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Volume I

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Preface

In the last decades there has been published a large number of books on functional analysis, either textbooks or books on a more advanced level, some of these with the main emphasis on normed vector spaces and others devoted to more general topological vector spaces. In most of these books the notion of a (partially) ordered vector space receives little or no attention, although a fair proportion of the standard examples of vector spaces is partially ordered. We recall that a (partially) ordered real vector space L is a vector space over the real numbers which is at the same time a partially ordered set such that the vector space structure and the order structure in L are compatible. This means that, for any pair of elements f and g in L satisfying $f \leqq g$, it follows that $f+h \leqq g+h$ holds for all h in L and $af \leqq ag$ holds for all real numbers $a \geqq 0$. If, in addition, the order structure in L is a lattice structure (i.e., if any pair of elements in L has a least upper bound with respect to the order), then L is called a vector lattice or also a Riesz space. Familiar examples are the vector space of all real continuous functions on a given topological space, all the spaces L_p $(p > 0)$ in integration theory, and certain linear subspaces of the vector space of all Hermitian operators on a Hilbert space. The present book is devoted to the theory of Riesz spaces. The first volume contains what may be called the algebraic part of the theory, whereas the second volume will be more analytic in character.

The theory of Riesz spaces was founded, independently, by F. Riesz, H. Freudenthal and L. V. Kantorovitch in the years around 1935, and it is interesting to observe now, more than thirty years later, the different methods of approach. F. Riesz was interested primarily in what is at present called the order dual space of a given ordered vector space, and he presented an extended version of his short 1928 Congress note ([1], International Mathematical Congress at Bologna) in a 1940 Annals of Mathematics paper [2], a translation of a 1937 Hungarian paper. H. Freudenthal, in 1936, proved a "spectral theorem" for Riesz spaces [1], the significance of which is illustrated by the fact that the Radon-Nikodym theorem in integration theory as well as the spectral theorem for Hermitian operators in Hilbert space are corollaries, although it was not until early in the fifties that a direct method was indicated for deriving the spectral theorem for Hermitian operators from the abstract spectral theorem. Finally, around 1935, L. V. Kantorovitch ([1], [2], [3]) began an extensive investigation of the algebraic and convergence properties of Riesz spaces, with applications to linear operator theory. He was soon joined, in Leningrad, by several other mathematicians, of which we mention A. G. Pinsker, A. I. Judin and B. Z. Vulikh. A few year later, between 1940 and 1944, important contributions to the subject were published by H. Nakano ([1], [2], [3], [4], [5]), T. Ogasawara [1], K. Yosida ([1], [2], [3]) in Japan and S. Kakutani and H. F. Bohnenblust (with papers about concrete representations of abstract L-spaces and M-spaces) in the United States. After this first period it has still lasted a relatively long time before the results and terminology of the various centers of research (mainly in Japan, the Soviet Union and the United States) began to grow together. As an illustration we mention the important paper by I. Amemiya ([2], 1953), an extension of earlier work by H. Nakano. On account of the "Nakano terminology" in this paper it is not immediately visible that we have to do here with prime ideal theory in Riesz spaces, to some extent similar to M. H. Stone's prime ideal theory in distributive lattices ([2], 1937). It is one of our objectives in the present book to collect the principal results, unify the terminology, and draw the reader's attention to similarities as the one mentioned above. As is often the case with work of this kind, this has led to certain new results (not always explicitly indicated as new in the text).

This is perhaps an appropriate point to say a few words about some other books devoted, in part or wholly, to ordered vector spaces, or, more specifically, to Riesz spaces. In the book by G. Birkhoff on lattice theory ([1], first edition in 1940, revised editions in 1948 and 1967), and also in the book by L. Fuchs on partially ordered algebraic structures ([1], 1966), the partially ordered vector spaces appear only as rather special examples. In the Birkhoff book, as indicated by the title, the principal interest is in general lattices without any other algebraic structure, and in the book by Fuchs a Riesz space is considered mainly as an example of the more general notion of a (not necessarily commutative) lattice group. The book by H. Nakano ([6], 1950) is pretty well restricted to the author's own research. There is also a book by T. Ogasawara (1948), the contents of which are not so easily accessible, because the book is written in Japanese, and no translation has been made. In the Soviet Union, in 1950, three mathematicians of the Leningrad school (L. V. Kantorovitch, B. Z. Vulikh and A. G. Pinsker) published a large monograph, called "Functional Analysis in Partially Ordered Spaces", and in 1961 B. Z. Vulikh [2] published a smaller and more modern textbook on the same subject which is very readable, but still to a large extent devoted mainly to the research done in the Soviet Union. Vulikh's textbook has been translated into English (1967). Finally, we mention the more recent books by A. L. Peressini (Ordered Topological Vector Spaces, 1967) and G. Jameson (Ordered Linear Spaces, 1970); the emphasis in both of these books is on topological ordered vector spaces (i.e., ordered vector spaces in which a topology is introduced such that the topology is compatible with the vector space structure as well as with the order structure). Spaces of this kind (in particular those in which the topology is generated by a Riesz norm) will be treated in the second volume of the present book.

