FUNCTIONAL ANALYSIS

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PREFACE

Functional analysis is the study of certain topological-algebraic structures and of the methods by which knowledge of these structures can be applied to analytic problems.

A good introductory text on this subject should include a presentation of its axiomatics (i.e., of the general theory of topological vector spaces), it should treat at least a few topics in some depth, and it should contain some interesting applications to other branches of mathematics. I hope that the present book meets these criteria.

The subject is huge and is growing rapidly. (The bibliography in volume I of [4] contains 96 pages and goes only to 1957.) In order to write a book of moderate size, it was therefore necessary to select certain areas and to ignore others. I fully realize that almost any expert who looks at the table of contents will find that some of his (and my) favorite topics are missing, but this seems unavoidable. It was not my intention to write an encyclopedic treatise. I wanted to write a book that would open the way to further exploration.

This is the reason for omitting many of the more esoteric topics that might have been included in the presentation of the general theory of topological vector spaces. For instance, there is no discussion of uniform spaces, of Moore-Smith convergence, of nets, or of filters. The notion of completeness occurs only in the context of metric spaces. Bornological spaces are not mentioned, nor are barreled ones. Duality is of course presented, but not in its utmost generality. Integration of vector-valued function is treated strictly as a tool; attention is confined to continuous integrands, with values in a Fréchet space.

Nevertheless, the material of Part 1 is fully adequate for almost all applications to concrete problems. And this is what ought to be stressed in such a course: The close interplay between the abstract and the concrete is not only the most useful aspect of the whole subject but also the most fascinating one.

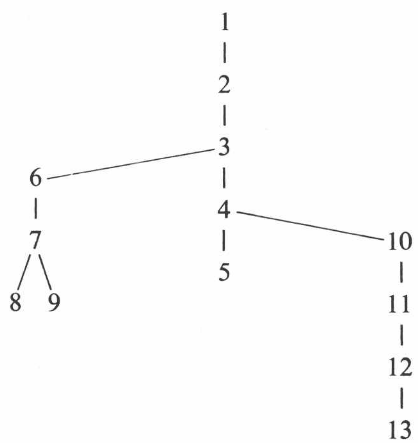
Here are some further features of the selected material. A fairly large part of the general theory is presented without the assumption of local convexity. The basic properties of compact operators are derived from the duality theory in Banach spaces. The Krein-Milman theorem on the existence of extreme points is used in several ways in

Chapter 5. The theory of distributions and Fourier transforms is worked out in fair detail and is applied (in two very brief chapters) to two problems in partial differential equations, as well as to Wiener's tauberian theorem and two of its applications. The spectral theorem is derived from the theory of Banach algebras (specifically, from the Gelfand-Naimark characterization of commutative B^*-algebras); this is perhaps not the shortest way, but it is an easy one. The symbolic calculus in Banach algebras is discussed in considerable detail; so are involutions and positive functionals. Several fairly recent results on Banach algebras that have not found their way into other textbooks as yet are included.

I assume familiarity with the theory of measure and Lebesgue integration (including such facts as the completeness of the L^p-spaces), with some basic properties of holomorphic functions (such as the general form of Cauchy's theorem, and Runge's theorem), and with the elementary topological background that goes with these two analytic topics. Some other topological facts are briefly presented in Appendix A. Almost no algebraic background is needed, beyond the knowledge of what a homomorphism is.

Historical references are gathered in Appendix B. Some of these refer to the original sources, and some to more recent books, papers, or expository articles in which further references can be found. There are, of course, many items that are not documented at all. In no case does the absence of a specific reference imply any claim to originality on my part.

Most of the applications are in Chapters 5, 8, and 9. Some are in Chapter 11 and in the more than 250 exercises; many of these are supplied with hints. The interdependence of the chapters is indicated in the following diagram.



This book grew out of a course that I have taught at the University of Wisconsin. I have had many fruitful conversations about various topics in it with some of my colleagues, especially with Patrick Ahern, Paul Rabinowitz, Daniel Shea, and Robert Turner. It is a pleasure to record my thanks to them.

Walter Rudin

CONTENTS

PART ONE

General Theory

1

TOPOLOGICAL VECTOR SPACES

Introduction

1.1 Many problems that analysts study are not primarily concerned with a single object such as a function, a measure, or an operator, but they deal instead with large classes of such objects. Most of the interesting classes that occur in this way turn out to be vector spaces, either with real scalars or with complex ones. Since limit processes play a role in every analytic problem (explicitly or implicitly), it should be no surprise that these vector spaces are supplied with metrics, or at least with topologies, that bear some natural relation to the objects of which the spaces are made up. The simplest and most important way of doing this is to introduce a *norm*. The resulting structure (defined below) is called a normed vector space, or a normed linear space, or simply a *normed space*.

Throughout this book, the term *vector space* will refer to a vector space over the complex field $\mathbb{C}$ or over the real field R. For the sake of completeness, detailed definitions are given in Section 1.4.

