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Smooth Ergodic Theory for Endomorphisms

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Comme souvenir de mon amitié avec le Professeur Y. -S. Sun, je voudrais mentionner que c'est lui qui m'a fait connaître la formule de Pesin pour la première fois de ma vie.

Min Qian

Preface

Smooth ergodic theory of deterministic dynamical systems deals with the study of dynamical behaviors relevant to certain invariant measures under differentiable mappings or flows. The relevance of invariant measures is that they describe the frequencies of visits for an orbit and hence they give a probabilistic description of the evolution of a dynamical system. The fact that the system is differentiable allows one to use techniques from analysis and geometry.

The study of transformations and their long-term behavior is ubiquitous in mathematics and the sciences. They arise not only in applications to the real world but also to diverse mathematical disciplines, including number theory, Lie groups, algorithms, Riemannian geometry, etc. Hence smooth ergodic theory is the meeting ground of many different ideas in pure and applied mathematics. It has witnessed a great progress since the pioneering works of Sinai, Ruelle and Bowen on Axiom A diffeomorphisms and of Pesin on non-uniformly hyperbolic systems, and now it becomes a well-developed field.

In this theory, among the major concepts are the notions of Lyapunov exponents and metric entropy. Lyapunov exponent describes the exponential rate of expansion or contraction in certain direction along an orbit. Obviously, positive Lyapunov exponents corresponds to the local instability of trajectories. One of the paradigms of dynamical systems is that the local instability of trajectories may lead to the stochastic behavior of the system. Metric entropy, introduced by Kolmogorov and Sinai, is a purely measure-theoretic invariant, which measures the complexity of the dynamical system generated by iterations of the transformation. It has been studied a good deal in abstract ergodic theory, see [75].

The relationship between these two concepts has always been an important problem. A fundamental result concerning this problem is Margulis-Ruelle inequality, which states that the metric entropy can be bounded from above by the sum of positive Lyapunov exponents (See Chapter II). More deep results can be obtained when the system exhibits certain hyperbolicity. The strongest hyperbolicity occurs in the important class of Axiom A systems. In the ergodic theory of Axiom A diffeomorphisms developed by Sinai [88], Ruelle [76] and Bowen [10, 11], it was shown that for an Axiom A attractor there is a unique invariant measure which is characterized by each of the following properties:

(1) The metric entropy is equal to the sum of positive Lypunov exponents.
(2) The conditional measures of the invariant measure on unstable manifolds are absolutely continuous with respect to the Lebesgue measures on these manifolds.
(3) Lebesgue almost every point in an open neighborhood of the attractor is generic to this measure.

Property (1) is now known as *Pesin's entropy formula* and property (2) is known as *SRB property* of the invariant measure. Each of these properties has been shown to be significant in its own right, but it is also remarkable that they are equivalent to each other in the case of an Axiom A attractor. In mid-seventies, in a series of papers Pesin developed a machinery to study non-uniformly hyperbolic systems [62, 63]. He obtained a general theorem on the existence and the absolute continuity of invariant families of stable and unstable manifolds of a smooth dynamical system, corresponding to its non-zero Lyapunov exponents. Meanwhile, he also studied the ergodic properties of smooth dynamical systems possessing an absolutely continuous invariant measure. The most striking result is that Pesin's entropy formula also holds in this case. Then it was conjectured by Ruelle and later on proved by Ledrappier, Strelcyn and Young that for an invariant measure of a C^2 diffeomorphism, Pesin's entropy formula holds if and only if it satisfies the SRB property [41, 42]. In other words, the equivalence of properties (1) and (2) can hold in a more general circumstance.

The above results have been successfully generalized to several frameworks. Among them are random iterations of diffeomorphisms and deterministic endomorphisms. For random diffeomorphisms, first initiated by Ledrappier and Young [44], Liu and Qian provided a systematic treatment on the subject [51]. However the results for deterministic system are still scattered in the literature.

The main purpose of this monograph is to summarize these results and to provide a *systematic* treatment on this aspect for *deterministic* systems. The novelty of our treatment lies in the fact that we directly consider endomorphisms throughout the monograph. The results for diffeomorphisms can be obtained as a special case. It is interesting to point out that the method developed to attack *Random Dynamical Systems* [38, 44] can be adapted to treat the endomorphism case. It turns out to be the inverse sequence approach known in the dynamical system theory but it has never been detailed into a systematic treatment as one can see in [44]. Therefore, this monograph gives convincing evidence how deterministic theory can be benefited by probabilistic consideration.