Readers who desire to restrict themselves at first to the topics of foremost importance (in particular the readers interested primarily in Freudenthal's spectral theorem and its applications) are advised in the introductory Chapter 1 to pay attention only to sections 1–3 and the very beginning of section 9, and then to omit in Chapter 2 everything after Theorem 16.4 as well as in Chapter 4 the sections 26 (on atoms) and 32 (on the Dedekind completion). Chapter 5 on prime ideals may then also be omitted completely, and in Chapter 6 (on Freudenthal's spectral theorem) it is enough to study only sections 38, 39, 40 until after Theorem 40.2, and then section 41. This will prepare the reader to understand all of Chapter 8 (on Hermitian and normal operators in Hilbert space), apart from a few isolated references to the spectral representation theory in Chapter 7. Finally, any reader who decides to take up also the study of prime ideals (Chapter 5) and spectral representations (Chapter 7) is advised to look first or simultaneously at sections 5–8 of Chapter 1.

In 1967 we prepared a "preprint" for the present volume which was distributed on a small scale. The preprint was much more concise; for example, most of Chapter 1 and all of the Chapters 5 and 7 were not included, and also the number of bibliographical references was very small.

A major role in the second volume will be played by the properties of linear mappings from one Riesz space into another, by the order dual space of a given Riesz space, and by topological Riesz spaces (in particular by normed Riesz spaces). The interplay between topological continuity and order continuity will be an important feature. Some of this material can be found in a series of notes published by us in the Proceedings of the Netherlands Academy of Science, Amsterdam, Vols. 66–68 (1963–65). A further remark is in order here. The spaces in sections 9–10, the normed Köthe spaces and the Orlicz spaces, serve to illustrate some phenomena discussed in the abstract theory. This will be the case more frequently in the second volume than in the first. After some hesitation we have decided, however, to maintain these sections in the first volume.

We express our thanks to the National Science Foundation of the U.S.A. and to the Netherlands Organization for the Advancement of Pure Research (Z.W.O.) for financial support at various stages during the preparation of this work. We also wish to thank Mrs. L. Decker and Mrs. A. Y. Hudson who, in Pasadena and Leiden respectively, gave us their valuable help in preparing the typed manuscripts of the present book as well as, back in 1967, of the preprint. Finally, we express our appreciation for the assistance and cooperation received from the staff at the North-Holland Publishing Company.

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CHAPTER 1

Distributive Lattices and Normed Function Spaces

We present a survey of several definitions and theorems (on distributive lattices, in particular Boolean algebras and Boolean rings, and on certain normed linear function spaces), which will be needed in what follows. Most of what is said in the sections 9 and 10 on function spaces is not immediately required.

1. Partial ordering

Let X be a non-empty set; the elements $x, y, \ldots$ of X will be called the *points* of X. The set of all ordered pairs (x, y) of points of X is called the *Cartesian product* of X by itself, and is denoted by $X \times X$.

By a *relation* in X we shall understand a non-empty subset of $X \times X$; the relation is sometimes denoted by R, and we shall write xRy whenever (x, y) is an element of the subset R of $X \times X$ which defines the relation. Well-known examples are the equivalence relations; R is called an *equivalence relation* whenever

(i) it follows from xRy and yRz that xRz (the relation is *transitive*),
(ii) xRx holds for all $x \in X$ (the relation is *reflexive*),
(iii) it follows from xRy that yRx (the relation is *symmetric*).

If the equivalence relation R has the property that the subset of $X \times X$ which defines R consists only of all points (x, x) for $x \in X$, then R is the relation of equality.

The relation R in X is called a *partial* ordering of X whenever R is transitive, reflexive and *anti-symmetric*, i.e., whenever

(i) it follows from xRy and yRz that xRz,
(ii) xRx holds for all $x \in X$,
(iii) it follows from xRy and yRx that $x = y$.

