Lecture Notes in Mathematics

Thomas Lorenz

Mutational Analysis

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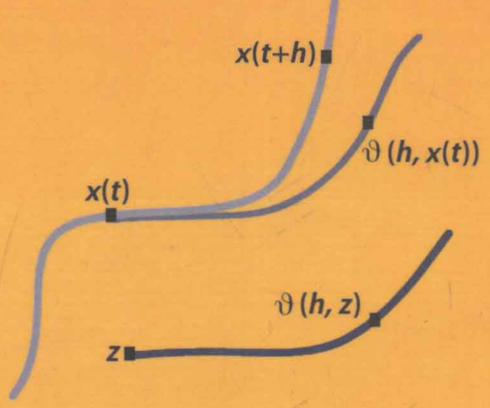
A Joint Framework for Cauchy Problems in and Beyond Vector Spaces



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A Joint Framework for Cauchy Problems In and Beyond Vector Spaces

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Preface

Differential problems should not be restricted to vector spaces in general.

The Main Goal of This Book

Ordinary differential equations play a central role in science. Newton's Second Law of Motion relating force, mass and acceleration is a very famous and old example formulated via derivatives. The theory of ordinary differential equations was extended from the finite-dimensional Euclidean space to (possibly infinite-dimensional) Banach spaces in the course of the twentieth century. These so-called *evolution equations* are based on strongly continuous semigroups.
For many applications, however, it is difficult to specify a suitable normed vector space. Shapes, for example, do not have an obvious linear structure if we dispense with any a priori assumptions about regularity and thus, we would like to describe them merely as compact subsets of the Euclidean space.

Hence, this book generalizes the classical theory of ordinary differential equations beyond the borders of vector spaces. It focuses on the well-posed Cauchy problem in any finite time interval.

In other words, states are evolving in a set (not necessarily a vector space) and, they determine their own evolution according to a given "rule" concerning their current "rate of change" — a form of feedback (possibly even with finite delay). In particular, the examples here do not have to be gradient systems in metric spaces.

The Driving Force of Generalization: Solutions via Euler Method

The step-by-step extension starts in metric spaces and ends up in nonempty sets that are merely supplied with suitable families of distance functions (not necessarily symmetric or satisfying the triangle inequality).
Solutions to the abstract Cauchy problem are usually constructed by means of the Euler method and so the key question for each step of conceptual generalization is: Which aspect of the a priori given structures can be still weakened so that the Euler method does not fail ?

Diverse Examples Have Always Given Directions ... Towards a Joint Framework.

In the 1990s, Jean-Pierre Aubin suggested what he called *mutational equations* and applied them to systems of ordinary differential equations and time-dependent compact subsets of $\mathbb{R}^N$ (equipped with the popular Pompeiu-Hausdorff metric). They are the starting point of this monograph.
Further examples, however, reveal that Aubin's a priori assumptions (about the additional structure of the metric space) are quite restrictive indeed. There is no obvious way for applying the original theory to semilinear evolution equations.

Our basic strategy to generalize mutational equations is simple: Consider several diverse examples successively and, whenever it does not fit in the respective mutational framework, then find some extension for overcoming this obstacle.

Mutational Analysis is definitely not just to establish another abstract term of solution though. Hence, it is an important step to check for each example individually whether there are relations to some more popular meaning (like classical, strong, weak or mild solution).

Here are some of the examples under consideration in this book:

- Feedback evolutions of nonempty compact subsets of $\mathbb{R}^N$
  Application to image segmentation
- Birth-and-growth processes of random closed sets (not necessarily convex)
- Semilinear evolution equations in arbitrary Banach spaces
- Nonlocal parabolic differential equations in noncylindrical domains
- Nonlinear transport equations for Radon measures on $\mathbb{R}^N$
- Structured population model with Radon measures on $\mathbb{R}_0^+$
- Stochastic ordinary differential equations with nonlocal sample dependence

In particular, these examples can now be coupled in systems immediately – due to the *joint* framework of Mutational Analysis. This possibility provides new tools for modelling in future.

