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Fumi-Yuki Maeda

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Fumi-Yuki MAEDA

February 1980

INTRODUCTION

The classical potential theory is, in a sense, a study of the Laplace equation $\Delta u=0$. It has been clarified that second order elliptic, and some parabolic, partial differential equations share many potential theoretic properties with the Laplace equation. An axiomatic potential theory tries to develop a unified method of treating these equations.

In an axiomatic potential theory, we start with defining a harmonic space $(X, \mathcal{H})$ or $(X, \mathcal{U})$, where X is locally compact Hausdorff space and $\mathcal{H}$ (resp. $\mathcal{U}$) is a sheaf of linear spaces of continuous functions (resp. convex cones of lower semicontinuous functions) which are called "harmonic" (resp. "hyperharmonic"). There are several different kinds of harmonic spaces so far introduced. Among them, the following three are the most well-established:

(a) Brelot's harmonic space $(X, \mathcal{H})$ (see [6], [7], [16], etc.);
(b) Harmonic spaces $(X, \mathcal{H})$ given in Bauer [1] and in Boboc-Constantinescu-Cornea [2];
(c) Harmonic space $(X, \mathcal{U})$ proposed in Constantinescu-Cornea [11].

On any of these harmonic spaces, we can naturally develop a theory of superharmonic functions and potentials, including the Perron-Wiener's method for Dirichlet problems, balayage theory and even integral representation of potentials; and thus a fairly large part of the classical potential theory is covered also by axiomatic theory.

There are, however, some important parts in the classical potential theory which involve the notion of Dirichlet integrals. Due to the fact that only topological notions and some order relations are involved in an axiomatic potential theory, it is impossible to define differentiation of functions without further structures on X. However, it appears that with some reasonable additional structure for $\mathcal{H}$ or $\mathcal{U}$, we can consider a notion corresponding to the gradient of functions on a harmonic space.

As an illustration, let us consider the case where X is an euclidean domain and the harmonic sheaf $\mathcal{H}$ is given by the solutions of the second order differential equation

$$Lu \equiv \Sigma\, a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \Sigma\, b_i \frac{\partial u}{\partial x_i} + cu = 0,$$

where a_{ij}, b_i, c are functions on X with certain regularity and (a_{ij}) is positive definite everywhere on X. Now, we have the following equality:

$$2\, \Sigma\, a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} = L(fg) - fLg - gLf + fgL1.$$

This shows that the function $\Sigma\, a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$ (which, by an abuse of terminology, we call mutual gradient of f and g) can be expressed in terms of L. Therefore, in an axiomatic theory, once a notion corresponding to the operator L is introduced, then mutual gradients of functions can be defined by the above equation.

The purpose of the present lectures is to define the notion of (mutual) gradients of functions on harmonic spaces following the above idea, to show that this notion enjoys some basic properties possessed by the form $\Sigma\, a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$ and to develop some theories involving the notion of Dirichlet integrals in the axiomatic setting.

As a matter of fact, we define the gradients of functions as measures, which we call gradient measures. The definition and the verification of basic properties of gradient measures can be carried out on general harmonic spaces in the sense of Constantinescu-Cornea [11]. Thus, in Part I, we give a theory on general harmonic spaces. Sections §1 and §2 are preparatory and almost all materials in these sections are taken from Part I of [11]. In §3, we give the definition of gradient measures and prove basic properties. This section is nearly identical with [26].

In order to obtain richer results, it becomes necessary for us to restrict ourselves to self-adjoint harmonic spaces. Self-adjointness of a harmonic space is defined by the existence of consistent system of symmetric Green functions (see §4 for details); its prototype is the space given by solutions of the equation of the form $\Delta u = cu$ (c: a function). Thus, in Part II and Part III, we develop our theory on self-adjoint harmonic spaces. The main theme of Part II is Green's formula. In §4, we study Green potentials and in §5 we establish

Green's formula for a harmonic function and a potential both with finite energy. Most of the materials in Part II are taken from [24] (and also [22], [23]), but in these lectures arrangements and proofs are often different from those in [24] and the final form of Green's formula is improved. Part III is devoted to the study of various spaces of Dirichlet-finite or energy-finite functions. Spaces of harmonic functions are mainly discussed in §6. In §7, we consider a functional completion to define those functions which correspond to continuous BLD-functions in the classical theory (cf. [12], [5]). Finally in §8, we shall show that some part of the theory of Royden boundary (cf. [29], [10]) can be also developed in the axiomatic theory and a Neumann problem can be discussed (cf. [19], [20] for the classical case).