If R is a partial ordering in X, we will usually write $x \leqq y$ (or, equivalently,

$y \geqq x$) for xRy. Elements x, y of X for which either $x \leqq y$ or $x \geqq y$ holds are said to be *comparable*; if neither $x \leqq y$ nor $x \geqq y$ holds, then x and y are said to be *incomparable*. If every two elements of X are comparable, the partial ordering is called a *linear ordering*. The other extreme case is that every two different elements of X are incomparable, and so the partial ordering states now only that $x \leqq y$ holds if and only if $x = y$.

If X is partially ordered and Y is a non-empty subset of X, then Y is partially ordered in a natural manner by the partial ordering which Y inherits from X. If the inherited partial ordering in Y is a linear ordering, then Y is said to be a *chain* in X.

If X is partially ordered, Y a non-empty subset of X, and $x_0 \in X$ satisfies $x_0 \geqq y$ for all $y \in Y$, then x_0 is called an *upper bound* of Y. If x_0 is an upper bound of Y such that $x_0 \leqq x_0'$ for any other upper bound x_0' of Y, then x_0 is called a *least upper bound* or *supremum* of Y. In this case x_0 is uniquely determined (in other words, any non-empty subset of X has at most one supremum). Indeed, if both x_0 and x_0' are suprema of Y, then $x_0 \leqq x_0'$ and $x_0' \leqq x_0$, and so $x_0 = x_0'$. If x_0 is the supremum of Y, this will be denoted by $x_0 = \sup Y$ or by

$$x_0 = \sup (y : y \in Y).$$

The notions of *lower bound* and *greatest lower bound* or *infimum* are defined similarly. Notation: $x_0 = \inf (y : y \in Y)$ if x_0 is the infimum of Y.

The element x_0 of the partially ordered set X is called a *maximal element* if it follows from $x \in X$ and $x_0 \leqq x$ that $x_0 = x$ (observe that this is not the same as requiring that $x_0 \geqq x$ holds for all $x \in X$). If there exists an element $x_0 \in X$ such that $x_0 \geqq x$ holds for all $x \in X$, then x_0 is called the *largest element* of X, and in this case x_0 is also a maximal element. Actually, x_0 is now the only maximal element of X. In the converse direction, if x_0 is the only maximal element of the partially ordered set X, then x_0 is not necessarily the largest element of X. Similar remarks hold for *minimal elements* and the possibly existing *smallest element*.

We recall the following well-known and frequently used lemma.

Zorn's lemma. *If every chain in the partially ordered set X has an upper bound, then X contains at least one maximal element.*

We proceed with some definitions.

Definition 1.1. *Let X be a partially ordered set.*

(i) *The set X is called order complete if every non-empty subset of X has a supremum and an infimum.*

(ii) *The set X is called Dedekind complete if every non-empty subset which is bounded from above has a supremum and every non-empty subset which is bounded from below has an infimum.*

(iii) *The set X is called Dedekind σ-complete if every non-empty finite or countable subset which is bounded from above has a supremum and every non-empty finite or countable subset which is bounded from below has an infimum.*

(iv) *The set X is called a lattice if every subset consisting of two elements has a supremum and an infimum.*

For Dedekind completeness a one-sided condition is sufficient, as follows.

Theorem 1.2. *The partially ordered set X is Dedekind complete if and only if every non-empty subset which is bounded from above has a supremum.*

Proof. Let every non-empty subset of X which is bounded from above have a supremum, and assume that Y is a subset of X which is bounded from below. We have to prove that $\inf Y$ exists. To this end, observe first that the set $L(Y)$ of all lower bounds of Y is non-empty and bounded from above, so $l_0 = \sup L(Y)$ exists. Since $l \leqq y$ holds for all $l \in L(Y)$ and all $y \in Y$, any $y \in Y$ is an upper bound of $L(Y)$, and so $l_0 \leqq y$. This shows that l_0 is a lower bound of Y, i.e., $l_0 = \sup L(Y)$ is itself a member of $L(Y)$. It is evident now that l_0 is the greatest lower bound of Y.

In a partially ordered set with a smallest and a largest element the notions of order completeness and Dedekind completeness are evidently identical. Conversely, of course, any order complete partially ordered set has a smallest and a largest element.