The monograph is organized as follows.

We will review some fundamental concepts in Chapter I. Since the whole monograph mainly deals with endomorphisms with the help of inverse limit space, we also provide the simple relations of entropies and Lyapunov exponents between the base dynamical system and the induced dynamical system on the inverse limit space.

Chapter II is devoted to the Margulis-Ruelle inequality. This inequality was first given by Margulis in the case of diffeomorphisms preserving a smooth measure.

The general statement is due to Ruelle [77]. Rigorous proofs are available in several books for the case of diffeomorphisms only, see [32] and [57]. We present a short and rigorous proof for the general C^1 maps in this chapter.

In Chapter III, we study the simplest case—expanding maps. Although Pesin's entropy formula is actually a consequence of the main theorem in Chapter VI, there are still some other nice results under weak conditions. This chapter follows from the work of H.-Y. Hu [27].

In Chapter IV, we study the strongest hyperbolic case, the ergodic theory for Axiom A endomorphisms. This chapter is from the work of Qian and Zhang [72].

Chapter V consists of the study of the structure of unstable manifolds. Since in general the unstable manifold at each point depends on the whole backward orbit, for different orbit there might be different unstable manifold at the same point. Therefore, there is no foliation structure of unstable manifolds in these case. We consider the structure of unstable manifolds in the inverse limit space. The source of this chapter is the work of S. Zhu [100] with slight modification.

In Chapter VI we extend Pesin's entropy formula to the general C^2 endomorphisms. This is done by Liu [46] recently in a different approach.

In Chapter VII we present a formulation of the SRB property for invariant measures of C^2 endomorphisms of a compact manifold via their inverse limit spaces, and then prove that this property is sufficient and necessary for the entropy formula. This is a non-invertible version of the main theorem of [42]. As a nontrivial corollary of this result, an invariant measure of a C^2 endomorphism has this SRB property if it is absolutely continuous with respect to the Lebesgue measure of the manifold. Invariant measures having this SRB property also exist on Axiom A attractors of C^2 endomorphisms. Comparing with the case of diffeomorphisms, the major difficulty arises from non-invertibility. To overcome this deficiency, the inverse limit space has to be introduced. Notice that, when the inverse limit space is introduced, one can compare a full orbit with a sample orbit from random iteration of diffeomorphisms. Keep this in mind, with some necessary modifications, many ideas and techniques developed for the random diffeomorphisms, for which a systematic treatment is now available in [51], can be applied to our present study. The result was given by Qian and Zhu in [73], and we provide a detailed presentation in this chapter.

In Chapter VIII, we study the ergodic hyperbolic attractors. This chapter follows from the work of Jiang and Qian [28].

Chapters IX and X may be viewed as the climax of this book. In Chapter IX, we present here a generalized entropy formula for any Borel probability measure invariant under a C^2 endomorphism. It is a non-invertible endomorphisms version of a formula obtained by Ledrappier and Young [43], hence covers theirs as a consequence. The generalized entropy formula relates closely to Eckmann-Ruelle conjecture for the endomorphism version; In Chapter X, we apply this entropy formula to hyperbolic measures preserved respectively by expanding maps and diffeomorphisms, proving Eckmann-Ruelle conjecture in these two situations. These two chapters are rewritten from the work of Qian and Xie [71] and Liu and Xie [54]. The proof of Eckmann-Ruelle conjecture (for diffeomorphisms) was first presented by Barreira et al [7] (see also [64, pp. 279–292]); our proof is slightly different from theirs and seems more accessible.

In Appendix A, we show that Pesin entropy formula still holds true for C^2 random endomorphisms if the sample measures of the invariant measure are smooth. This result covers those obtained by Pesin [63] for C^2 diffeomorphisms, Liu [46] for C^2 endomorphisms, Ledrappier and Young [44] for i.i.d. random diffeomorphisms and Liu [48] for two-sided stationary random endomorphisms.

In Appendix B, we present a large deviation theorem, it was included as one chapter in the first draft of the manuscript. Since the presentation is not self-contained, we prefer to shift it to the end as Appendix B. This part of the manuscript was prepared by Y. Zhao, see [53].

