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Vector-Valued Optimization Problems in Control Theory

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VECTOR-VALUED OPTIMIZATION PROBLEMS IN CONTROL THEORY

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Introduction

The science of control began to emerge almost one and a half centuries ago. Some of the important stages in its development are discussed in [1]. To compose a survey of the basic directions in control theory, it is necessary to organize a book by an historical plan. We restrict ourselves to considering only the important role of Soviet scholars in the development and formation of this comparatively young science—on the basis of which was created high-quality and high-accuracy automatic control systems; we recommend the survey works [2,3] to the interested reader. In these works a good account is given of the contributions of Soviet scholars to the development of control theory and its extensions.

One of the most important directions in control theory is the development of methods for analyzing the quality of control processes. The fact is that contemporary progress in science and technology very sharply presents scientists and engineers with the problem of creating even more perfect automatic control systems. Up to fifty years ago the requirement on such a system was only to insure stability. More recently, the requirements have been increased and consist of the creation of higher quality systems that are ranked according to one or the other of several indices of quality. The formulation of such questions has produced an especially vital scientific field—the theory of optimal control.

Historically, the theory of optimal control processes was used for problems concerning the optimal control of aircraft maneuvers. One of the first publications in which the character of this new scientific area was exhibited was the work of D. E. Okhotsimskii [4], presenting a variant of a problem related to a ballistic rocket. In this work the regimes of mass expenditure for which some characteristics of the motion are extremal were studied. In speaking about the formation of optimal control theory, we must note the great service of A. A. Fel'dbaum in posing the general problem of minimal time [5] and in studying several of its particular cases.

Fruitful development of optimal control theory began in the 1950s. New mathematical methods appeared for the calculation of optimal controls. In this connection we note the maximum principle of L. S. Pontryagin [6,7], Bellman's method of dynamic programming [8,9], methods of functional analysis [10–17], the method of analytic construction of optimal regulators [18], methods of operations research [19–27], and so on. Later the theory of optimal processes was developed for application to discrete systems [28–32], and there appeared numerical methods for computing optimal controls based upon the ideas of successive analysis of variants and dynamic programming [33].

The fundamental results in the theory of optimal control, obtained by L. S. Pontryagin, N. N. Krasovskii, and R. Bellman, were extensively developed in the works of the Soviet scholars V. G. Boltyanskii, A. G. Butkovskii, R. Gabasov, R. V. Gamkrelidze, Yu. V. Germeier, A. Ya. Dubovitskii, V. I. Zubov, Yu. N. Ivanov, F. M. Kirillova, A. A. Krasovskii, V. G. Krotov, A. M. Letov, A. I. Lur'e, A. A. Miliutin, E. F. Mischenko, N. N. Moiseev, A. A. Pervozvanskii, B. N. Pshennichnii, B. S. Razumikin, L. I. Rozonoer, V. V. Tokarev, V. A. Troitskii, A. A. Fel'dbaum, G. L. Kharatishvili, Ya. Z. Tsypkin and the foreign scholars M. Athans, A. Balakrishnan, L. Zadeh, R. Kalman, R. Kulikowski, J. Kurzweil, J. LaSalle, G. Leitmann, A. Miele, L. Neustadt, S. Chang, and many others.

Optimal control theory deals with a set of questions that are surveyed in [34–36] and also in [6–32]; more recent works are treated in [43].

We note that in all of the above-mentioned publications the problems are considered from the viewpoint of the optimization of a fixed scalar functional given in advance. However, in many practical problems it is not sufficient to consider the optimization of control systems with a single scalar criterion [39,47]. Therefore, further development of control methods for problems with vector-valued criteria is needed.

In one of the new areas of optimal control theory, the theory of differential games, problems of the pursuit of one controlled object by another are studied; in these problems features appear pertaining to the attainability of different, conflicting objectives. In problems of pursuit the partners tend to select strategies in order that the chosen criteria give different values, which are in conflict. In other, more general problems in differential games, the partners have their own indices of quality that are different, but depend upon the strategy of each player; the players tend to choose strategies that are best in the sense of their own indices of quality [11,12,20].