Presentations of these lectures are almost self-contained. The biggest exception is that we use without proof the existence of Green functions and the integral representation theorem for potentials on Brelot's harmonic spaces. For these one may refer to [16] and [11]. Some examples are given without detailed explanations. In the Appendix, networks are studied as examples of harmonic spaces.

Terminology and notation

Given a topological space X and a subset A of X, we denote by $\overline{A}$ the closure of A, $\overset{\circ}{A}$ the interior of A and ∂A the boundary of A. For two sets A,B, $A\setminus B$ means the difference set. The family of all open subsets of X is denoted by $\mathcal{O}_X$. A connected open set is called a domain. By a function, we shall always mean an extended real valued function. A continuous function will mean a finite-valued one. The set of all continuous functions on X is denoted by $\mathbf{C}(X)$, and the set of all $f\in \mathbf{C}(X)$ having compact supports in X is denoted by $\mathbf{C}_0(X)$. The support of f is denoted by Supp f. Given a set $A\subset X$ and a class $\mathcal{F}$ of functions on A, we say that $\mathcal{F}$ separates points of A if for any $x,y\in A$, $x\neq y$, there are $f,g\in\mathcal{F}$ satisfying $f(x)g(y)\neq f(y)g(x)$ (with convention $0\cdot\infty=\infty\cdot 0=0$). For two classes $\mathcal{F}_1$, $\mathcal{F}_2$ of finite valued functions, $\mathcal{F}_1-\mathcal{F}_2=\{f_1-f_2 \mid f_1\in\mathcal{F}_1, f_2\in\mathcal{F}_2\}$. For a class $\mathcal{F}$ of functions, $\mathcal{F}^+=\{f\in\mathcal{F} \mid f\geq 0\}$.

For a locally compact space X, a measure on X will mean a (signed) real Radon measure on X. The set of all measures on X is denoted by $\mathcal{M}(X)$. For $\mu\in\mathcal{M}(X)$, μ^+ and μ^- denotes the positive part and the negative part of μ, and $|\mu| = \mu^+ + \mu^-$. For $\mu \in \mathcal{M}(X)$ and $f \in \mathcal{C}(X)$, $f\mu$ is the measure defined by $(f\mu)(\varphi) = \mu(f\varphi)$ for $\varphi \in \mathcal{C}_o(X)$. Restriction of a function or a measure to a set A is denoted by $\cdot|A$.

By a <u>sheaf</u> of functions on X (resp. a sheaf of measures on X), we mean a mapping Φ defined on $\mathcal{O}_X$ satisfying the following three conditions:

(a) for any $U \in \mathcal{O}_X$, $\Phi(U)$ is a set of functions (resp. measures) on U;

(b) if $U, V \in \mathcal{O}_X$, $U \subset V$ and $\varphi \in \Phi(V)$, then $\varphi|V \in \Phi(U)$;

(c) if $(U_\iota)_{\iota \in I}$ is a subfamily of $\mathcal{O}_X$, φ is a function (resp. measure) on $\bigcup_{\iota \in I} U_\iota$ and if $\varphi|U_\iota \in \Phi(U_\iota)$ for all $\iota \in I$, then $\varphi \in \Phi(\bigcup_{\iota \in I} U_\iota)$.

The mapping $\mathcal{M}$: $U \mapsto \mathcal{M}(U)$ is a sheaf, which is called <u>the sheaf of measures</u> on X.

For a locally compact space X with a countable base, a sequence $\{U_n\}$ of relatively compact open sets U_n such that $\overline{U}_n \subset U_{n+1}$ for each n and $\bigcup U_n = X$ is called an <u>exhaustion</u> of X.

CONTENTS

PART I THEORY ON GENERAL HARMONIC SPACES

§1. Harmonic spaces

In this section, we first give the definition of harmonic spaces in the sense of Constantinescu-Cornea [11]. Then, we shall show that Brelot's harmonic spaces and Bauer-Boboc-Constantinescu-Cornea's harmonic spaces are special cases.

Throughout, the base space X is assumed to be a locally compact (Hausdorff) space with countable base.