We would like to thank Prof. P.-D. Liu for very useful discussions. We are deeply indebted to Prof. Ledrappier for his helpful comments on Qian and Xie's work [71]. S. Zhu also wishes to thank Profs. Anatole Katok, Yakov Pesin, Luis Barreira and Wenxian Shen for many helpful conversations during his visit at Pennsylvania State University and Aub ur n University. During the long editing period of this monograph, M. Qian was partially support by the 973 Fund of China for Nonlinear Science and the NSFC of China; and J.-S. Xie is partially support by NSFC Grant No. 10701026.

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Contents

Chapter I
Preliminaries

In this part we review some necessary concepts and results from ergodic theory, which will be frequently used in this monograph.

Throughout this book, M is an m_0-dimensional, smooth, compact and connected Riemannian manifold without boundary. We use $f \in C^r(O,M)$ to denote a C^r map from O to M, where O is an open subset of M, and we call f a C^r *endomorphism* on M if $f \in C^r(M,M)$. We use Tf to denote the tangent map induced by f when $r \geq 1$.

For any compact metrizable space X and continuous map $T : X \to X$, We use $\mathscr{M}_T(X)$ to denote the set of T-invariant Borel probability measures on X.

I.1 Metric Entropy

Let X be a compact metrizable space, $T : X \to X$ a continuous map on X, and μ a T-invariant Borel probability measure on X.

For any finite partition $\eta = \{C_i\}$ of X, define the *entropy* of η by

$$H_\mu(\eta) = -\sum_i \mu(C_i) \log \mu(C_i).$$

Let

$$h_\mu(T,\eta) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\eta \wedge T^{-1}\eta \wedge \cdots \wedge T^{-n+1}\eta).$$

Then define the *metric entropy* of T with respect to μ as

$$h_\mu(T) = \sup\{h_\mu(T,\eta) : \eta \text{ is a finite partition of } X\}.$$

For properties of the metric entropy, we refer the reader to [92].

M. Qian et al., *Smooth Ergodic Theory for Endomorphisms*, Lecture Notes in Mathematics 1978, DOI 10.1007/978-3-642-01954-8_I,

I.2 Multiplicative Ergodic Theorem

From Oseledec's theorem we have the following version of Multiplicative Ergodic Theorem for differentiable maps [92].

Theorem I.2.1 *Let f be a C^1 endomorphism on M. Then there exists a Borel subset $\Gamma \subset M$ with $f\Gamma \subset \Gamma$ and $\mu(\Gamma) = 1$ for any $\mu \in \mathscr{M}_f(M)$. Moreover, the following properties hold.*

(1) *There is a measurable integer function $r : \Gamma \to \mathbb{Z}^+$ with $r \circ f = r$.*
(2) *For any $x \in \Gamma$, there are real numbers*

$$+\infty > \lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{r(x)}(x) \geq -\infty,$$

where $\lambda_{r(x)}(x)$ could be $-\infty$.
(3) *If $x \in \Gamma$, there are linear subspaces*

$$V^{(0)}(x) = T_xM \supset V^{(1)}(x) \supset \cdots \supset V^{(r(x))}(x) = \{0\}$$

of T_xM.
(4) *If $x \in \Gamma$ and $1 \leq i \leq r(x)$, then*

$$\lim_{n\to\infty} \frac{1}{n} \log\left|T_x f^n \xi\right| = \lambda_i(x)$$

for all $\xi \in V^{(i-1)}(x) \backslash V^{(i)}(x)$. Moreover,

$$\lim_{n\to\infty} \frac{1}{n} \log\left|\det(T_x f^n)\right| = \sum_{i=1}^{r(x)} \lambda_i(x) m_i(x),$$

where $m_i(x) = \dim V^{(i-1)}(x) - \dim V^{(i)}(x)$ for all $1 \leq i \leq r(x)$.
(5) *$\lambda_i(x)$ is measurably defined on $\{x \in \Gamma \mid r(x) \geq i\}$ and f-invariant, i.e. $\lambda_i(fx) = \lambda_i(x)$.*
(6) *$T_x f\left(V^{(i)}(x)\right) \subset V^{(i)}(fx)$ if $i \geq 0$.*

The numbers $\{\lambda_i(x)\}_{i=1}^{r(x)}$, given by Theorem I.2.1 are called the *Lyapunov exponents* of f at x, and $m_i(x)$ is called the *multiplicity* of $\lambda_i(x)$.