The Structure of This Extended Book ... for the Sake of the Reader

This monograph is written as a synthesis of two aims: first, the reader should have quick access to the results of individual interest and second, all mathematical conclusions are presented in detail so that they are sufficiently comprehensible.

Each chapter is elaborated in a quite self-contained way so that the reader has the opportunity to select freely according to the examples of personal interest. Hence some arguments typical for mutational analysis might appear rather frequently, but they are always adapted to the respective framework. Moreover, the proofs are usually collected at the end of each subsection so that they can be skipped easily if wanted. References to results elsewhere in the monograph are usually supplied with page numbers. Each example contains a table that summarizes the choice of basic sets, distances etc. and indicates where to find the main results.

The introductory Chapter 0 summarizes the essential notions and motivates the generalizations in this book. Many of the subsequent conclusions have their origins in §§ 1.1 – 1.6 and so these subsections facilitate understanding the modifications later.

Experience has already taught that such a monograph cannot be written free from any errors or mistakes. I would like to apologize in advance and hope that the gist of both the approach and examples is clear. Comments are very welcome.

Heidelberg and Frankfurt, January 2010 Thomas Lorenz

Acknowledgments

This monograph would not have been completed if I had not benefited from the harmony and the support in my vicinity. Both the scientific and the private aspect are closely related in this context.

Prof. Willi Jäger has been my academic teacher since my very first semester at Heidelberg University. Infected by the "virus" of analysis, I followed his courses, full of insights into mathematical relations. As a part of his scientific support, he drew my attention to set-valued maps quite early and gave me the opportunity to gain experience very autonomously. I would like to express my deep gratitude to Prof. Jäger.
Moreover, I am deeply indebted to Prof. Jean–Pierre Aubin and Hélène Frankowska. Their mathematical influence on me started quite early — as a consequence of their monographs. During three stays at CREA of Ecole Polytechnique in Paris, I benefited from collaborating with them and meeting several colleagues sharing my mathematical interests partly.
Furthermore, I would like to thank all my friends, collaborators and colleagues for the inspiring discussions and observations over time. This list (in alphabetical order) is neither complete nor a representative sample, of course: Zvi Artstein, Robert Baier, Bruno Becker, Hans Belzer, Christel Brüschke, Eva Crück, Roland Dinkel, Herbert-Werner Diskut, Tzanko Donchev, Matthias Gerdts, Piotr Gwiazda, Peter E. Kloeden, Roger Kömpf, Stephan Luckhaus, Anna Marciniak-Czochra, Reinhard Mohr, Jerzy Motyl, José Alberto Murillo Hernández, Janosch Rieger, Ina Scheid, Ursula Schmitt, Roland Schnaubelt, Oliver Schnürer, Jens Starke, Angela Stevens, Martha Stocker, Christina Surulescu, Manfred Taufertshöfer, Friedrich Tomi, Edelgard Weiß-Böhme, Kurt Wolber.

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Contents

Chapter 0
Introduction

Think beyond vector spaces !

0.1 Diverse Evolutions Come Together Under the Same Roof

Many applications consist of diverse components and thus, their mathematical description as functions often starts with long preliminaries (like restrictive assumptions about regularity).
However, *shapes and images are basically sets, not even smooth* (Aubin [10]). This observation leads to the question how to specify models in which both real- or vector-valued functions and shapes are involved. The components usually depend on time and have a huge amount of influence over each other. Consider e.g.

→ A bacterial colony is growing in a nonhomogeneous nutrient broth. For the bacteria, both speed and direction of expansion depend on the nutrient concentration close to the boundary in particular. On the other hand, the nutrient concentration is changing due to consumption and diffusion. (Further applications of set-valued flows in biological modeling are sketched in [57].)

→ A chemical reaction in a liquid is endothermic and depends strongly on the dissolved catalyst. However, this catalyst is forming crystals due to temperature decreasing.

