Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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H.R. Miller D.C. Ravenel (Eds)

Algebraic Topology

Proceedings, Seattle 1985

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Proceedings of a Workshop held at
the University of Washington, Seattle, 1985

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Preface

The University of Washington hosted a Workshop in Algebraic Topology during the 1984-85 academic year. The program had several components. Four topics courses were offered:

Doug Ravenel, "Im J and the EHP sequence."

Emmanuel Dror Farjoun, "Homotopy and homology theory for diagrams of spaces."

Fred Cohen, "Some aspects of classical homotopy theory."

Mark Mahowald, "The Kervaire invariant."

Some 29 topologists visited Seattle for varying periods of time, mainly in the Winter and Spring quarters. A very active seminar resulted; a schedule follows on pages v and vi.

We gratefully acknowledge support from several sources. The Department of Mathematics focused its resources on it: Fred Cohen, Emmanuel Dror Farjoun, and Mark Mahowald were given visiting positions; four quarters of topics courses with very low nominal enrollment were offered; and the recently donated Milliman endowment underwrote the visit by J. Frank Adams as well as a number of others. The N.S.F. provided travel and per diem support under grant number DMS-8407234.

In this volume we have collected lecture notes from the courses of Fred Cohen and Emmanuel Dror Farjoun, together with papers submitted by participants in the workshop. All have been refereed; we wish to thank the referees for their assistance.

This volume is dedicated to Seattle's children, Helen Beatrice Whalen Cohen and Meier Amir Farjoun.

Schedule of Talks

Jan.	11	D.C. Ravenel	What I know about $BP_*\Omega^2S^{2n+1}$
	16	N.J. Kuhn	Stable splittings of BG
	25	A. Pearlman	Bordism with singularities
Feb.	1	B. McQuistan	$bo_*\Omega^2S^3$
	8	M.J. Hopkins	Nilpotence in stable homotopy, I
	8	S.A. Mitchell	Loop groups, I
	12	M.J. Hopkins	II
	15	S.A. Mitchell	II
	15	M.J. Hopkins	III
	20	M.J. Hopkins	IV
	22	D. Sjerve	Actions of finite groups on Riemann surfaces
	22	J.H. Smith	Splittings constructed from the symmetric group, I
	26	M.J. Hopkins	V
	26	J.H. Smith	II
Mar.	1	M.J. Hopkins	VI
	1	J.H. Smith	III
	4	M.J. Hopkins	VII
	8	M.J. Hopkins	VIII
	12	M.J. Hopkins	IX
	12	J.R. Harper	Cohomology operations and cup products
	15	M.J. Hopkins	X
	15	D. Waggoner	$H_*(\Omega^2SU(n))$
	22	J.R. Harper	Co H-spaces and self-maps
Apr.	3	J.F. Adams	Extensions of the Segal conjecture
	5	J.F. Adams	Equivariant analogues of the Adams spectral sequence
	8	A. Zabrodsky	Fixed points and homotopy fixed points
	10	J.F. Adams	Classifying spaces revisited
	10	N. Yagita	$BP_*(X)$ and $H_*(X;\mathbf{Z}/p)$
	12	J.F. Adams	Problems in the topology of Lie groups and finite H-spaces with classifying spaces.
	15	E. Devinatz	Extended powers in the Adams spectral sequence and the proof of the nilpotence conjecture
	17	J.D.S. Jones	Cyclic homology, I
	19	J.D.S. Jones	II
	22	J.D.S. Jones	III
	24	H. Mui	Homology operations derived from modular coinvariants

	25	R.L. Cohen	The cyclic groups, loop spaces, and K-theory, I
	26	H. Mui	II
	26	R.L. Cohen	II
	29	R.L. Cohen	III
	30	R.L. Cohen	IV
May	8	J.A. Neisendorfer	Moore variations on a theme of Selick
	13	J.A. Neisendorfer	On maps between classifying spaces, I
	15	J.A. Neisendorfer	II
	17	L.G. Lewis	Calculating the RO(G) - graded ordinary homology of a point
	20	C.-F. Bödigheimer	Configuration spaces: general ideas
	22	C.-F. Bödigheimer	Configuration spaces: examples
	22	S.B. Priddy	Invariant theory and characteristic classes
	29	D.J. Pengelley	$H_*(BO)$, I
		V. Giambalvo	$H_*(BO)$, II
	31	M.A. Guest	Loop groups
		C.W. Wilkerson	Aguadé's work on rings of invariants
June	3	M.J. Hopkins	The potence of nilpotence
	4	D.M. Davis	The spectrum $(P \wedge BP\langle 2\rangle)_{-\infty}$
	5	D.M. Davis	The stable homtopy types of stunted real projective spaces
	5	W.M. Singer	The Λ-algebra and the homology of the symmetric groups
	10	R.D. Thompson	Products of unstable elements suspending to ImJ,I
	12	R.D. Thompson	II
	12	J.C. Moore	Lemaire's proof of Halperin's theorem, and the application of McGibbon and Wilkerson
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	14	F. Raymond	Manifolds on which only tori can act
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Contents