In many cases, we require that system (M, f, μ) satisfies the following integrability condition

$$\log\left|\det(T_x f)\right| \in L^1(M, \mu). \tag{I.1}$$

By Multiplicative Ergodic Theorem, under condition (I.1) we have

$$\int_M \log\left|\det(T_x f)\right| d\mu(x) = \int_\Gamma \sum_{i=1}^{r(x)} \lambda_i(x) m_i(x)\, d\mu(x). \tag{I.2}$$

Define

$$\Gamma_\infty = \left\{ x \in \Gamma \;\middle|\; T_x f \text{ is degenerate or } \lambda_{r(x)}(x) = -\infty \right\}.$$

The integrability condition (I.1) and identity (I.2) imply that

$$\mu(\Gamma_\infty) = 0. \tag{I.3}$$

Let

$$\Gamma' = \Gamma \setminus \bigcup_{n=0}^{\infty} f^{-n}(\Gamma_\infty). \tag{I.4}$$

It is easy to see that $f(\Gamma') \subset \Gamma'$ and for any $x \in \Gamma'$, $T_x f$ is an isomorphism and $\lambda_{r(x)}(x) > -\infty$. From (I.3) we have $\mu(\Gamma') = 1$.

For $x \in M$ and $1 \leq k \leq m_0$, let $(T_x M)^{\wedge k}$ be the k^{th}-*exterior power space* of $T_x M$, namely, $(T_x M)^{\wedge k}$ is the linear space of all linear combinations of elements in $\{\xi_1 \wedge \ldots \wedge \xi_k : \xi_i \in T_x M, 1 \leq i \leq k\}$ in which the following relations hold:

(1) for all $\alpha, \beta \in \mathbb{R}$ and $1 \leq i \leq k$,

$$\begin{aligned} \xi_1 \wedge \cdots \wedge (\alpha \xi_i + \beta \xi_i') \wedge \cdots \wedge \xi_k = {} & \alpha \xi_1 \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_k \\ & + \beta \xi_1 \wedge \cdots \wedge \xi_i' \wedge \cdots \wedge \xi_k \end{aligned}$$

(2) for all $1 \leq i, j \leq k$,

$$\xi_1 \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_j \wedge \cdots \wedge \xi_k = -\xi_1 \wedge \cdots \wedge \xi_j \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_k$$

Obviously, if $\{\xi_i : 1 \leq i \leq m_0\}$ is a basis of $T_x M$, then $\{\xi_{i_1} \wedge \cdots \wedge \xi_{i_k} : 1 \leq i_1 \leq \cdots \leq i_k \leq m_0\}$ is a basis of $(T_x M)^{\wedge k}$. Now, if $\{e_i : 1 \leq i \leq m_0\}$ is an orthonormal basis of $T_x M$, then by letting

$$< e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{j_1} \wedge \cdots \wedge e_{j_k} > \stackrel{\text{def}}{=} \begin{cases} 1 \text{ if } (i_1, \cdots, i_k) = (j_1, \cdots, j_k) \\ 0 \qquad \text{otherwise} \end{cases}$$

we can define an inner product $< \cdot, \cdot >$ on $(T_x M)^{\wedge k}$, and it is clearly independent of the choice of the orthonormal basis $\{e_i : 1 \leq i \leq m_0\}$. We shall denote also by $|\cdot|$ the norm on $(T_x M)^{\wedge k}$ induced by this inner product.

If $f : M \to M$ is a C^1 map, we define for $x \in M$ and $1 \leq k \leq m_0$

$$\begin{aligned} (T_x f)^{\wedge k} : (T_x M)^{\wedge k} & \to (T_{fx} M)^{\wedge k} \\ \xi_1 \wedge \cdots \wedge \xi_k & \mapsto (T_x f \xi_1) \wedge \cdots \wedge (T_x f \xi_k) \end{aligned}$$

and define

$$|(T_x f)^{\wedge}| \stackrel{\text{def}}{=} 1 + \sum_{k=1}^{m_0} |(T_x f)^{\wedge k}|.$$

Then an important conclusion from [77] gives

Proposition I.2.2 *Let (M, f, μ) be given. Then we have*

$$\lim_{n\to+\infty} \frac{1}{n} \log |(T_x f^n)^\wedge| = \sum_i \lambda_i(x)^+ m_i(x), \quad \mu - a.e.$$

and

$$\lim_{n\to+\infty} \frac{1}{n} \int \log |(T_x f^n)^\wedge| d\mu = \int \sum_i \lambda_i(x)^+ m_i(x) d\mu.$$

