$. However, functions of the form $f = \varphi(f_1, \dots, f_k)$ (for all k, φ) belong to C and are dense in C.

We see that (1.7) does not depend on the choice of generators f_k of a given C.

The second (equivalent) description of our structure is the 'joint compactification' of $\Omega_1, \Omega_2, \dots$ I mean a pair (K, α) such that

$$\begin{gathered} K \text{ is a metrizable compact topological space,} \\ \alpha : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to K \text{ is a map,} \\ \text{the image } \alpha(\Omega_1 \uplus \Omega_2 \uplus \dots) \text{ is dense in } K. \end{gathered} \tag{1.8}$$

Every joint compactification (K, α) determines a set C satisfying (1.2). Namely,

$$C = \alpha^{-1}\big(C(K)\big)\,;$$

that is, observables $f \in C$ are, by definition, functions of the form

$$f = g \circ \alpha, \text{ that is, } f(\omega) = g(\alpha(\omega)), \quad g \in C(K)\,.$$

The Banach algebra C is basically the same as the Banach algebra $C(K)$ of all continuous functions on K.

Every C satisfying (1.2) corresponds to some joint compactification. Proof: C is generated by some f_k such that $|f_k(\omega)| \le 1$ for all k, ω. We introduce

$$\alpha(\omega) = \big(f_1(\omega), f_2(\omega), \dots\big) \in [-1,1]^\infty ,$$
$$K \text{ is the closure of } \alpha(\Omega_1 \uplus \Omega_2 \uplus \dots) \text{ in } [-1,1]^\infty ;$$

clearly, (K, α) is a joint compactification. Coordinate functions on K generate $C(K)$, therefore f_k generate $\alpha^{-1}\big(C(K)\big)$, hence $\alpha^{-1}\big(C(K)\big) = C$.

Finiteness of each Ω_n is not essential. The same holds for arbitrary probability spaces $(\Omega_n, \mathcal{F}_n, P_n)$. Of course, instead of $l_\infty(\Omega_1 \uplus \Omega_2 \uplus \dots)$ we use $L_\infty(\Omega_1 \uplus \Omega_2 \uplus \dots)$, and the map $\alpha : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to K$ must be measurable. It sends the given measure P_n on Ω_n into a measure $\alpha(P_n)$ (denoted also by $P_n \circ \alpha^{-1}$) on K. If measures $\alpha(P_n)$ weakly converge, we get the limiting model (Ω, P) by taking $\Omega = K$ and $P = \lim_{n\to\infty} \alpha(P_n)$.

1.3 Examples of High Symmetry

Example 1.6. Let Ω_n be the set of all permutations $\omega : \{1, \dots, n\} \to \{1, \dots, n\}$, each permutation having the same probability $(1/n!)$;

$$f : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to \mathbb{R} \text{ is defined by}$$
$$f(\omega) = \big|\{k : \omega(k) = k\}\big| ;$$

that is, the number of fixed points of a random permutation. Though f is not bounded, which happens quite often, in order to embed it into the framework of Sect. 1.2, we make it bounded by some homeomorphism from $\mathbb{R}$ to a bounded interval (say, $\omega \mapsto \arctan f(\omega)$). The distribution of $f(\cdot)$ on Ω_n converges (for $n \to \infty$) to the Poisson distribution $P(1)$. Thus, the limiting model exists; however, it is scanty: just $P(1)$.

We may enrich the model by introducing

$$f_u(\omega) = \big|\{k < un : \omega(k) = k\}\big| ;$$

for instance, $f_{0.5}(\cdot)$ is the number of fixed points among the first half of $\{1, \dots, n\}$. The parameter u could run over $[0,1]$, but we need a countable set of functions; thus we restrict u to, say, rational points of $[0,1]$. Now the limiting model is the Poisson process.

Each finite model here is invariant under permutations. Functions f_u seem to break the invariance, but the latter survives in their increments, and turns in the limit into invariance of the Poisson process (or rather, its derivative, the point process) under all measure preserving transformations of $[0,1]$.

Note also that *independent* increments in the limit emerge from *dependent* increments in finite models.

We feel that all these $f_u(\cdot)$ catch only a small part of the information contained in the permutation. You may think about more information, say, cycles of length $1, 2, \dots$ (and what about length $n/2$?)

Example 1.7. Let Ω_n be the set of all graphs over $\{1,\dots,n\}$. That is, each $\omega \in \Omega_n$ is a subset of the set $\binom{\{1,\dots,n\}}{2}$ of all unordered pairs (treated as edges, while $1,\dots,n$ are vertices); the probability of ω is $p_n^{|\omega|}(1-p_n)^{n(n-1)/2-|\omega|}$, where $|\omega|$ is the number of edges. That is, every edge is present with probability p_n, independently of others. Define $f(\omega)$ as the number of isolated vertices. The limiting model exists if (and only if) there exists a limit $\lim_n n(1-p_n)^{n-1} = \lambda \in [0,\infty)$;[1] the Poisson distribution $P(\lambda)$ exhausts the limiting model.

A Poisson process may be obtained in the same way as before.

You may also count small connected components which are more complicated than single points.

Note that the finite model contains a lot of independence (namely, $n(n-1)/2$ independent random variables); the limiting model (Poisson process) also contains a lot of independence (namely, independent increments). However, we feel that independence is not inherited; rather, the independence of finite models is lost in the limiting procedure, and a new independence emerges.

Example 1.8. Let $\Omega_n = \{-1,+1\}^n$ with uniform measure, and $f_n : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to \mathbb{R}$ be defined by

$$f_u(\omega) = \frac{1}{\sqrt{n}} \sum_{k<un} \tau_k(\omega)\,;$$

as before, $\tau_1,\dots,\tau_n$ are the coordinates, that is, $\omega = \bigl(\tau_1(\omega),\dots,\tau_n(\omega)\bigr)$ and u runs over rational points of $[0,1]$. The limiting model is the Brownian motion, of course.

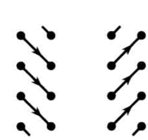
Similarly to Example 1.6, each finite model is invariant under permutations. The invariance survives in increments of functions f_k, and in the limit, the white noise (the derivative of the Brownian motion) is invariant under all measure preserving transformations of $[0,1]$.

A general argument of Sect. 6.3 will show that a high symmetry model cannot lead to a nonclassical scaling limit.

1.4 Example of Low Symmetry

Example 1.8 may be rewritten via the composition of random maps

$$\alpha_-,\alpha_+ : \mathbb{Z} \to \mathbb{Z}\,,$$
$$\alpha_-(k) = k-1\,,\quad \alpha_+(k) = k+1\,;$$
$$\alpha_\omega = \alpha_{\tau_n(\omega)} \circ \dots \alpha_{\tau_1(\omega)}\,;$$



[1] Formally, the limiting model exists also for $\lambda = \infty$, since the range of f is compactified.