A course in some aspects of classical homotopy theory

F.R. Cohen*

These notes are based on a course in classical homotopy theory given during the 1985 emphasis year in Topology at the University of Washington. The material here is expository and expresses some prejudices of the author; all of it is either in the literature or known to the experts.

The main direction of these notes is based on the Whitehead product and the classical distributivity law. The divisibility of the Whitehead product, the so-called strong form of the Kervaire invariant, comes up in several places where we study the difference between the H-space squaring map and the loopings of the degree 2 map on $\Omega^q S^n$. We study the relation to P. Selick's theorem on the odd primary homotopy groups of S^3 and allied 2-primary decompositions. A smattering of information is given about function spaces together with some remarks about related work of Dickson.

These notes are neither comprehensive nor complete; they are an exposition of some interesting aspects of classical homotopy theory.

We would like to thank Ed Curtis, Haynes Miller, Doug Ravenel, and Steve Mitchell for their kind hospitality; Joe Neisendorfer, Frank Peterson, Paul Selick, Kathleen Whalen, and Helen Beatrice Whalen Cohen for their conversations and help.

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§1: Whitehead products and Samelson products

Throughout this paper spaces are simply-connected unless otherwise stated. Let X and Y be compactly generated Hausdorff spaces with non-degenerate base-point $*$[St] and let F denote the homotopy theoretic fibre of the inclusion $i: X\vee Y \to X\times Y$. Since $\Omega(i)$ is a principal fibration with a cross-section, it follows that $\Omega(X\vee Y)$ is homotopy equivalent to $\Omega X\times\Omega Y\times\Omega F$. Thus $\pi_q(X\vee Y)$ is isomorphic to $\pi_q X\oplus\pi_q Y\oplus\pi_q F$. If $X=S^k$ and $Y=S^n$, then F is $(n+k-2)$-connected and $\pi_{n+k-1}F$ is isomorphic to $\mathbb{Z}$ by inspection of the Serre spectral sequence for i. The Whitehead product $[\iota_k,\iota_n]$ is the element in $\pi_{n+k-1}(S^k\vee S^n)$ given by the image of a choice of generator for $\pi_{n+k-1}F$[Wh]. Notice that $[\iota_k,\iota_n]$ has been defined up to a sign which will be made precise shortly. Furthermore, since the composite

$$S^{n+k-1} \xrightarrow[{[\iota_k,\iota_n]}]{} S^k\vee S^n \to S^k\times S^n$$

gives a long exact sequence in homology by the above calculation, the following proposition is immediate.

Proposition 1.1. The Whitehead product $[\iota_k,\iota_n]$ is the attaching map of the top cell in $S^k\times S^n$ to $S^k\vee S^n$.

Next notice that there is an induced map

$$[\ ,\ ]: \pi_k X\otimes\pi_n X \to \pi_{n+k-1}X$$

with $[\ ,\ ]$ defined to be the composite

$$S^{n+k-1} \xrightarrow{[\iota_k,\iota_n]} S^k\vee S^n \xrightarrow{\alpha\vee\beta} X\vee X \xrightarrow{\text{fold}} X$$

for α in $\pi_k X$ and β in $\pi_n X$.

Useful properties of the Whitehead product are given below and proven in [H_2,MU,NT,T_4,W_2].

Proposition 1.2. The Whitehead product

$$[\ ,\ ]: \pi_k X \otimes \pi_n X \to \pi_{k+n-1} X$$

satisfies the following properties where degree $(\iota_q)=q$.

(1) It is bilinear.

(2) $[\iota_k, \iota_n] = (-1)^{kn}[\iota_n, \iota_k]$.

(3) $(-1)^{jn}[[\iota_j,\iota_k],\iota_n]+(-1)^{jk}[[\iota_k,\iota_n],\iota_j]+(-1)^{kn}[[\iota_n,\iota_j],\iota_k] = 0$.