1-1. Definition of harmonic spaces (cf. [11])

A sheaf $\mathcal{U}$ of functions on X is called a hyperharmonic sheaf if for any $U \in \mathcal{O}_X$, $\mathcal{U}(U)$ is a convex cone of lower semicontinuous $]-\infty,+\infty]$-valued functions on U.

Given a hyperharmonic sheaf $\mathcal{U}$, we define

$$\mathcal{H}_{\mathcal{U}}(U) = \mathcal{U}(U) \cap -\mathcal{U}(U)$$

for each $U \in \mathcal{O}_X$. $\mathcal{H}_{\mathcal{U}}(U)$ is a linear space of continuous functions on U, and $\mathcal{H}_{\mathcal{U}}$ is a sheaf of functions on X, which is called the harmonic sheaf associated with $\mathcal{U}$.

An open set $U \in \mathcal{O}_X$ is called an MP-set for $\mathcal{U}$ if the following minimum principle is valid:

If $f \in \mathcal{U}(U)$, $f \geq 0$ on $U \setminus K$ for some compact set K in X and $\liminf_{x \to \xi,\ x \in U} f(x) \geq 0$ for every $\xi \in \partial U$, then $f \geq 0$ on U.

Let U be an MP-set. For a function φ on ∂U, we define

$$\overline{\mathcal{U}}_{\varphi}^{U} = \left\{ u \in \mathcal{U}(U) \;\middle|\; \begin{array}{l} u \text{ is lower bounded on } U, \\ u \geq 0 \text{ on } U\setminus K \text{ for some compact set } K \text{ in } X, \\ \liminf_{x\to\xi,\ x\in U} u(x) \geq \varphi(\xi) \text{ for every } \xi \in \partial U \end{array} \right\}$$

and $\underline{\mathcal{U}}_{\varphi}^{U} = -\overline{\mathcal{U}}_{-\varphi}^{U}$. Put

$$\overline{H}_{\varphi}^{U} = \inf \overline{\mathcal{U}}_{\varphi}^{U} \quad \text{and} \quad \underline{H}_{\varphi}^{U} = \sup \underline{\mathcal{U}}_{\varphi}^{U}$$

(if $\overline{\mathcal{U}}_\varphi^U = \emptyset$, then $\overline{H}_\varphi^U \equiv +\infty$; if $\underline{\mathcal{U}}_\varphi^U = \emptyset$, then $\underline{H}_\varphi^U \equiv -\infty$). Then, from the definitions, the following properties are easily seen:

$$-\overline{H}_\varphi^U = \underline{H}_{-\varphi}^U , \quad \underline{H}_\varphi^U \le \overline{H}_\varphi^U,$$

$$\overline{H}_{\alpha\varphi} = \alpha\overline{H}_\varphi \quad \text{if } \alpha \text{ is a constant and } \alpha \ge 0,$$

$$\varphi \le \psi \text{ on } \partial U \text{ implies } \overline{H}_\varphi^U \le \overline{H}_\psi^U \text{ and } \underline{H}_\varphi^U \le \underline{H}_\psi^U$$

$$\overline{H}_{\varphi+\psi}^U \le \overline{H}_\varphi^U + \overline{H}_\psi^U, \text{ provided that } +\infty-\infty \text{ or } -\infty+\infty$$

does not occur.

A function φ on ∂U is called <u>resolutive</u> (for U, with respect to $\mathcal{U}$) if $\overline{H}_\varphi^U = \underline{H}_\varphi^U$ and it belongs to $\mathcal{H}_\mathcal{U}(U)$. In this case we denote $\overline{H}_\varphi^U = \underline{H}_\varphi^U$ by H_φ^U. A non-empty open set $U \in \mathcal{O}_X$ is called a <u>resolutive set</u> (with respect to $\mathcal{U}$) if it is an MP-set and every $\varphi \in \mathcal{C}_0(\partial U)$ is resolutive. If U is a resolutive set, then for each $x \in U$ the map $\varphi \mapsto H_\varphi^U(x)$ is a positive linear functional on $\mathcal{C}_0(\partial U)$. Hence, there exists a non-negative measure μ_x^U on ∂U such that

$$H_\varphi^U(x) = \int \varphi d\mu_x^U \quad \text{for all } \varphi \in \mathcal{C}_0(\partial U).$$

This measure μ_x^U is called the <u>harmonic measure</u> of U at x (with respect to $\mathcal{U}$). For a function φ on ∂U, we define $\mu^U\varphi$ by

$$(\mu^U\varphi)(x) = \int^* \varphi d\mu_x^U.$$

In particular, $\mu^U\varphi = H_\varphi^U$ if $\varphi \in \mathcal{C}_0(\partial U)$.