2. Lattices

Let X be a lattice. We shall denote the supremum of the set consisting of the elements $x, y \in X$ by $\sup (x, y)$, or by $x \vee y$ if this is notationally more convenient. Similarly, the infimum of the set consisting of x and y will be denoted by $\inf (x, y)$ or by $x \wedge y$. By induction it follows easily that in a lattice every finite subset has a supremum and an infimum. If the elements in the finite subset are $x_1, \ldots, x_n$, its supremum is denoted by $\sup (x_1, \ldots, x_n)$ or $x_1 \vee \ldots \vee x_n$ or $\vee_{i=1}^n x_i$, and its infimum by $\inf (x_1, \ldots, x_n)$ or $x_1 \wedge \ldots \wedge x_n$ or $\wedge_{i=1}^n x_i$.

Definition 2.1. *The lattice X is called distributive if*

$$x \wedge (y_1 \vee y_2) = (x \wedge y_1) \vee (x \wedge y_2)$$

holds for all $x, y_1, y_2 \in X$.

We can interchange suprema and infima in this definition, as shown by the following theorem.

Theorem 2.2. *The lattice X is distributive if and only if*

$$x \vee (y_1 \wedge y_2) = (x \vee y_1) \wedge (x \vee y_2)$$

holds for all $x, y_1, y_2 \in X$.

Proof. Assume that X is a distributive lattice, and denote by l and r the left hand side and right hand side respectively of the formula to be proved. Also, write $z = x \vee y_1$. Then

$$\begin{aligned} r &= z \wedge (x \vee y_2) = (z \wedge x) \vee (z \wedge y_2) \\ &= \{(x \vee y_1) \wedge x\} \vee \{(x \vee y_1) \wedge y_2\} \\ &= (x \wedge x) \vee (y_1 \wedge x) \vee (x \wedge y_2) \vee (y_1 \wedge y_2) \\ &= x \vee (y_1 \wedge x) \vee (x \wedge y_2) \vee (y_1 \wedge y_2) = x \vee (y_1 \wedge y_2) = l, \end{aligned}$$

where we have used in the last line that $y_1 \wedge x \leqq x$ and $x \wedge y_2 \leqq x$. The proof in the converse direction is similar.

Theorem 2.3. *In order that the lattice X is distributive, it is necessary and sufficient that $x \wedge y_1 \leqq z$ and $x \wedge y_2 \leqq z$ implies $x \wedge (y_1 \vee y_2) \leqq z$.*

Proof. The necessity is evident from the definition of distributivity. Conversely, let X be a lattice such that $x \wedge y_1 \leqq z$ and $x \wedge y_2 \leqq z$ implies $x \wedge (y_1 \vee y_2) \leqq z$. We have to prove that

$$x \wedge (y_1 \vee y_2) = (x \wedge y_1) \vee (x \wedge y_2)$$

holds for $x, y_1, y_2 \in X$. Denoting the left hand side and right hand side of the formula to be proved by l and r respectively, we have $x \wedge y_1 \leqq l$ and $x \wedge y_2 \leqq l$, so $r \leqq l$. On the other hand, since $x \wedge y_1 \leqq r$ and $x \wedge y_2 \leqq r$, we have by hypothesis that $x \wedge (y_1 \vee y_2) \leqq r$, i.e., $l \leqq r$. It follows that $l = r$.

If a lattice X has a smallest and (or) a largest element, these are sometimes called the *null* and (or) the *unit* of X; we shall denote the null and the unit by θ and e respectively. If X is a distributive lattice with null and unit, and if $x, x' \in X$ satisfy $x \wedge x' = \theta$ and $x \vee x' = e$, then x' is called a *complement* of x. Of course, x is now also a complement of x'. Note the possibility that X consists of only one element; now $\theta = e$.

Theorem 2.4. *If the element x in the distributive lattice X with null and unit* *has a complement x', then x' is uniquely determined (in other words, every element in X has at most one complement).*

Proof. Assume that $x^{\sim}$ is also a complement of x. Then

$$x^{\sim} = x^{\sim} \vee \theta = x^{\sim} \vee (x \wedge x') = (x^{\sim} \vee x) \wedge (x^{\sim} \vee x') = e \wedge (x^{\sim} \vee x') = x^{\sim} \vee x'.$$

Similarly $x' = x' \vee x^{\sim}$. Hence $x^{\sim} = x'$.

If X is a lattice with null θ, and $x \wedge y = \theta$ for the elements x, y of X, then x and y are called *disjoint elements.* If Y is a non-empty subset of X, the set of all $x \in X$ such that x is disjoint from all $y \in Y$ is called the *disjoint complement* of Y, and this set is denoted by Y^d.

