Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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continued on page 231

# PREFACE

Our intention in this mongraph is to survey a number of topics related to the study of the continuity and differentiation properties of real functions, in certain generalized senses. There is now a relatively large literature devoted to such subtle concerns but which is accessible and known only to specialists. Since the ideas are essentially simple and the techniques required fairly elementary this literature should be easily absorbed by any interested mathematician, and it is hoped that the presentation here is sufficiently readable and the exposition adequately clear for this purpose.

Probably the reader needs only a familiarity with the usual basics of real analysis (measure, category, density, etc.) in order to follow the arguments. This material is readily available in a variety of textbooks. A better preparation would be to master the books

S.Saks, Theory of the integral,

and

A.M.Bruckner, Differentiation of real functions,

(references [33] and [209] in the bibliography) that most analysts who work in this particular set of topics would surely consider fundamental to our subject. The present monograph continues certain concerns that arise in each of these works.

Part of this material was presented in a series of seminars at the University of California at Santa Barbara in the spring of 1984, during the special year in Real Analysis that was held there. I am particularly grateful and certainly indebted to the participants in that seminar who offered much helpful criticism and indicated numerous improvements. What remains is, doubtless, flawed but much less so than it would have been without the opportunity to meet with so many fine analysts.

In the first chapter is presented a general structure (called here a local system of sets) that can be used to formulate a variety of general notions of limit, continuity, derivative, etc. for real functions. The reason we have chosen this abstract framework is to enable us to clarify and codify the type of arguments that appear in the study. The greater generality itself is not of much interest; the real intention is to lay bare the underlying analysis. Thus it will appear that almost all of the arguments used in the subject reduce to a few general themes, most notably 'intersection conditions' and various thickness conditions (porosity and density usually). All the basic concepts and arguments to be used in the rest of the work are introduced in the first chapter.

The second chapter is a review of the classical material on real cluster sets, from the perspective established in the first chapter. Again the basic arguments here, and elsewhere, will involve appropriate intersection conditions. This cluster set material is attractive and elementary, but does not seem to have been presented in any text to date leaving the interested reader to search through a large number of early references. By restricting ourselves to real cluster sets (i.e. cluster sets for functions of one real variable) we can present an apparently complete survey of the known results.

Chapter three contains a brief account of some general notions of continuity for real functions.

Chapter four gives an introduction to the notion of total variation for a real function. This presentation allows us to include some very classical material on functions of bounded variation, VBG* functions, singular functions, Lebesgue-Stieltjes measures, etc. from a perspective that is not well known and which allows a unified and simple treatment of some apparently diverse ideas.

In Chapter five we have given an account of several classes of monotonicity theorem. This should perhaps be read in conjunction with a study of Chapter XI of Bruckner's monograph ([33,pp.173-198]).

Chapters six and seven are devoted to a number of questions whose theme is the relationship that must hold among different types of generalized derivatives. This includes the well-known Denjoy-Young-Saks theorem and a variety of lesser known variants, both classical and recent.

Finally an Appendix is included that contains a survey on the notion of set porosity. This material too is not well known and can be found, so far, only scattered in the literature. As porosity computations and language appear in many instances in real analysis, this material should be of some use either as a point of reference or as an introduction to the concepts.

There are many more topics that could have been included and which would fit naturally within the framework that we are using. The properties of derivatives and extreme derivates in a generalized sense are currently being studied by some researchers. However this appears still to be in the early stages of development and we have chosen not to report on it. The interested reader should consult the article Bruckner, O'Malley and Thomson [43].

The bibilography contains many articles related to our concerns here, even if not explicitly discussed. It should not be considered complete, however, and the authors whose works I have not mentioned will forgive my oversight.

The notation used is mostly standard nowadays. Thus $A \cup B$, $A \cap B$, and $A \setminus B$ denote the usual union, intersection and difference of the sets $A$ and $B$, while $\mathbb{R}$ denotes the set of real numbers and $\overline{A}$ the closure of the set $A$ in $\mathbb{R}$. However the peculiarities of the word processor have led to the somewhat old fashioned notation

$$\sum_{k=m}^{n} A_k \quad \text{and} \quad \prod_{k=m}^{n} A_k$$

for the union and intersection of a sequence of sets $\{A_k\}$. This notation should present no difficulties. Other special notations are explained in the text and may be found in the index.