The peculiarity of differential games rests in the fact that for optimization of each index of quality there is a characteristic control resource. In control theory the situation is different. The rule of consumption of control resource is chosen so that all performance indices of the system take on their best values. However, the selection of a control law is effected by optimizing only one of the chosen criteria, which, as noted above, is not sufficient in many practical situations.

The lack of control rules that are well defined through conditions of optimizing a scalar criterion leads to the following: The selection and optimization of a given functional answers to only one of the requirements of the control system, while other requirements, which are often equally important, are ignored. The totality of all requirements may be accounted for by a collection of functionals forming a vector of criterion functions, i.e., a vector-valued criterion. The formulation of requirements for automatic systems in the form of a set of optimality criteria apparently must be empirically arrived at, after which a definite control rule may be determined by application of mathematical techniques [6–33].

Very naturally there arise mathematical problems requiring the simultaneous optimization of a collection of functionals, each of which measures a definite aspect of the system. Important problems of simultaneous optimization of two or more functionals are repeatedly noted by Soviet and foreign scholars [37,47,48].

The current work is devoted exactly to this problem. In this work we study problems of programming optimal trajectories, the analytic construction of regulators, and linear and nonlinear programming under vector-valued criteria. For the solution of all these problems, we employ a single approach based on the definition of an ideal (utopian) point and on the introduction of a norm into the space of optimizing functionals.

Naturally, the choice of a norm in the space of optimizing functionals determines a definite solution of the problem. We study the above-mentioned problems for the particular case in which we choose an euclidean space of functionals. Such a step, i.e., the choice of a quadratic measure, results in a situation that in many practical problems leads to exceptionally good results, and also we obtain a more transparent form of the mathematical presentation. The influence of the measure on the solution of the problem must be studied separately for each concrete problem. Generally, it is reasonable to choose a quadratic metric and to study the question of the stability of the solution to the problem under variation of the measure.

In the first chapter of this monograph we give a survey of the existing literature on the problem of vector optimization and more explicitly consider a number of works that are similar in theme. We give an assessment of each of these works.

In the second chapter we formulate the general mathematical statement of the problem of optimizing vector functionals, based on the idea of defining an ideal point in a space of optimizing functionals and approximating the values of the quality indices of the system to this point. We discuss the expediency of such an approach to the solution of the problem using a concrete example.

The third chapter is devoted to a discussion of the problem of the existence of a solution to the problem of optimizing vector functionals presented in the spirit of Chapter 2. We introduce necessary conditions for the existence of such a solution in problems of programming optimal trajectories. These conditions are illustrated with the concrete example of flight for an autopilot aircraft.

In Chapter 4, on the basis of the Pontryagin maximum principle formulated for the problem of Mayer, we solve the general problem of programming optimal trajectories for vector-valued criteria. To illustrate the method we solve the problem of optimal flight of a rocket to a given point.

In Chapter 5 we discuss the problem of analytic construction of optimal regulators for vector-valued criteria. Along with the general problem, we give a solution to the linear problem of analytic construction for vectors whose components are integrals of quadratic indices.

Chapter 6 is devoted to problems of operations research for vector goal functions. The general problems of linear and nonlinear programming are considered, questions of existence of a solution under vector goals are studied, compromise solutions are discussed, and concrete examples of planning for metallurgical operations are solved.

In Chapter 7 of the monograph, we solve a practical problem from the domain of nuclear reactor installations. With the help of the above methods we solve a problem of determining optimal parameters of the heating machinery of an atomic power station from the point of view of minimizing both weight and volume of condensers, as well as the cost of cooling the system. The solution consists of a number of results. The results of this chapter were obtained together with A. N. Yoseliani, A. A. Michailovizh, and V. B. Nesterenko.

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CHAPTER I

A SURVEY OF OPTIMIZATION PROBLEMS WITH VECTOR CRITERIA

1. General Survey of the Problem

The problem of optimization of a vector-valued criterion arises in connection with the solution of problems in the areas of planning and organization of production. However, recently it has rapidly spread to dynamical control systems. Currently, the problem of optimizing vector-valued criteria has become a central part of control theory and great attention is given to it in the design and construction of modern automatic control systems.

A little history.

The first formulation of a problem of optimizing vector-valued criteria was given in 1896 in a publication of the economist Pareto (49). More than a half a century later questions of multicriteria optimization were touched upon in the publications (50,51), but essential developments of the problem were not carried out.