If X is a lattice with null θ, and Z is a subset of X with the property that $z_1, z_2 \in Z$ implies $z_1 \vee z_2 \in Z$ and $z \in Z$ implies $z' \in Z$ for all z' satisfying $z' \leqq z$, then Z is called an *ideal* in X. The condition that $z \in Z$ implies $z' \in Z$ for all $z' \leqq z$ is equivalent to $z \wedge x \in Z$ for all $z \in Z$, $x \in X$. Note that the set $\{\theta\}$, i.e., the set consisting of θ only, is an ideal. Also, if Y is a non-empty subset of the distributive lattice X with null, then the disjoint complement of Y is evidently an ideal in X.

3. Boolean algebras

By definition, a *Boolean algebra* is a distributive lattice with null and unit such that every element in the lattice has a complement. By Theorem 2.4 the complement is uniquely determined. Note the possibility that the Boolean algebra consists of only one element.

Theorem 3.1. *If x and y are elements in the Boolean algebra X such that $x \leqq y$, then*

$$X_{x,y} = (z\colon x \leqq z \leqq y),$$

with the partial ordering inherited from X, is also a Boolean algebra with x as null and y as unit.

Proof. We denote the null and the unit of X by θ and e respectively. Evidently, $X_{x,y}$ is a distributive lattice (with respect to the partial ordering inherited from X) with x as null and y as unit. It remains to show that every $z \in X_{x,y}$ has a complement in $X_{x,y}$. Let z' be the complement of z in X, and set $z^* = (z' \wedge y) \vee x$. Then

$$z \wedge z^* = z \wedge \{(z' \wedge y) \vee x\} = \{z \wedge (z' \wedge y)\} \vee (z \wedge x) = \theta \vee x = x$$

and

$$z \vee z^* = z \vee (z' \wedge y) \vee x = z \vee (z' \wedge y) = (z \vee z') \wedge (z \vee y) = e \wedge y = y,$$

which shows that z^* is the complement of z in $X_{x,y}$.

Theorem 3.2. *For any x in the Boolean algebra X, denote the complement of x by x'. Then the following holds.*

(i) *For any x, the element x' is the largest element in X disjoint from x.*

(ii) *If $x \leqq y$, then $x' \geqq y'$.*

(iii) *We have $(x \vee y)' = x' \wedge y'$ for all x, y in X (and hence $(x \wedge y)' = x' \vee y'$).*

(iv) *If $\{x_\tau : \tau \in \{\tau\}\}$ is an indexed subset of X such that $x = \sup x_\tau$ exists, then $x' = \inf x'_\tau$.*

Proof. (i) We have $x \wedge x' = \theta$ and $x \vee x' = e$. Assume that $x^\sim$ is also disjoint from x, so $x \wedge x^\sim = \theta$. Then $y = x' \vee x^\sim$ satisfies

$$x \vee y = x \vee x' \vee x^\sim = e$$

and

$$x \wedge y = x \wedge (x' \vee x^\sim) = (x \wedge x') \vee (x \wedge x^\sim) = \theta.$$

Hence $y = x'$, i.e., $x' \vee x^\sim = x'$. It follows that $x^\sim \leqq x'$, which shows that x' is the largest element disjoint from x.

(ii) Let $x \leqq y$. Then $y' \wedge x \leqq y' \wedge y = \theta$, so y' is disjoint from x. It follows now by means of (i) that $y' \leqq x'$.

(iii) It follows from $x \vee y \geqq x$ that $(x \vee y)' \leqq x'$. Similarly $(x \vee y)' \leqq y'$, and so $(x \vee y)' \leqq x' \wedge y'$. We have also

$$(x' \wedge y') \wedge (x \vee y) = (x' \wedge y' \wedge x) \vee (x' \wedge y' \wedge y) = \theta \vee \theta = \theta,$$

so $x' \wedge y'$ is disjoint from $x \vee y$. This implies, by part (i), that $x' \wedge y' \leqq (x \vee y)'$. The final result is, therefore, that

$$(x \vee y)' = x' \wedge y'.$$

(iv) Evidently, x' is a lower bound of the set of all x'_τ. Let y be another lower bound, i.e., $y \leqq x'_\tau$ for all τ. Then $y' \geqq x''_\tau = x_\tau$ for all τ, so $y' \geqq x$. It follows that $y'' \leqq x'$, i.e., $y \leqq x'$. This shows that $x' = \inf x'_\tau$.

We list some examples. Given the non-empty point set X, the collection Γ of subsets of X is called a *ring* whenever it follows from $A, B \in \Gamma$ that $A \cup B \in \Gamma$ and $A - B \in \Gamma$, where $A - B$ denotes the set theoretic difference of A and B. It can easily be verified that in this case finite unions and finite