→ In image segmentation, a computer is to detect the region belonging to one and the same object. An example of a so-called region growing method (presented here in § 1.10) is based on constructing time-dependent compact segments so that an error functional is decreasing in the course of time. So far, smoothing effects on the image within the current segment are not taken into account. Basically speaking, it is an example how to extend Lyapunov methods to shape optimization. Further examples can be found in [58, 71].

→ In dynamic economic theory, the results of control theory form the mathematical basis for important conclusions (e.g. [11]). Coalitions of economic agents, technological progress and social effects due to migration, however, have an important impact on the dynamic process that is difficult to quantify by vector-valued functions. Thus, some parameters ought to be described as sets of permissible values and, these subsets might depend on current and former states.

Our goal consists in a joint framework for Cauchy problems of maybe completely different types. In particular, examples of evolving shapes motivate the substantial aspect that we dispense with any (additional) linear structure whenever possible. In other words, the key question here is how to extend ordinary differential equations beyond vector spaces.

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Why We Need a "Nonvectorial" Approach to Evolving Subsets of $\mathbb{R}^N$

In regard to time-dependent subsets of the Euclidean space $\mathbb{R}^N$, several formulations in vector spaces have already been suggested and, they have proved to be very useful. Each of these "detours" via a vector space, however, has conceptual constraints for analytical (but not geometric) reasons. This observation strengthens our interest in describing shape dynamics on the basis of distances (not vectors).

Osher and Sethian, for example, devised new numerical algorithms for fronts propagating with curvature-dependent speed in 1988 [149]. Describing these fronts as level sets of a real-valued auxiliary function leads to equations of motion which resemble Hamilton-Jacobi equations with parabolic right-hand sides. As an essential advantage, their numerical methods can handle topological merging and breaking naturally.
Meanwhile this level set approach has a solid analytical base in the form of viscosity solutions introduced by Crandall and Lions (see e.g. [51, 52], [43, 44], [24, 175]). The viscosity approach, however, has two constraints due to the parabolic maximum principle as its conceptual starting point:

(1.) All these geometric evolutions have to obey the so-called *inclusion principle*, i.e., whenever an initial set contains another initial subset, this inclusion is always preserved while evolving.
De Giorgi even suggested to use this inclusion principle for constructing subsolutions and supersolutions whose values are sets with nonsmooth boundaries — similarly to Perron's method for elliptic partial differential equations [54], [28, 29]. Cardaliaguet extended this notion to set evolutions depending on their nonlocal properties [36, 37, 38]. However, there is no obvious way how to apply these concepts to the easy example that the normal velocity at the boundary is $\frac{1}{1 + \text{set diameter}} > 0$.

(2.) There is no popular theory for the existence of viscosity solutions to *systems* so far.

Replacing viscosity solutions by weak (distributional) solutions to the equations of motion, we always have to neglect any influence of subsets with measure 0.
The distance from a given subset might provide a suitable alternative to the characteristic function of this set, but in general, the distance is just Lipschitz continuous. The choice of the function space is directly related to the regularity of the topological boundary. Delfour and Zolésio pointed out that the *oriented distance function* is often a more appropriate way to characterize a closed subset $K \subset \mathbb{R}^N$, i.e.

$$\mathbb{R}^N \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} \operatorname{dist}(x, K) \overset{\text{Def.}}{=} \inf\{|x-y| \,:\, y \in K\} & \text{if } x \in \mathbb{R}^N \setminus K \\ -\operatorname{dist}(x, \partial K) & \text{if } x \in K. \end{cases}$$

If its restriction to a neighborhood of the topological boundary ∂K belongs to the Sobolev space $W^{2,p}_{\text{loc}}$ with $p > N$, for example, then the well-known embedding theorem of Sobolev implies immediately that the set K is of class $C^{1,\alpha}$ [55, § 5.6.3].

0.2 Some Introductory Examples

0.2.1 A Region Growing Method of Image Segmentation

An important problem of computer vision is the detection of image segments which belong to the same object. Meanwhile many concepts have been developed to find their boundaries on grey-scale images. We mention only few earlier approaches for clarifying the differing aspects here.