The fundamental work of von Neumann and Morgenstern (19) on game theory and economic behavior, published in 1944, laid the basis for the concept of a solution in the case of several conflicting criteria. An account of these ideas is given in (21).

In 1963 Professor Lotfi Zadeh published the article (52), which first presented the question of designing control systems which are optimal relative to several performance indices. In spite of the fact that the paper did not contain a rigorous mathematical solution of the general problem of vector optimization, the work must be reckoned as the beginning of a new, more global concept of optimization.

Next, as a continuation of the arguments of (52), in the article (53), a concrete example, using simple linear programming, was given to show the behavior of systems subject to optimization with respect to different objective functions.

In 1964, V. Nelson (48) formulated the general multicriteria optimal control problem for dynamical systems. However, he reduced the problem to one in which only one of the scalar criteria was optimized under isoperimetric constraints on the values of the other functionals. Here already is expressed the idea of optimization of a functional of the solution of a problem which is obtained as the solution of an optimization problem using other functionals not connected with the first. In this work, a simplified aircraft problem is considered for the minimization of fuel expenditure under a flight-time constraint.

In 1966 Professor Sheldon Chang, in his general theory of optimal processes (54) developed necessary conditions for so-called non-improvable solutions in dynamical problems of multicriterion optimization. At the same time, a series of works (55-58) were presented dealing with the determination of necessary conditions for the existence of solutions for multicriteria optimization problems in finite-dimensional and in linear topological spaces. Here, the distinctive approach of "scalarization" was used, i.e. the reduction of the vector optimization problem to a family of optimization problems with a scalar performance index. By scalarization is meant the representation of a vector criterion by a linear combination of its components with strictly positive (real) coefficients.

In the work (59), published in 1967, the ideas of Nelson (48) about optimization of a system with respect to a given scalar criterion were developed, determining another scalar performance index from the optimization conditions. Such an approach obtained the name optimization of an ordered sequence of criteria.

In the same year, Frederick Waltz published the article (60) which in agreement with the ideas of L. Zadeh on the problem of optimization of vector-valued functionals, presented a method for its solution based upon the principle of hierarchical ranking. The principle consists of the following: determination of a numerical matrix. Each element of this matrix Φ_{ij} defines a number which is obtained by calculating the value of the the j^{th} functional. Next, by estimating the gain and loss of different performance indices, we choose the principal criterion of the system and, by the same token, we establish a hierarchical subordination of the criteria.

The problem of multicriteria optimization was widely developed in the 1970s. Here we note both the work of foreign scholars (61-101) and the local work (102-135).

The problem of vector optimization in simplified form consists of the following: the behavior of a system is characterized by an n-dimensional vector $x = \{x_1,...,x_n\}$, $x \in X \subset E^n$ and is measured by a k-dimensional vector function $I(x) = \{I_1(x),...,I_k(x)\}$, the components of which represent given real functions of x. It is required to determine a point $x^0 \in X$, optimizing in some sense the values of the functions $I_1(x),...,I_k(x)$.

The solution methods in the above-mentioned works for the problem of vector optimization may be classified into the following groups:

1) optimization of a hierarchical sequence of performance indices,

2) determination of a set of unimprovable points,

3) determination of a solution based upon one form of compromise or another.

The method of <u>optimization of a heirarchical sequence of performance criteria</u> is based upon the introduction of a preference ordering of the given criteria so that the preference order is reduced to an ordering of scalar criteria. After the establishment of such an order, assumed to be $I_1(x), I_2(x),...,I_k(x)$, the solution $x^0 \in X$ is determined as that point satisfying the relations

$$I_1(x^0) = \min_{x \in X_0 \subset X} I_1(x),$$

$$I_2(x^0) = \min_{x \in X_1 \subset X_0} I_2(x),$$

$$I_3(x^0) = \min_{x \varepsilon X_2 \subset X_1} I_3(x),$$

. . . .

$$I_k(x^0) = \min_{x \varepsilon X_{k-1} \subset X_{k-2}} I_k(x).$$

Here the set $X_i \subset X_{i-1}$ $(i=1,2,\ldots,k-1)$ is defined as

$$X_i = \{x : I_i(x) = \min_{x \varepsilon X_{i-1}} I_i(x)\}.$$

As was noted, the idea of such an approach to the solution of vector optimization problems was proposed in (48). The works (59, 60,102) were carried out in this direction and we shall devote special attention to them later. Here we note only that in (60), on the basis of engineering considerations, the principal criterion was established while the remaining criteria were declared hierarchically subordinate to it.