Instead of deriving these directly, we use methods of G.W. Whitehead [W_2] and H. Samelson [Sa] after listing the following corollary.

Corollary 1.3.

(1) $[\iota_{2n+1}, \iota_{2n+1}]$ has order 2 in $\pi_{4n+1}S^{2n+1}$.

(2) $[\iota_{2n+1}[\iota_{2n+1}, \iota_{2n+1}]] = 0$ in $\pi_{6n+1}S^{2n+1}$.

(3) $3[\iota_{2n}[\iota_{2n}, \iota_{2n}]] = 0$ in $\pi_{6n-2}S^{2n}$.

(4) $[\iota_{2n}[\iota_{2n}[\iota_{2n}, \iota_{2n}]]] = 0$ in $\pi_{8n-3}S^{2n}$.

Remarks 1.4. The element $[\iota_{2n}, \iota_{2n}]$ has infinite order in $\pi_{4n-1}S^{2n}$ as we shall see later. That $[\iota_{2n+1}, \iota_{2n+1}]$ is non-zero in $\pi_{4n+1}S^{2n+1}$ if $n \neq 0,1,3$ follows from work of J.F. Adams [A1]. The so-called "strong form of the Kervaire invariant" is the question whether $[\iota_{2n+1}, \iota_{2n+1}]$ is divisible by 2 when $n=2^k-1$. The element $[\iota_{2n}[\iota_{2n}, \iota_{2n}]]$ is discussed by Toda [T1].

Next assume that G is a loop space, ΩA. Consider the group of pointed homotopy classes of maps $[X_1 \times X_2, \Omega A]$ which is isomorphic to $[\Sigma(X_1 \times X_2), A]$. Since $\Sigma(X_1 \times X_2)$ is homotopy equivalent to $\Sigma X_1 \vee \Sigma X_2 \vee \Sigma(X_1 \wedge X_2)$, and $X_1 \vee X_2 \to X_1 \times X_2 \to X_1 \wedge X_2$ is a cofibration, there is a short exact sequence of groups which is split as sets,

$$1 \to [X_1 \wedge X_2, \Omega A] \to [X_1 \times X_2, \Omega A] \to [X_1 \vee X_2, \Omega A] \to 1.$$

Next, observe that there is a map $\bar{c}\colon \Omega A\times\Omega A \to \Omega A$ given by $\bar{c}(f,g) = ((f\cdot g)f^{-1})g^{-1}$ which when restricted to $\Omega A\vee\Omega A$ is null-homotopic. Thus there is a map $c\colon \Omega A\wedge\Omega A \to \Omega A$ which is unique up to homotopy and which gives a homotopy commutative diagram

$$\begin{array}{ccc} \Omega A\times\Omega A & \xrightarrow{\ \bar{c}\ } & \Omega A \\ \text{quotient}\downarrow & \nearrow_{c} & \\ \Omega A\wedge\Omega A & & \end{array} .$$

Define the Samelson product

$$<\ ,\ >\colon \pi_p\Omega A\otimes\pi_q\Omega A \to \pi_{p+q}\Omega A$$

to be given by the composite

$$S^p\wedge S^q \xrightarrow{\alpha\wedge\beta} \Omega A\wedge\Omega A \xrightarrow{\ c\ } \Omega A.$$

Proposition 1.5. The Samelson product satisfies the following properties.

(1) It is bilinear.

(2) $<\alpha,\beta> = (-1)^{pq+1}<\beta,\alpha>$ for $\alpha\in\pi_p\Omega A$ and $\beta\in\pi_q\Omega A$.

(3) $<\alpha<\beta,\gamma>> = <<\alpha,\beta>,\gamma> + (-1)^{pq}<\beta<\alpha,\gamma>>$ for $\alpha\in\pi_p\Omega A$, $\beta\in\pi_q\Omega A$, and $\gamma\in\pi_r\Omega A$.

Consider the suspension $E\colon X \to \Omega\Sigma X$ and the induced map $ad\colon X\wedge X \to \Omega\Sigma X$ given by $c\cdot(E\wedge E)$. Inductively define $ad^k\colon \wedge_k X \to \Omega\Sigma X$ by $c\cdot(E\wedge ad^{k-1})$. Recall that with field coefficients, $H_*\Omega\Sigma X$ is isomorphic to the tensor algebra $T(\bar{H}_*X)$ as an algebra; the diagonal is induced by the diagonal for X[BS]. Assume that homology groups are taken with field coefficients in the following lemma.