I.3 Inverse Limit Space

Let X be a compact metric space. For any continuous map T on X, let X^T denote the subset of $X^{\mathbb{Z}}$ consisting of all full orbits, i.e.,

$$X^T = \left\{ \tilde{x} = \{x_i\}_{i\in\mathbb{Z}} \,\middle|\, x_i \in X,\ Tx_i = x_{i+1},\ \forall i \in \mathbb{Z} \right\}.$$

Obviously, X^T is a closed subset of $X^{\mathbb{Z}}$ (endowed with the product topology and the metric $d(\tilde{x}, \tilde{y}) = \sum_{i=-\infty}^{+\infty} 2^{-|i|} d(x_i, y_i)$ for $\tilde{x} = \{x_i\}_{i\in\mathbb{Z}}$, $\tilde{y} = \{y_i\}_{i\in\mathbb{Z}} \in X^{\mathbb{Z}}$). X^T is called the *inverse limit space* of system (X, T). Let p denote the natural projection from X^T to X, i.e.,

$$p(\tilde{x}) = x_0, \quad \forall \tilde{x} \in X^T,$$

and $\theta : X^T \to X^T$ as the shift homeomorphism. Clearly the following diagram commutes,

$$\begin{array}{ccc} X^T & \xrightarrow{\theta} & X^T \\ p\big\downarrow & & \big\downarrow p \\ X & \xrightarrow[T]{} & X \end{array}$$

i.e. $p \circ \theta = T \circ p$.

It is a basic knowledge that p induces a continuous map from $\mathcal{M}_\theta(X^T)$ to $\mathcal{M}_T(X)$, usually still denoted by p, i.e. for any θ-invariant Borel probability measure $\tilde{\mu}$ on X^T, p maps it to a T-invariant Borel probability measure $p\tilde{\mu}$ on X defined by

$$p\tilde{\mu}(\varphi) = \tilde{\mu}(\varphi \circ p), \quad \forall \varphi \in C(X).$$

The following proposition guarantees that p is a bijection between $\mathcal{M}_\theta(X^T)$ and $\mathcal{M}_T(X)$.

Proposition I.3.1 *Let T be a continuous map on X. For any T-invariant Borel probability measure μ on X, there exists a unique θ-invariant Borel probability measure $\tilde{\mu}$ on X^T such that $p\tilde{\mu} = \mu$.*

Before providing the proof of the above proposition, we first introduce two elementary lemmas.

Lemma I.3.2 *Let X and Y be two compact metrizable spaces, and $h : X \to Y$ a continuous surjective map. Then for any Borel probability measure μ on Y, there exists a Borel probability measure ν on X such that $h\nu = \mu$.*

Proof. Let

$$W = \{\psi \in C(X) \mid \exists \varphi \in C(Y) \text{ such that } \psi = \varphi \circ h\}.$$

Obviously W is a linear subspace of $C(X)$. Define a bounded linear functional L on W as follows,

$$L\psi = \mu(\varphi), \quad \text{where} \quad \varphi \in C(Y) \text{ such that } \psi = \varphi \circ h.$$

It is easy to see that L is a positive bounded linear functional with $L1 = 1$. By a modification of the Hahn-Banach Theorem L can be extended to a positive bounded linear functional on $C(X)$ preserving the property $L1 = 1$. Then Rieze Representation Theorem implies that there is a Borel probability measure ν on X such that $L\psi = \nu(\psi)$ for all $\psi \in C(X)$. It is easy to verify that $h\nu = \mu$. □

Lemma I.3.3 *Let X and Y be two compact metrizable spaces, and $T : X \to X$ and $S : Y \to Y$ measurable mappings on corresponding spaces. Suppose there is a continuous surjective map $h : X \to Y$ such that $S \circ h = h \circ T$. Then for any S-invariant Borel probability measure μ on Y, there is a T-invariant Borel probability measure ν on X such that $h\nu = \mu$.*

Proof. From Lemma I.3.2, there is a Borel probability measure ν_0 on X such that $h\nu_0 = \mu$. Let

$$\nu_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i \nu_0,$$

and suppose that $\nu_{n_k} \to \nu$ as $n_k \to +\infty$. It is then easy to see that $\nu \in \mathscr{M}_T(X)$ and $h\nu = \mu$. □

We are now ready to prove Proposition I.3.1.