A pair $(X, \mathcal{U})$ of a locally compact space X (with countable base) and a hyperharmonic sheaf $\mathcal{U}$ on X is called a <u>harmonic space</u> if the following four axioms are satisfied:

(P)(Axiom of positivity): For each $x \in X$, there is $U \in \mathcal{O}_X$ and $h \in \mathcal{H}_\mathcal{U}(U)$ such that $x \in U$ and $h(x) \neq 0$.

(R)(Axiom of resolutivity): The resolutive sets with respect to $\mathcal{U}$ form a base of the topology of X.

(C)(Axiom of completeness): For any open set U, a lower semicontinuous $]-\infty,+\infty]$-valued function u on U belongs to $\mathcal{U}(U)$ if, for any relatively compact resolutive set V such that $\overline{V} \subset U$, $\mu^V u \leq u$ on V.

(BC) (Bauer convergence property): For any $U \in \mathcal{O}_X$, if $\{u_n\}$ is a monotone increasing sequence of functions in $\mathcal{H}_{\mathcal{U}}(U)$ and if it is locally uniformly bounded on U, then the limit function $u = \lim_{n\to\infty} u_n$ belongs to $\mathcal{H}_{\mathcal{U}}(U)$.

Remark 1.1. In Axiom (P), the condition $h(x) \neq 0$ may be replaced by $h(x) > 0$. Furthermore, by choosing U small enough, we may require $h > 0$ on U, or even on $\overline{U}$.

Remark 1.2. By (C) and the fact that $\mathcal{U}$ is a sheaf, we have the following: For $U \in \mathcal{O}_X$ and a lower semicontinuous $]-\infty,+\infty]$-valued function u on U, if every $x \in U$ has an open neighborhood V_x such that, whenever V is a relatively compact resolutive set with $\overline{V} \subset V_x$, $\mu^V u \leq u$ on V, then $u \in \mathcal{U}(U)$.

Given a harmonic space $(X, \mathcal{U})$, functions in $\mathcal{H}_{\mathcal{U}}(U)$ are called harmonic on U and functions in $\mathcal{U}(U)$ are called hyperharmonic on U. If $-u$ is hyperharmonic on U, then u is called hypoharmonic on U.

Let $Y \in \mathcal{O}_X$ $(Y \neq \emptyset)$ and let $f \in \mathcal{C}(Y)$ be strictly positive on Y. For each $U \in \mathcal{O}_Y$ put

$$\mathcal{U}_{Y,f}(U) = \{u/f \mid u \in \mathcal{U}(U)\}.$$

Then, $\mathcal{U}_{Y,f}$ is a hyperharmonic sheaf on Y and $(Y, \mathcal{U}_{Y,f})$ is a harmonic space. In case Y=X, $\mathcal{U}_{Y,f}$ will be denoted by $\mathcal{U}_f$; in case $f \equiv 1$, $\mathcal{U}_{Y,f}$ will be denoted by $\mathcal{U}_Y$ and $(Y, \mathcal{U}_Y)$ is called the restriction of $(X, \mathcal{U})$ to Y.

1-2. Brelot's harmonic spaces (cf. [6], [7], [11;Chap.3])

A pair $(X, \mathcal{H})$ of a locally compact space X and a sheaf $\mathcal{H}$ of functions on X is called a Brelot's harmonic space if it satisfies the following three axioms:

Axiom 1. For any $U \in \mathcal{O}_X$, $\mathcal{H}(U)$ is a linear subspace of $\mathcal{C}(U)$.

Axiom 2. Regular domains (with respect to $\mathcal{H}$) form a base of the topology of X.

Here, a domain V in X is called <u>regular</u> with respect to $\mathcal{H}$ if it is relatively compact, $\partial V \neq \phi$ and for each $\varphi \in \mathcal{C}(\partial V)$ there is a unique $u \in \mathcal{C}(\overline{V})$ such that $u|\partial V = \varphi$ and $u|V \in \mathcal{H}(V)$, and such that $\varphi \geq 0$ implies $u \geq 0$.

Axiom 3. If U is a domain in X, $\{u_n\}$ is a monotone increasing sequence of functions in $\mathcal{H}(U)$ and $\{u_n(x_o)\}$ is bounded for some $x_o \in U$, then $u = \lim_{n\to\infty} u_n$ belongs to $\mathcal{H}(U)$.