Application of the given method becomes less effective for the solution of the majority of practical problems since optimization with respect to the first, most important criterion already leads to a unique optimal solution and everything is reduced to optimization only relative to the first criterion.

Several numerical problems connected with the choice of the optimal structure of hierarchical control systems are discussed in (103).

The largest collection of work is devoted to the method of determination of the set of unimprovable points. A point $x^0 \varepsilon X$ is called unimprovable in X relative to I(x), if among all $x \varepsilon X$ there does not exist a point $\bar{x}$ such that $I_\alpha(\bar{x}) \leq I_\alpha(x^0)$, $\alpha = 1,2,\ldots,k$, with at least one of the inequalities being strict. Here, at first we note the work (52), in which the definition of an unimprovable point is first given, together with a discussion of the problem. In (61) several properties of the set of unimprovable points are studied in the general problem of dynamical systems with k performance criteria__ and it is shown that the problem reduces to the minimization of a linear form of the components of the vector I(x) with constant weighting coefficients, i.e. to the minimization of the expression

$$I_\lambda = \sum_{\alpha=1}^{k} \lambda_\alpha I_\alpha(x),$$

in which $\lambda_\alpha > 0$, $\alpha = 1,2,\ldots,k$, $\sum_{\alpha=1}^{k} \lambda_\alpha = 1$.

Under the name "The General Theorem of Optimal Control", in (54) a theorem on necessary conditions for unimprovable solutions in the general optimal control problem with vector-valued criteria was given. Later, we shall formulate this theorem. The results of (54) were generalized to linear topological spaces in (55-58). Necessary and sufficient conditions for the optimality of a control for differentiable criteria functions were given in (62,63).

Necessary and sufficient conditions for the existence of an unimprovable solution in mathematical programming problems and the separation of dominant solutions from the set of unimprovable points are dealt with in the series of works (65-68).

From the important works dealing with operations research problems, we should isolate works in which the problem of admissibility of solutions in complex situations are studied. One of the characterisitc features in the theory of admissible solutions is the presence of a large number of criteria of the utility of the system and the impossibility of comparing these criteria because of the complexity of the system. Several questions connected with the relative dependence of the criteria are studied in (69).

Since the admissibility of a solution must be realized on the basis of a comparison of a set of alternatives, i.e. a choice of points from a set of unimprovable points, in many practical situations it is expedient to employ man-machine procedures. The idea of using man to effect the solution in one or another form was expressed in (70). Already in this paper a method was proposed based upon obtaining information from humans at several levels in the solution process and using this information at higher levels.

The process is: presentation of intermediate information to the person about the solution, machine computation using additional information supplied by the person and repetition of this cycle until an acceptable best-possible solution is obtained. Exhaustive surveys of the man-machine procedure for acceptance of a solution is given in (104,105).

A method allowing the consideration of incommensurable criteria was presented in (71). The idea of the method consists in a detection of successively less-disputed relations between the alternatives and in a declaration of their incommensurability.

With the goal of obtaining information needed for justifying an admissible solution, in (106,107) it was proposed to effect a decomposition of the set of states into subsets for each of which an alternative solution is optimal. The role of the measure of information in problems of admissible solutions is discussed in (108-110).

Admissible solutions in complex situations are also dealt with in the works (111-119). In the works (120-124) are studied some problems arising in the admissibility of solutions in the control of large systems. Exhaustive surveys on the basic problems and methods of vector optimization are given in (125,126).

Methods of physical modeling in mathematical programming and economics are presented in (127,128). The effective determination of the equilibria of linear economic models, consisting of economic systems having characteristic goals and budgets, has been carried out using these methods. A very important result of these investigations is the establishment of the existence of a global goal function, having the physical meaning of system entropy, and the determination of recurrence relations for the optimal distribution of resources between systems.

In game theory, the method of unimprovable points is called a Pareto-optimal solution. Currently, the term "Pareto-optimal" is widely used in the theory of vector optimization. The properties