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CONTENTS

## CHAPTER ONE: LOCAL SYSTEMS



## CHAPTER TWO: CLUSTER SETS



## CHAPTER THREE: CONTINUITY

CHAPTER FOUR: VARIATION OF A FUNCTION



CHAPTER FIVE: MONOTONICITY



CHAPTER SIX: RELATIONS AMONG DERIVATES



CHAPTER SEVEN: THE DENJOY-YOUNG RELATIONS

APPENDIX: SET POROSITY

# CHAPTER ONE
## LOCAL SYSTEMS

### §1. Introduction.

The basic concepts of elementary analysis, (limits, continuity, derivatives, etc.) have undergone numerous generalizations in order to provide deeper insights into the structure of real functions. The beginning analysis student soon learns that all of the notions of 'limit' that are introduced in a first course may be applied in a one sided version. Thus functions may have right or left hand limits, may be right or left continuous, and have right or left hand derivatives. Many authors have carried these ideas further by considering much more delicate refinements. The most important of the generalizations in this spirit has been the approximate limit and various related notions.

We shall give in this introductory section a definition of the approximate extreme limits that will help motivate the abstract structure that will be introduced in this chapter. Recall that the usual extreme limits of a function $f$ at a point $x_0$, may be written in the following three equivalent formulations; each expresses essentially the same computation but suggests a different perspective on the limit operation. We may define $\limsup_{x\to x_0} f(x)$ as

$$\inf\{\, y : \{\, t : t = x_0 \text{ or } f(t) < y \,\} \text{ is a neighbourhood of } x_0 \,\},$$

or as

$$\inf\{\, y : x_0 \text{ is not a limit point of } \{\, t : t = x_0 \text{ or } f(t) > y \,\}\,\},$$

or, again equivalently,

$$\sup\{\, y : x_0 \text{ is a limit point of } \{\, t : t = x_0 \text{ or } f(t) > y \,\}\,\}.$$

Symmetrically we may define $\liminf_{x\to x_0} f(x)$ as

$$\sup\{\, y : \{\, t : t = x_0 \text{ or } f(t) > y \,\} \text{ is a neighbourhood of } x_0 \,\},$$

or

$$\sup\{\, y : x_0 \text{ is not a limit point of } \{\, t : t = x_0 \text{ or } f(t) < y \,\}\,\},$$

or

$$\inf\{\, y : x_0 \text{ is a limit point of } \{\, t : t = x_0 \text{ or } f(t) < y \,\}\,\}.$$

These expressions have led numerous authors to a number of generalizations of the notion of a limit for a real function. One of the earliest and most useful of these has been the concept of an approximate limit introduced by Denjoy (under that name) and by Khintchine (under the name 'asymptotic limit'). The definitions above for the extreme limits use in the computations the idea that some set is either very thin at $x_0$ (does not have $x_0$ as a limit point) or is very fat at $x_0$ (is a full neighbourhood of $x_0$). The approximate limits are defined in a similar way but take thinness and thickness in the sense of density; thus the thinness is taken as density zero (that is upper, outer density zero) and the thickness is taken as full density 1 (that is lower, inner density 1).

For an arbitrary function f at a point $x_0$ one defines $\text{ap-lim sup}_{x \to x_0} f(x)$ as

$$\inf\{ y : \{ t : t = x_0 \text{ or } f(t) < y \} \text{ has (inner) density 1 at } x_0 \},$$

or

$$\inf\{ y : \{ t : t = x_0 \text{ or } f(t) > y \} \text{ has density zero at } x_0 \},$$

or

$$\sup\{ y : \{ t : t = x_0 \text{ or } f(t) > y \} \text{ has positive density at } x_0 \},$$

and with similar definitions for the lower approximate limit.

Once these definitions had been sufficiently studied it was natural that there would appear studies in which some analogous generalization would be used. Thus one can consider a number of alternative concepts that might might be used for the thinness and thickness notions here, in place of density.