If V is a regular domain and $\varphi \in \mathcal{C}(\partial V)$, the function $u \in \mathcal{C}(\overline{V})$ satisfying $u|\partial V = \varphi$ and $u|V \in \mathcal{H}(V)$ is denoted by H^U_φ. Then the mapping $\varphi \mapsto H^U_\varphi(x)$ is positive linear on $\mathcal{C}(\partial V)$, so that the harmonic measure μ^V_x of V at $x \in V$ is defined as in the case of resolutive sets, and we define μ^V similarly.

Let $U \in \mathcal{O}_X$ and let u be a lower semicontinuous $]-\infty,+\infty]$-valued function on U. u is called locally hyperharmonic on U (with respect to $\mathcal{H}$) if every $x \in U$ has an open neighborhood V_x such that, whenever V is a regular domain with $\overline{V} \subset V_x$, $\mu^V u \leq u$ on V. Let $\mathcal{U}_{\mathcal{H}}(U)$ be the class of all locally hyperharmonic functions on U (with respect to $\mathcal{H}$). Then it is easy to see that $\mathcal{U}_{\mathcal{H}}$ is a hyperharmonic sheaf on X and $\mathcal{H}_{\mathcal{U}_{\mathcal{H}}} = \mathcal{H}$.

<u>Lemma 1.1.</u> Let $(X, \mathcal{H})$ be a Brelot's harmonic space, $U \in \mathcal{O}_X$ is a domain and $u \in \mathcal{H}(U)$. If $u \geq 0$ on U and $u(x_o) = 0$ for some $x_o \in U$, then $u = 0$.

<u>Proof.</u> Let $u_n = nu$ $(n=1,2,\ldots)$. Then $u_n \in \mathcal{H}(U)$, $\{u_n\}$ is monotone increasing and $\{u_n(x_o)\}$ is bounded. Hence, $\lim_{n\to\infty} u_n \in \mathcal{H}(U)$ by Axiom 3, which implies $u(x) = 0$ for all $x \in U$.

<u>Lemma 1.2.</u> Let $(X, \mathcal{H})$ be a Brelot's harmonic space, V be a regular domain and W be an open set such that $\partial V \cap W \neq \emptyset$. Then $\mu^V_x(\partial V \cap W) > 0$ for all $x \in V$.

<u>Proof.</u> Choose $\varphi \in \mathcal{C}(\partial V)$ such that $0 \leq \varphi \leq 1$ on ∂V, Supp $\varphi \subset \partial V \cap W$ and $\varphi \not\equiv 0$. By the above lemma, $H^V_\varphi(x) > 0$ for all $x \in V$. Hence, $\mu^V_x(\partial V \cap W) \geq H^V_\varphi(x) > 0$.

Lemma 1.3. Let $(X, \mathcal{H})$ be a Brelot's harmonic space, U be a domain in X and $u \in \mathcal{U}_{\mathcal{H}}(U)$. If $u \equiv +\infty$ on a non-empty open set $W \subset U$, then $u \equiv +\infty$ on U.

Proof. Let $U' = \{x \in U \mid u \equiv +\infty$ on a neighborhoof of $x\}$. Then U' is non-empty and open. Suppose $U' \neq U$. Let U_1 be a connected component of U'. Since U is connected, $\partial U' \cap U \neq \emptyset$. Let $x_1 \in \partial U_1 \cap U$. Choose an open set V_1 such that $x_1 \in V_1$ and $\mu^V u \leq u$ for all regular domain V with $\overline{V} \subset V_1$. Choose $y_1 \in V_1 \cap U_1$ and choose a regular domain V such that $x_1 \in V$ and $\overline{V} \subset V_1 \setminus \{y\}$. Since U_1 is connected, $\partial V \cap U_1 \neq \emptyset$. Since $u \equiv +\infty$ on $\partial V \cap U_1$, the previous lemma implies $u(x) \geq \mu^V u(x) = +\infty$ for all $x \in V$. Therefore, $x_1 \in U'$, which is a contradiction. Thus $U' = U$, and the lemma is proved.

Proposition 1.1. Let $(X, \mathcal{H})$ be a Brelot's harmonic space, U be a domain in X, $u \in \mathcal{U}_{\mathcal{H}}(U)$ and $u \geq 0$ on U. If $u(x_o) = 0$ for some $x_o \in U$, then $u = 0$.