Lemma 1.6. (1) $ad_*(x\otimes y) = x\otimes y-(-1)^{|x||y|}y\otimes x+\sum_{k>2} z_1\otimes\cdots\otimes z_k$.

(2) If x and y are primitive, $ad_*(x\otimes y) = x\otimes y-(-1)^{|x||y|}y\otimes x$.

Thus consider the fundamental classes ι_k and ι_n in $\pi_*\Omega\Sigma(S^k \vee S^n)$. By Lemma 1.6, the Hurewicz image of $<\iota_k,\iota_n>$ is the primitive $\iota_k \otimes \iota_n - (-1)^{kn}\iota_n \otimes \iota_k$. But by inspection of the Serre spectral sequence, this last element is a generator of $H_{k+n}\Omega F \cong \pi_{k+n}\Omega F$. Since ΩF is $(k+n-1)$-connected, $<\iota_k,\iota_n>$ is the adjoint of the Whitehead product up to a sign. Define the sign of $[\iota_{k+1},\iota_{n+1}]$ by setting $[\iota_{k+1},\iota_{n+1}]$ adjoint to $(-1)^k < \iota_k,\iota_n>$. The details in 1.2 are deleted.

Notice that the Hurewicz image of $<\iota_{2n-1},\iota_{2n-1}>$ is $2(\iota_{2n-1})^2$ and this element has infinite order in $H_{4n-2}(\Omega S^{2n};\mathbb{Z})$.

A good reference for homotopy with coefficients is [N1].

<u>Proof of Proposition 1.5</u>: First check that $<k\alpha,\beta> = k<\alpha,\beta>$. Notice that $k<\alpha,\beta>$ is given by the composite $S^p \wedge S^q \xrightarrow{\alpha\wedge\beta} \Omega A \wedge \Omega A \xrightarrow{c} \Omega A \xrightarrow{k} \Omega A$ where k denotes the kth power map. Thus k is the composite $\Omega A \xrightarrow{\Delta^k} \times_k \Omega A \xrightarrow{\mu_k} \Omega A$ where $\Delta^k(x) = (x,\ldots,x)$ and $\mu_k(x_1,\ldots,x_k) = (x_1(x_2(\cdots(x_{k-1},x_k)\cdots))$. Since $\Delta^k : S^n \to \times_k S^n$ factors through the inclusion of the bouquet $\vee_k S^n$ in $\times_k S^n$, there is a homotopy commutative diagram

$$\begin{array}{ccccc}
 & & S^p\wedge S^q & \xrightarrow{c\cdot(\alpha\wedge\beta)} & \Omega A \\
 & \swarrow & \downarrow \Delta^k & & \downarrow \Delta^k \\
\vee_k(S^p\wedge S^q) & \to & \times_k S^p\wedge S^q & \xrightarrow{[c\cdot(\alpha\wedge\beta)]^k} & \times_k \Omega A \\
\downarrow \text{fold} & & & & \downarrow \mu_k \\
S^p\wedge S^q & & \xrightarrow{c\cdot(\alpha\wedge\beta)} & & \Omega A
\end{array}$$

The left hand composite from $S^p \wedge S^q$ to itself is degree k and thus statement (1) follows.

We next prove (2). Let α and β be in elements in $\pi_p\Omega A$ and $\pi_q\Omega A$ respectively. Then $<\alpha,\beta>$ is the composite $S^p \wedge S^q \xrightarrow{\alpha\wedge\beta} \Omega A \wedge \Omega A \xrightarrow{c} \Omega A$. But observe that by the definitions, $\langle\alpha,\beta\rangle$ is homotopic to the composite $S^p\wedge S^q \xrightarrow{\text{switch}} S^q\wedge S^p \xrightarrow{\beta\wedge\alpha} \Omega A\wedge\Omega A \overset{c}{\to} \Omega A \xrightarrow{-1} \Omega A$. Thus (2) follows.