With one sided versions, a spectrum of density type properties (upper, outer, inner, lower, various density values in [0,1], etc.) and with further notions replacing density there has evolved a large literature devoted to the investigation of subtle properties of real functions within the language of certain of these generalized limits. The methods, even when they are similar, have not been systematically described, and the results often appear in a scattered fashion in the literature, and are frequently duplicated. It has become difficult to keep track of the results that have been obtained, the interrelations, and the general patterns.

In this chapter we outline a general structure which can be used to unify and simplify this study, and which will allow us to survey a broad range of results. The abstract notion of a 'local system' will merely replace, formally, the above informal ideas relating to thickness and thinness. The duality between the two notions of thick and thin (expressed above in the observation that an extreme limit is defined as either an inf or a sup relative either to the thick or the thin notion) will be formalized too, and systematically used as a genuine dual notion.

The basic ideas are derived from many sources. The structure represents a mild generalization of the notion of limit used in topology, but adapted to the needs of certain problems in real analysis. Other authors have used similar structures. See, for example, Császár [52], Jędrzejewski [129], Tevy and Bruteanu [230], Świątkowski [226], and Zajíček [268].

## §2. Local systems.

The framework that we shall use for our general notions of limit, continuity, derivation, etc. is a modification of the concept of a filter that is used in topology. Many authors have taken the definition of a filter and relaxed it in various ways, as for example in the notion of 'sieve' or 'quasi-filter'. The setting we use is particularly convenient for expressing a large class of ideas in classical real analysis.

(2.1) DEFINITION. By a local system we mean a family $\mathbf{S}$ such that at each point $x \in \mathbb{R}$ there is given a nonempty collection of sets $\mathbf{S}(x)$ with the following properties :

(i) $\{x\} \notin \mathbf{S}(x)$,

(ii) if $S \in \mathbf{S}(x)$ then $x \in S$,

(iii) if $S_1 \in \mathbf{S}(x)$ and $S_2 \supset S_2$ then $S_2 \in \mathbf{S}(x)$,

(iv) if $S \in \mathbf{S}(x)$ and $\delta > 0$ then

$$S \cap (x - \delta, x + \delta) \in \mathbf{S}(x) .$$

Such a system can be used for a variety of generalized notions and it is this notion that is to be exploited extensively throughout this work. In this section, by way of an introduction, we will present only some of the more basic ideas. If $\mathbf{S}$ is a local system then one can define a notion, relative to $\mathbf{S}$, of a limit of a function $f$ at a point $x$.

(2.2) DEFINITION. Let $\mathbf{S}$ be a local system, let $f$ be a real function and let $x$ be a point of $\mathbb{R}$. Then an $(\mathbf{S})$-limit of $f$ at $x$ is defined as any extended real number $c$,

$$(\mathbf{S})\text{-}\lim_{y \to x} f(y) = c$$

for which it is true that the set

$$\{ t : t = x \text{ or } f(t) \in U_c \}$$

belongs to $\mathbf{S}(x)$ for every neighbourhood $U_c$ of $c$.

There is no requirement here that a limit be unique, and indeed in many applications it is the family of such numbers that is of interest. In such studies we shall denote the collection of all $\mathbf{S}$-limits by the expression

$$(\mathbf{S})\text{-}\Lambda(f, x)$$

and refer to this collection by the term $\mathbf{S}$-cluster set.

The extreme limits relative to a system $\mathbf{S}$ at a point $x$ are defined as

$$(\mathbf{S})\text{-}\limsup_{y \to x} f(y) = \inf \{ y : \{ t : t = x \text{ or } f(t) < y \} \in \mathbf{S}(x) \}$$

and

$$(\mathbf{S})\text{-}\liminf_{y \to x} f(y) = \sup \{ y : \{ t : t = x \text{ or } f(t) > y \} \in \mathbf{S}(x) \} .$$

The example which follows will help make the intention of this definition clear as well as to exhibit the scope of its application. The two systems that are introduced in this example, $\mathbf{S}_0$ and $\mathbf{S}_\infty$, will be used frequently in the sequel and we will use this notation throughout our study.