1.2 Normed spaces A vector space X is said to be a *normed space* if to every $x \in X$ there is associated a nonnegative real number $\|x\|$, called the *norm* of x, in such a way that

(*a*) $\|x + y\| \le \|x\| + \|y\|$ for all x and y in X,
(*b*) $\|\alpha x\| = |\alpha| \, \|x\|$ if $x \in X$ and α is a scalar,
(*c*) $\|x\| > 0$ if $x \neq 0$.

The word "norm" is also used to denote the *function* that maps x to $\|x\|$.

Every normed space may be regarded as a metric space, in which the distance $d(x, y)$ between x and y is $\|x - y\|$. The relevant properties of d are:

(*i*) $0 \le d(x, y) < \infty$ for all x and y,
(*ii*) $d(x, y) = 0$ if and only if $x = y$,
(*iii*) $d(x, y) = d(y, x)$ for all x and y,
(*iv*) $d(x, z) \le d(x, y) + d(y, z)$ for all x, y, z.

In any metric space, the *open ball* with center at x and radius r is the set

$$B_r(x) = \{y : d(x, y) < r\}.$$

In particular, if X is a normed space, the sets

$$B_1(0) = \{x : \|x\| < 1\} \qquad \text{and} \qquad \bar{B}_1(0) = \{x : \|x\| \le 1\}$$

are the *open unit ball* and the *closed unit ball* of X, respectively.

By declaring a subset of a metric space to be open if and only if it is a (possibly empty) union of open balls, a *topology* is obtained. (See Section 1.5.) It is quite easy to verify that the vector space operations (addition and scalar multiplication) are continuous in this topology, if the metric is derived from a norm, as above.

A *Banach space* is a normed space which is *complete* in the metric defined by its norm; this means that every Cauchy sequence is required to converge.

1.3 Many of the best-known function spaces are Banach spaces. Let us mention just a few types: spaces of continuous functions on compact spaces; the familiar L^p-spaces that occur in integration theory; Hilbert spaces — the closest relatives of euclidean spaces; certain spaces of differentiable functions; spaces of continuous linear mappings from one Banach space into another; Banach algebras. All of these will occur later on in the text.

But there are also many important spaces that do not fit into this framework. Here are some examples:

(*a*) $C(\Omega)$, the space of all continuous complex functions on some open set Ω in a euclidean space R^n.
(*b*) $H(\Omega)$, the space of all holomorphic functions in some open set Ω in the complex plane.

(*c*) C_K^∞, the space of all infinitely differentiable complex functions on R^n that vanish outside some fixed compact set K with nonempty interior.

(*d*) The test function spaces used in the theory of distributions, and the distributions themselves.

These spaces carry natural topologies that cannot be induced by norms, as we shall see later. They, as well as the normed spaces, are examples of *topological vector spaces*, a concept that pervades all of functional analysis.

After this brief attempt at motivation, here are the detailed definitions, followed (in Section 1.9) by a preview of some of the results of Chapter 1.

1.4 Vector spaces The letters R and $\mathcal{C}$ will always denote the field of real numbers and the field of complex numbers, respectively. For the moment, let Φ stand for either R or $\mathcal{C}$. A *scalar* is a member of the *scalar field* Φ. A *vector space over* Φ is a set X, whose elements are called vectors, and in which two operations, *addition* and *scalar multiplication*, are defined, with the following familiar algebraic properties:

(*a*) To every pair of vectors x and y corresponds a vector $x + y$, in such a way that

$$x + y = y + x \qquad \text{and} \qquad x + (y + z) = (x + y) + z;$$

X contains a unique vector 0 (the *zero vector* or *origin* of X) such that $x + 0 = x$ for every $x \in X$; and to each $x \in X$ corresponds a unique vector $-x$ such that $x + (-x) = 0$.

(*b*) To every pair (α, x) with $\alpha \in \Phi$ and $x \in X$ corresponds a vector αx, in such a way that

$$1x = x, \qquad \alpha(\beta x) = (\alpha\beta)x,$$

and such that the two distributive laws

$$\alpha(x + y) = \alpha x + \alpha y, \qquad (\alpha + \beta)x = \alpha x + \beta x$$

hold.

The symbol 0 will of course also be used for the zero element of the scalar field.

A *real vector space* is one for which $\Phi = R$; a *complex vector space* is one for which $\Phi = \mathcal{C}$. Any statement about vector spaces in which the scalar field is not explicitly mentioned is to be understood to apply to both of these cases.

If X is a vector space, $A \subset X$, $B \subset X$, $x \in X$, and $\lambda \in \Phi$, the following notations will be used:

$$x + A = \{x + a : a \in A\},$$

$$x - A = \{x - a : a \in A\},$$

$$A + B = \{a + b : a \in A,\ b \in B\},$$

$$\lambda A = \{\lambda a : a \in A\}.$$

In particular (taking $\lambda = -1$), $-A$ denotes the set of all additive inverses of members of A.