Next recall that if Λ is a group with x,y,z in Λ, then $[x[yz]]\cdot[y[zx]]\cdot[z[xy]] \equiv 1$ modulo commutators of length at least 4[Z]. Consider pointed maps $\alpha\colon S^p \to \Omega A$, $\beta\colon S^q \to \Omega A$ and $\gamma\colon S^t \to \Omega A$. Then the composite $S^p\times S^q\times S^t \to \Omega A$ represented by $[\alpha[\beta\gamma]]+[\beta[\gamma,\alpha]]+[\gamma[\alpha,\beta]]$ is 0 in the group $[S^p\times S^q\times S^t,\Omega A]$ because the diagonal $S^n \to S^n\times S^n$ is null-homotopic in $S^n\wedge S^n$. Now observe that $[\beta[\gamma,\alpha]]$ and $[\gamma[\alpha,\beta]]$ are represented by

$$S^p\wedge S^q\wedge S^t \xrightarrow{\sigma_1} S^q\wedge S^t\wedge S^p \xrightarrow{\beta\wedge\gamma\wedge\alpha} \textstyle\bigwedge_3 \Omega A \xrightarrow{ad^2} \Omega A, \quad \text{and}$$
$$S^p\wedge S^q\wedge S^t \xrightarrow{\sigma_2} S^t\wedge S^p\wedge S^q \xrightarrow{\gamma\wedge\alpha\wedge\beta} \textstyle\bigwedge_3 \Omega A \xrightarrow{ad^2} \Omega A$$

respectively where σ_i is the indicated permutation of coordinates. Thus the following equation

$$[\alpha[\beta\gamma]]+(-1)^{p(q+t)}[\beta[\gamma,\alpha]]+(-1)^{t(p+q)}[\gamma[\alpha,\beta]] = 0$$

is satisfied in the group $[S^{p+q+t},\Omega A]$. Proposition 1.5(3) follows.

<u>Proof of Lemma 1.6:</u> By definition, the following diagram homotopy commutes where $\pi\colon X\times X\to X\wedge X$ is the natural projection:

$$\begin{array}{ccccccc}
X\times X & \xrightarrow{E\times E} & (\Omega\Sigma X)^2 & \xrightarrow{\Delta\times\Delta} & (\Omega\Sigma X)^2\times(\Omega\Sigma X)^2 & \xrightarrow{(1\times -1)\times(1\times -1)} & (\Omega\Sigma X)^4 \\
 & & & & & & \downarrow{\scriptstyle 1\times\text{switch}\times 1} \\
\downarrow{\scriptstyle\pi} & & & & & & (\Omega\Sigma X)^4 \\
 & & & & & & \downarrow{\scriptstyle\mu_4} \\
X\wedge X & & & \xrightarrow[\quad ad \quad]{} & & & \Omega\Sigma X
\end{array}$$

Thus $ad_*(x\otimes y) = \Sigma(-1)^{|x''||y'|}x'\otimes y'\otimes \chi(x'')\otimes\chi(y')$ where $\Delta z = \Sigma z'\otimes z''$ is the coproduct and $\chi=(-1)_*$. Furthermore, it follows from the definition [MM] that $\chi(1)=-1$ and $\Sigma x'\chi(x")=0$ if $|x|>0$. Hence $\chi(x)=-x$ + decomposable elements. The formula in 1.6(1) follows. Notice that if x is primitive, then $\chi(x)=-x$ and so formula (2) follows.

§2: The Hilton-Milnor theorem

Before stating one form of the Hilton-Milnor theorem [H_1,Mr], we point out that it gives a partial description of the group $[\Sigma A, \Sigma X \vee \Sigma Y]$. For example, let $[k]: S^n \to S^n$ denote the degree k map. Since the map $[k]$ is given by $S^n \xrightarrow{\text{pinch}} \underset{k}{\vee} S^n \xrightarrow{\text{fold}} S^n$ if $k \geq 1$, one can use the Hilton-Milnor theorem to study the effect of $[k]$ on the homotopy groups of S^n by factoring the map through the homotopy groups of $\underset{k}{\vee} S^n$.

The Hilton-Milnor theorem gives a specific product decomposition for $\Omega\Sigma(X \vee Y)$. Let $X^{[k]}$ denote the k-fold smash product $\underbrace{X \wedge \cdots \wedge X}_{k}$. Namely there is a homotopy equivalence

$$\theta: \Omega\Sigma X \times \Omega\Sigma(Y \underset{k \geq 1}{\vee} (X^{[k]} \wedge Y)) \to \Omega\Sigma(X \vee Y).$$

The usual statement of the Hilton-Milnor theorem is obtained by iterating the above decomposition to exhibit a specific homotopy equivalence between $\Omega\Sigma(X \vee Y)$ and the weak product $\prod_\alpha \Omega\Sigma(Z_\alpha)$ where Z_α is a smash product of copies of X and Y. We will not need this further precision here. However, it is useful to have a precise description of the map θ.