The early methods use real-valued "detectors" to check if a point belongs to the contours or not. These criteria mostly depend on large changes of the grey values that are reflected by their gradients. For finding the segments of the same objects, the detected points have to be combined to boundaries, but the algorithms of each step lose more information about the image and, errors can hardly be corrected.
For avoiding this weakness, other methods are based on approximations that are improved in some sense while time is increasing. Active contour models (*snakes*) belong to the popular examples that have been implemented efficiently (e.g. [39, 98, 185]). Restricted to two dimensions, they describe each contour as a Jordan curve that is deformed for minimizing some energy functional. These curves are to approximate the solution of a variational problem while time is increasing.

Many algorithms of image segmentation rely on analytical concepts that use a priori assumptions about regularity. Snakes (in their classical form), for example, are described as Jordan curves that are even twice continuously differentiable. Therefore edges can be found only in some smoothed shapes. Furthermore it is impossible to change the topological properties of the resulting segment.
Meanwhile there have been several suggestions to overcome such weaknesses. Level set methods represent probably the most popular approach [148, 172]. Many of these ideas follow former directions and develop abstract generalizations which are to bridge the gaps. Level set methods, for example, use viscosity solutions of (generalized) Hamilton-Jacobi equations as mentioned before.

On the Way to an Approach (Just) by Means of Set-Valued Analysis

Our goal here is a (hopefully rather simple) region growing method – just on the basis of evolving compact subsets of $\mathbb{R}^N$, i.e. in comparison with many preceding approaches, there are:

- no a priori restrictions on the regularity of final contours and
- no parameterization of boundaries while expanding.

Indeed, searching for the (connected) image segment of an object, the basic graphical notion is only to decide which points belong to the segment. If we omit any additional conditions on regularity we want to detect a compact subset of $\mathbb{R}^N$ and so, the approximations depending on time are described as a set-valued map which associates each time $t \in [0,T[$ with a nonempty compact subset $K(t) \subset \mathbb{R}^N$:

$$K(\cdot) : [0,T[\longrightarrow \mathscr{K}(\mathbb{R}^N).$$

For quantifying the "quality" of the approximations we need a real-valued functional of compact subsets of $\mathbb{R}^N$. We prefer regarding it as "measurement of error" to interpreting it as "energy". The variance of grey values $G|_M$ (restricted to a subset $M \subset \mathbb{R}^N$ with positive Lebesgue measure), for example, gives a quantitative impression of their oscillation in M. More generally, we consider

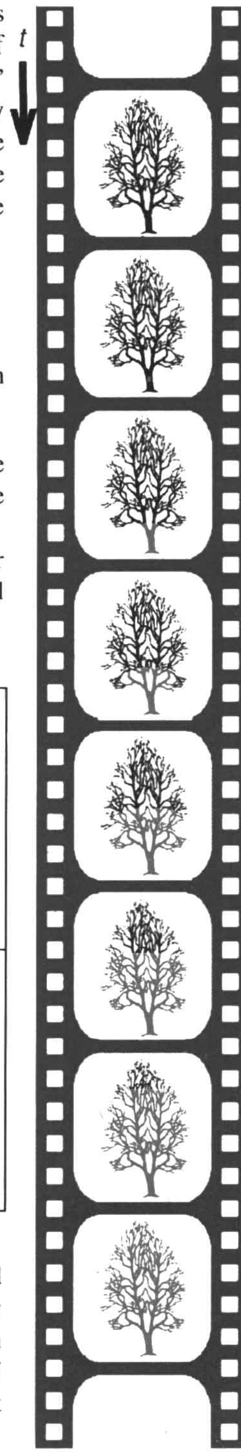
$$\begin{aligned} \Phi : \mathcal{K}(\mathbb{R}^N) &\longrightarrow \mathbb{R}, \\ M &\longmapsto \psi\Big(\mathcal{L}^N(M),\ \int_M G\,dx,\ \int_M G^2\,dx\Big) \end{aligned}$$

with a function $\psi \in C_c^2(]0,\infty[\times \mathbb{R}^2, \mathbb{R})$. The composition

$$\Phi \circ K : \ [0,T[\ \longrightarrow \mathbb{R}$$

is a usual real-valued function which ought to decrease for improving the approximation $K(t) \subset \mathbb{R}^N$ in the course of time.