A word of warning: With these conventions, it may happen that $2A \neq A + A$ (Exercise 1).

A set $Y \subset X$ is called a *subspace* of X if Y is itself a vector space (with respect to the same operations, of course). One checks easily that this happens if and only if $0 \in Y$ and

$$\alpha Y + \beta Y \subset Y$$

for all scalars α and β.

A set $C \subset X$ is said to be *convex* if

$$tC + (1 - t)C \subset C \qquad (0 \leq t \leq 1).$$

In other words, it is required that C should contain $tx + (1 - t)y$ if $x \in C$, $y \in C$, and $0 \leq t \leq 1$.

A set $B \subset X$ is said to be *balanced* if $\alpha B \subset B$ for every $\alpha \in \Phi$ with $|\alpha| \leq 1$.

A vector space X has *dimension n* (dim $X = n$) if X has a *basis* $\{u_1, \ldots, u_n\}$. This means that every $x \in X$ has a unique representation of the form

$$x = \alpha_1 u_1 + \cdots + \alpha_n u_n \qquad (\alpha_i \in \Phi).$$

If dim $X = n$ for some n, X is said to have *finite dimension.* If $X = \{0\}$, then dim $X = 0$.

Example If $X = \mathbb{C}$ (a one-dimensional vector space over the scalar field $\mathbb{C}$), the balanced sets are: $\mathbb{C}$, the empty set $\varnothing$, and every circular disc (open or closed) centered at 0. If $X = R^2$ (a two-dimensional vector space over the scalar field R), there are many more balanced sets; any line segment with midpoint at (0, 0) will do. The point is that in spite of the well-known and obvious identification of $\mathbb{C}$ with R^2, these two are entirely different as far as their vector space structure is concerned.

1.5 Topological spaces A *topological space* is a set S in which a collection τ of subsets (called *open sets*) has been specified, with the following properties: S is open, $\varnothing$ is open, the intersection of any two open sets is open, and the union of every collection of open sets is open. Such a collection τ is called a *topology on S.* When clarity seems to demand it, the topological space corresponding to the topology τ will be written (S, τ) rather than S.

Here is some of the standard vocabulary that will be used, if S and τ are as above.

A set $E \subset S$ is *closed* if and only if its complement is open. The *closure* $\bar{E}$ of E is the intersection of all closed sets that contain E. The *interior* E° of E is the union of all open sets that are subsets of E. A *neighborhood* of a point $p \in S$ is any open set that contains p. (S, τ) is a *Hausdorff space*, and τ is a *Hausdorff topology*, if distinct points of S have disjoint neighborhoods. A set $K \subset S$ is *compact* if every open cover of K has a finite subcover. A collection $\tau' \subset \tau$ is a *base* for τ if every member of τ (that is, every open set) is a union of members of τ'. A collection γ of neighborhoods of a point $p \in S$ is a *local base at p* if every neighborhood of p contains a member of γ.

If $E \subset S$ and if σ is the collection of all intersections $E \cap V$, with $V \in \tau$, then σ is a topology on E, as is easily verified; we call this the topology that E *inherits* from S.

If a topology τ is induced by a metric d (see Section 1.2) we say that d and τ are *compatible* with each other.

A sequence $\{x_n\}$ in a Hausdorff space X *converges* to a point $x \in X$ (or: $\lim_{n\to\infty} x_n = x$) if every neighborhood of x contains all but finitely many of the points x_n.

1.6 Topological vector spaces Suppose τ is a topology on a vector space X such that

(*a*) *every point of X is a closed set, and*
(*b*) *the vector space operations are continuous with respect to* τ.

Under these conditions, τ is said to be a *vector topology* on X, and X is a *topological vector space*.

Here is a more precise way of stating (*a*): For every $x \in X$, the set $\{x\}$ which has x as its only member is a closed set.

In many texts, (*a*) is omitted from the definition of a topological vector space. Since (*a*) is satisfied in almost every application, and since most theorems of interest require (*a*) in their hypotheses, it seems best to include it in the axioms. [Theorem 1.12 will show that (*a*) and (*b*) together imply that τ is a Hausdorff topology.]

To say that addition is *continuous* means, by definition, that the mapping

$$(x, y) \to x + y$$

of the cartesian product $X \times X$ into X is continuous: If $x_i \in X$ for $i = 1, 2$, and if V is a neighborhood of $x_1 + x_2$, there should exist neighborhoods V_i of x_i such that

$$V_1 + V_2 \subset V.$$

Similarly, the assumption that scalar multiplication is continuous means that the mapping

$$(\alpha, x) \to \alpha x$$

of $\Phi \times X$ into X is continuous: If $x \in X$, α is a scalar, and V is a neighborhood of αx, then for some $r > 0$ and some neighborhood W of x we have $\beta W \subset V$ whenever $|\beta - \alpha| < r$.