There are canonical maps $E_X: X \to \Omega\Sigma(X \vee Y)$ and $E_Y: Y \to \Omega\Sigma(X \vee Y)$. Recall the map $c: \Omega A \wedge \Omega A \to \Omega A$ of section 1 inducing the Samelson product. Inductively define maps $ad^k: X^{[k]} \wedge Y \to \Omega\Sigma(X \vee Y)$ by setting $ad^1 = c \cdot (E_X \wedge E_Y)$ and $ad^{k+1} = c \cdot (E_X \wedge ad^k)$. Thus there is a map $ad: Y \underset{k \geq 1}{\vee} (X^{[k]} \wedge Y) \to \Omega\Sigma(X \vee Y)$ which is given by E_Y on Y and by ad^k on $X^{[k]} \wedge Y$. Let $\Omega(\lambda): \Omega\Sigma(Y \underset{k \geq 1}{\vee} (X^{[k]} \wedge Y)) \to \Omega\Sigma(X \vee Y)$ denote the canonical multiplicative extension of ad. Define θ to be the composite

$$\Omega\Sigma X \times \Omega\Sigma(Y \underset{k \geq 1}{\vee} (X^{[k]} \wedge Y)) \xrightarrow{\Omega(i_X) \times \Omega(\lambda)} [\Omega\Sigma(X \vee Y)]^2 \xrightarrow{\mu_2} \Omega\Sigma(X \vee Y)$$

where i_X is the natural inclusion $\Sigma X \to \Sigma X \vee \Sigma Y$.

Theorem 2.1[H_1,Mr,P]. The map θ is a homotopy equivalence.

An immediate consequence is

Proposition 2.2. The homotopy theoretic fibre of the inclusion $\Sigma X \vee \Sigma Y \subset \Sigma X \times \Sigma Y$ is $\Sigma(\Omega\Sigma X)\wedge(\Omega\Sigma Y)$.

That the homotopy theoretic fibre of the inclusion of $A \vee B$ in $A\times B$ is $\Sigma(\Omega A)\wedge(\Omega B)$ for simply-connected A and B is given in [G].

There are several proofs of this theorem. We reproduce the quick proof in [G_1] which does not specifically give the map θ and which is based on the following where we assume that A and B are simply-connected.

Proposition 2.3. The homotopy theoretic fibre of the pinch map $p\colon A\vee B \to A$ is the half-smash product $B\times\Omega A/{*}\times\Omega A = B \ltimes \Omega A$.

Proof. Recall that if $f\colon X \to A$ is any map, then there is a map $\tilde{f}\colon \tilde{X} \to A$ which is a fibration and $\tilde{X}$ is homotopy equivalent to X; the space $\tilde{X}$ is $\{(x,g) \mid x\in X,\ g\colon I \to A,\ g(o)=f(x)\}$ and $\tilde{f}(x,g)=g(1)\}$. Apply this to the pinch map $p\colon A\vee B \to A$ to get $\tilde{p}\colon \widetilde{A\vee B} \to A$. The fibre of $\tilde{p}$,F, is the space $\{(x,g) \mid x\in A\vee B,\ p(x)=g(o),\ \text{and } g(1)={*}\}$ where $*$ is the base-point in A.

Write $F_A=\{(x,g) \mid x\in A,\ p(x)=g(o),\ g(1)={*}\}$ and $F_B = \{(x,g) \mid x\in B,\ p(x)=g(o),\ g(1)={*}\}$. Notice that (1) $F = F_A \cup F_B$, (2) F_B is homeomorphic to $B\times\Omega A$, and (3) F_A is homeomorphic to the path space PA. Next observe that $F_A \cap F_B$ is $\{(x,g) \mid x\in A\cap B,\ p(x)=g(o),\ g(1)={*}\}$ which is ΩA. Thus F is homeomorphic to $(B\times\Omega A)\cup_{\Omega A} PA$. Since PA is contractible and (F,PA) is an NDR pair, the quotient map $F \to F/PA$ is a homotopy equivalence. But F/PA is homeomorphic to $(B\times\Omega A)\cup_{\Omega A} PA/PA$ and this latter space is homeomorphic to $B\times\Omega A/{*}\times\Omega A$. The proposition follows.

Next one has

Lemma 2.4. $\Sigma A\times B/{*}\times B$ is homotopy equivalent to $A\wedge(\Sigma B\vee S^1)$.