Proof. The set $U^+ = \{x \in U \mid u(x) > 0\}$ is open. Suppose $U^+ \neq \emptyset$. Let $u_\infty = \lim_{n\to\infty} nu$. Clearly, $u_\infty \in \mathcal{U}_{\mathcal{H}}(U)$. Since $u_\infty \equiv +\infty$ on U_1, the previous lemma implies $u_\infty \equiv +\infty$, i.e., $u(x) > 0$ for all $x \in U$.

Proposition 1.2. (Minimum principle) Let $(X, \mathcal{H})$ be a Brelot's harmonic space, $U \in \mathcal{O}_X$ and suppose there is $u_o \in \mathcal{U}_{\mathcal{H}}(U) \cap \mathcal{C}(U)$ such that $\inf_U u_o > 0$. Then U is an MP-set with respect to $\mathcal{U}_{\mathcal{H}}$.

Proof: Let $u \in \mathcal{U}_{\mathcal{H}}(U)$ and suppose $u \geq 0$ on $U \setminus K$ for some compact set K in X and $\liminf_{x\to\xi, x\in U} u(x) \geq 0$ for every $\xi \in \partial U$. Put $\alpha \equiv \inf_U (u/u_o)$. Suppose $\alpha < 0$. Then, by the lower semicontinuity of u/u_o and the boundary condition for u, we see that there is $x_o \in U$ such that $\alpha = u(x_o)/u_o(x_o)$. The function $v = u - \alpha u_o$ belongs to $\mathcal{U}_{\mathcal{H}}(U)$, $v \geq 0$ on U and $v(x_o) = 0$. Hence, by Proposition 1.1, $u = \alpha u_o$ on the component U' of U which contains x_o. Since $\alpha < 0$, this fact contradicts our boundary condition for u.

Theorem 1.1. If $(X, \mathcal{H})$ is a Brelot's harmonic space, then $(X, \mathcal{U}_{\mathcal{H}})$ is a harmonic space (in the sense of [11]). Furthermore, any locally hyperharmonic functions are hyperharmonic, i.e., if $u \in \mathcal{U}_{\mathcal{H}}(U)$, then $\mu^V u \leq u$ on V for any regular domain V such that $\overline{V} \subset U$.

Proof. Let V be a regular domain and let $u_o = H_1^V$. Then $u_o \in \mathcal{H}(V)$ and $u_o > 0$ on $\overline{V}$ by Lemma 1.1. Hence, by Axiom 2, Axiom (P) is satisfied. Furthermore, by Proposition 2, we see that any regular domain is an MP-set. Then it is easy to see that $H_\varphi^V = \overline{H}_\varphi^V = \underline{H}_\varphi^V$ for $\varphi \in \mathcal{C}(\partial V)$ for a regular domain V. Thus, a regular domain is resolutive with respect to $\mathcal{U}_{\mathcal{H}}$, and hence Axiom 2 implies Axiom (R). Axiom (C) is an immediate consequence of the definition of $\mathcal{U}_{\mathcal{H}}$ and Axiom (BC) is a weaker form of Axiom 3. The last assertion of the theorem follows from the fact that every regular domain is an MP-set with respect to $\mathcal{U}_{\mathcal{H}}$.

1-3. Bauer-Boboc-Constantinescu-Cornea's harmonic space (cf. [1], [2], [11,Chap.3])

Let X be a locally compact space and $\mathcal{H}$ a sheaf of functions on X satisfying Axioms 1 and 2 of Brelot. As in the case of Brelot's harmonic space, let $\mathcal{U}_{\mathcal{H}}$ be the sheaf of locally hyperharmonic functions with respect to $\mathcal{H}$. Let $\mathcal{H}^*(U)$ be the set of all hyperharmonic functions on U, i.e.,

$$\mathcal{H}^*(U) = \left\{ u \;\middle|\; \begin{array}{l} \text{lower semicontinuous }]-\infty,+\infty]\text{-valued,} \\ \mu^V u \leq u \text{ for } \underline{\text{all}} \text{ regular domain } V \text{ with } \overline{V} \subset U \end{array} \right\}.$$

The pair $(X, \mathcal{H})$ is called a Bauer-Boboc-Constantinescu-Cornea's (or, simply, Bauer's; cf. [11; Chap.3]) harmonic space if, in addition to Axioms 1 and 2 of Brelot, it satisfies Axioms (P), (BC) and the following (S):

(S) For any $x \in X$, there is an open neighborhood V of x for which $\mathcal{H}^*(V)$ separates points of V.