(2.3) Example. Let $S_0$ denote the system defined at each point x as

$$S_0(x) = \{ S : S \text{ contains an open interval about the point } x \}$$

so that each $S_0(x)$ is precisely the neighbourhood filter at the point x.

We define a closely associated system $S_\infty$ defined at each point x as

$$S_\infty(x) = \{ S : S \text{ contains } x \text{ and has } x \text{ as an accumulation point} \} .$$

The limits defined above then can be easily seen to have the following properties :

$$(S_0)\text{-}\lim_{y \to x} f(y) = \lim_{y \to x} f(y)$$

where the limit is taken in the usual sense; and (more dramatically)

$$(S_\infty)\text{-}\lim_{y \to x} f(y) = c$$

if and only if there is at least one sequence $x_n \to x$ $(x_n \neq x)$ such that the sequence $f(x_n)$ converges to c. In particular it is clear that the limits in the $S_\infty$ sense are normally not unique.

For the extreme limits relative to these two systems we have a remarkable property : these extreme limits are again just the usual extreme limits but with an interesting reversal for the system $S_\infty$.

One has

$$(S_0)\text{-}\limsup_{y \to x} f(y) = (S_\infty)\text{-}\liminf_{y \to x} f(y) = \limsup_{y \to x} f(y) ,$$

and

$$(S_0)\text{-}\liminf_{y \to x} f(y) = (S_\infty)\text{-}\limsup_{y \to x} f(y) = \liminf_{y \to x} f(y) .$$

This feature of the two systems offers us a duality for local systems that we will exploit in section 4. The unusual feature of these limits can be made more intuitive by reviewing the material in the introduction. It is an elementary fact that an extreme limit may be viewed as a sup-inf or equally well as an inf-sup.

The systems $S_0$ and $S_\infty$ do not merely serve as illustrations of the theory, but actually lie at the two extremes permitted by the definition. We may express this in an easily proved lemma.

(2.4) LEMMA. For any local system S, and at any point x, one has invariably the set inclusions

$$S_0(x) \subset S(x) \subset S_\infty(x) .$$

PROOF. By the way that a local system S has been defined it is clear that, at any point x, each neighbourhood $(x - \delta, x + \delta)$ must belong to S(x) and that each set $S \in S(x)$ must have x as a limit point. This is precisely what the two set inclusions say.

## §3. Partial order.

It is natural in this setting to introduce a partial order on the family of local systems, so that we have a lattice structure available. Although we have no great need of an elaborate formal treatment of such structures, it is nonetheless convenient to introduce this language and to exploit it to some degree in the theory.

(3.1) DEFINITION. Let $S_1$ and $S_2$ be two local systems. We will write $S_1 \ll S_2$ if, at every point $x$, there is the inclusion $S_1(x) \subset S_2(x)$.

It is clear that this relation is a partial order on the family of local systems that allows us to rewrite lemma (2.4) above as asserting the relation

$$S_0 \ll S \ll S_\infty .$$

Thus there are two extremes in the partial order. In fact the partial order has the structure of a lattice with the following definitions of the lattice operations.

(3.2) DEFINITION. Let $S_1$ and $S_2$ be two local systems. Then we define the systems $S_1 \vee S_2$ and $S_1 \wedge S_2$ by writing at each point $x$,

$$[S_1 \vee S_2](x) = S_1(x) \cup S_2(x) ,$$

and

$$(S_1 \wedge S_2](x) = S_1(x) \cap S_2(x) .$$

It is easy to check that the terms $S_1 \vee S_2$ and $S_1 \wedge S_2$ are themselves local systems, and that one has the partial order relations

$$S_1 \wedge S_2 \ll S_i \ll S_1 \vee S_2 \qquad (i = 1, 2) .$$

(3.3) Example. Let $S_0$ denote the usual neighbourhood system. We can define right and left versions of this by writing

$$S_0^+(x) = \{ U : U \text{ a right neighbourhood of } x \}$$

and

$$S_0^-(x) = \{ U : U \text{ a left neighbourhood of } x \} .$$

Then the systems $S_0$, $S_0^+$, and $S_0^-$ are related by the assertion

$$S_0 = S_0^+ \wedge S_0^- .$$

Interpreted in terms of limits these lattice operations permit the following expressions.