Proof of Proposition I.3.1. Let $X_0 = \bigcap_{n=0}^{\infty} T^n(X)$. Obviously X_0 is a compact subset of X, and $T(X_0) = X_0$, $\mu(X_0) = 1$ for any $\mu \in \mathscr{M}_T(X)$. Therefore $X^T = X_0^T$ and $p : X_0^T \to X_0$ is continuous and surjective. As a consequence of Lemma I.3.3, there is $\tilde{\mu} \in \mathscr{M}_\theta(X^T)$ such that $p\tilde{\mu} = \mu$. Since X^T is a compact subset of $X^{\mathbb{Z}}$, $\tilde{\mu}$ can be uniquely determined by its values on all cylinder sets. For any Borel subsets A_0, $A_1, \ldots, A_n \subset M$, we have

$$\tilde{\mu}([A_0, A_1, \ldots, A_n]) = \mu(A_0 \bigcap T^{-1}A_1 \bigcap \cdots \bigcap T^{-n}A_n),$$

where

$$[A_0, A_1, \ldots, A_n] = \{\tilde{x} \in X^T \mid x_i \in A_i, i = 0, 1, \ldots, n\}$$

is a cylinder set in X^T. This ensures that $\tilde{\mu}$ is uniquely determined by μ. The proof is completed. □

Remark I.1. In the circumstances of Proposition I.3.1, it is not hard to see that $(X^T, \theta, \tilde{\mu})$ is ergodic if and only if (X, T, μ) is ergodic.

The following proposition provides the relationship between the entropies of these two systems.

Proposition I.3.4 *Let $T : X \to X$ be a continuous map on the compact metric space X with an invariant Borel probability measure μ. Let X^T be the inverse limit space of (X, T), θ the shift homeomorphism and $\tilde{\mu}$ the θ-invariant Borel probability measure on X^T such that $p\tilde{\mu} = \mu$. Then*

$$h_\mu(T) = h_{\tilde{\mu}}(\theta). \tag{I.5}$$

Proof. For each $n \in \mathbb{N}$, take a maximal $1/n$-separated set E_n of X. (Recall that a subset E of a metric space (X, d) is an *ε-separated set* of X iff $d(x,y) \geq \varepsilon$ for any distinct points $x, y \in E$. It is called a *maximal ε-separated set* of X if in addition E is maximal, i.e., for any point $x \notin E$ and $y \in E$, $d(x,y) < \varepsilon$. Given a transform $T : X \hookleftarrow$ and a positive integer n, one can define a new metric d_n as

$$d_n(x,y) := \max\left\{d(T^k x, T^k y) : 0 \leq k \leq n\right\}.$$

Then an ε-separated set of (X, d_n) is called an *(n, ε)-separating set* of X.) We define a measurable finite partition $\xi_n = \left\{\xi_n(x) \mid x \in E_n\right\}$ of X such that $\xi_n(x) \subset \overline{\mathrm{Int}(\xi_n(x))}$ and $\mathrm{Int}(\xi_n(x)) = \left\{y \in X \mid d(y,x) < d(y,x_i) \text{ if } x \neq x_i \in E_n\right\}$ for every $x \in E_n$. Clearly $\mathrm{Diam}\,\xi_n \leq 1/n$. By Theorem 8.3 of [92],

$$h_\mu(T) = \lim_{n \to \infty} h_\mu(T, \xi_n). \tag{I.6}$$

Using ξ_n, we may construct a measurable finite partition η_n of X^T by

$$\eta_n = \bigvee_{i=-n}^{n} \theta^i(p^{-1}\xi_n).$$

It is easy to see $\mathrm{Diam}\,\eta_n \to 0$ as $n \to \infty$, thus

$$h_{\tilde{\mu}}(\theta) = \lim_{n \to \infty} h_{\tilde{\mu}}(\theta, \eta_n). \tag{I.7}$$

Notice that θ is invertible, by Theorem 4.12 (vii) of [92] we have

$$h_{\tilde{\mu}}(\theta, \eta_n) = h_{\tilde{\mu}}(\theta, p^{-1}\xi_n) = h_\mu(T, \xi_n).$$

This together with (I.6) and (I.7) yields that identity (I.5) holds. □

In the previous proposition, we see that the entropies of these two systems are in fact identical. Now we consider the relationship between the Lyapunov exponents of these two systems.