It can be shown that Brelot's harmonic space is a Bauer's harmonic space; we postpone its proof to §2 (Remark 2.2). Here, we shall show that Bauer's harmonic space is a harmonic space (in the sense of [11]).

<u>Lemma 1.4.</u> (Bauer) Let Y be a compact set and $\mathcal{F}$ be a family of lower semicontinuous $]-\infty,+\infty]$-valued functions on Y. Suppose $\mathcal{F}$ separates points of Y and there is $g \in \mathcal{F}$ which is continuous and strictly positive on Y. If $f \in \mathcal{F}$ and $f(x) < 0$ for some $x \in Y$, then there exists $x_o \in Y$ such that $f(x_o) < 0$ and the unit point mass ε_{x_o} at x_o is the only non-negative measure μ on Y satisfying

$$\int u \, d\mu \leq u(x_o) \qquad \text{for all } u \in \mathcal{F}.$$

<u>Proof.</u> Put $\alpha = -\inf_Y (f/g)$. Then $\alpha > 0$ and $f+\alpha g \geq 0$ on Y. Since f/g is lower semicontinuous on the compact set Y,

$$K = \{y \in Y \mid f(y) + \alpha g(y) = 0\}$$

is non-empty. Obviously, K is a compact set and $f < 0$ on K. For each $y \in Y$ put

$$\mathcal{M}_y = \{\mu \in \mathcal{M}^+(Y) \mid \int u \, d\mu \leq u(y) \quad \text{for all } u \in \mathcal{F}\}$$

and

$$\mathcal{O} = \left\{ A \subset Y \;\middle|\; \begin{array}{l} A \neq \phi, \text{ compact,} \\ \text{if } y \in A \text{ and } \mu \in \mathcal{M}_y \text{ then } \mu(Y \backslash A) = 0 \end{array} \right\}.$$

If $y \in K$ and $\mu \in \mathcal{M}_y$, then $0 \leq \int (f + \alpha g) d\mu \leq (f + \alpha g)(y) = 0$, so that $f + \alpha g = 0$ μ-a.e., i.e., $\mu(Y \backslash K)=0$. Hence $K \in \mathcal{O}$. If we consider the converse inclusion relation in $\mathcal{O}$, then it is inductive; in fact if $\mathcal{L} \subset \mathcal{O}$ is linearly ordered, then $A_o = \cap \mathcal{L}$ belongs to $\mathcal{O}$. Therefore, by Zorn's lemma, there is a minimal set $\tilde{A}$ in $\mathcal{O}$ which is contained in K. We shall show that $\tilde{A}$ consists of a single point. Let $u \in \mathcal{F}$ and $u \equiv +\infty$ on $\tilde{A}$. Then $(u + \beta f)(x') < 0$ for some $\beta > 0$ and $x' \in \tilde{A}$. Let $\gamma = -\inf_A[(u + \beta f)/g]$. Then $\gamma > 0$, $u + \beta f + g\gamma \geq 0$ on $\tilde{A}$ and $A' = \{y \in \tilde{A} \mid u(y) + \beta f(y) + \gamma g(y) = 0\}$ is non-empty. By the same argument as for K, we see that $A' \in \mathcal{O}$. Since $\tilde{A}$ is minimal, $A' = \tilde{A}$, which means that $u = -\beta f - \gamma g =$ $= (\alpha\beta - \gamma)g$ on $\tilde{A}$. Thus, every $u \in \mathcal{F}$ is proportional to g on $\tilde{A}$. Since $\mathcal{F}$ separates points of Y, it follows that $\tilde{A}$ consists of a single point: $\tilde{A} = \{x_o\}$. If $\mu \in \mathcal{M}_{x_o}$ then $\mu(Y \backslash \{x_o\}) = 0$, so that $\mu = c\varepsilon_{x_o}$ for some $c \geq 0$. Since $\int u \, d\mu \leq u(x_o)$ for all $u \in \mathcal{F}$ and $f(x_o) < 0$, $g(x_o) > 0$, we see that $c = 1$. Therefore, $\mu = \varepsilon_{x_o}$, i.e., $\mathcal{M}_{x_o} = \{\varepsilon_{x_o}\}$.