Finally, the aim of a *region growing* method (in a stricter sense) can be formulated as the following mathematical problem:



Given:	function of grey values $G \in C_c^0(\mathbb{R}^N)$, $N \geq 2$ error functional $\Phi : \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathbb{R}$ s.t. $\Phi(M) = \psi\Big(\mathcal{L}^N(M),\ \int_M G\,dx,\ \int_M G^2\,dx\Big)$ with some $\psi \in C_c^2(]0,\infty[\times \mathbb{R}^2, \mathbb{R})$, initial set $K_0 \in \mathcal{K}(\mathbb{R}^N)$.
Wanted:	$K(\cdot) : [0,T[\longrightarrow \mathcal{K}(\mathbb{R}^N)$ $(T \in]0,\infty])$: (i) $K(0) = K_0$ (ii) $K(s) \subset K(t)$ whenever $s \leq t$ (iii) $K(\cdot)$ continuous w.r.t. Hausdorff metric (iv) $\Phi \circ K(\cdot) : [0,T[\longrightarrow \mathbb{R}$ nonincreasing (v) $M := \bigcup_{0 \leq t < T} K(t)$ is "critical" w.r.t. Φ

The term of a "critical" set in $\mathbb{R}^N$ remains to be specified precisely. Intuitively we are looking for a (not necessarily closed) set $M \subset \mathbb{R}^N$ which cannot be "improved" in an obvious way by decreasing $\Phi \circ K(\cdot)$ and thus, M is the final candidate for the wanted image segment surrounding K_0.

The ansatz for $K(t)$ is based on the notion of prescribing the *speed* of set expansion (but not the direction of the corresponding *velocity*). We can easily avoid restrictions of regularity if this speed function is not specified just on the boundary $\partial K(t)$, but on the whole space $\mathbb{R}^N$. Then for a function $c: [0,T[\times \mathbb{R}^N \longrightarrow [0,\infty[$ given, the initial compact set $K_0 \in \mathscr{K}(\mathbb{R}^N)$ is deformed to

$$K(t) := \{x(t) \in \mathbb{R}^N \mid \exists\, x(\cdot) \in W^{1,1}([0,t],\mathbb{R}^N) : x(0) \in K_0, \\ |x'(s)| \leq c(s,x(s)) \text{ for } \mathscr{L}^1\text{-almost every } s \in [0,t]\}.$$

In other words, this is the *reachable set* of K_0 and the differential inclusion

$$x'(\cdot) \in \mathbb{B}_{c(\cdot,x(\cdot))}(0) \quad \text{a.e. in } [0,t].$$

Here $\mathbb{B}_{c(s,x(s))}(0) \subset \mathbb{R}^N$ denotes the closed ball with center at 0 and radius $c(s,x(s))$. The key criterion for constructing $c(\cdot,\cdot)$ is that the real-valued composition

$$[0,T[\longrightarrow \mathbb{R}, \qquad t \longmapsto \Phi(K(t)) = \psi\Big(\mathscr{L}^N(K(t)), \int_{K(t)} G\,dx, \int_{K(t)} G^2\,dx\Big)$$

should be decreasing. Reynolds Transport Theorem for differential inclusions (in § A.6 on page 476 ff.) provides sufficient conditions on $c(\cdot,\cdot)$ such that each time-dependent argument

$$[0,T[\longrightarrow \mathbb{R}, \qquad t \longmapsto \int_{K(t)} G^k\,dx \qquad (k=0,1,2)$$

is absolutely continuous with the (weak) derivative

$$\tfrac{d}{dt} \int_{K(t)} G^k\,dx = \int_{\partial K(t)} G(x)^k\, c(t,x)\, d\mathscr{H}^{N-1}x$$

Here $\mathscr{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$. Now the chain rule for absolutely continuous functions provides the weak derivative of the relevant composition

$$\tfrac{d}{dt}\,\Phi(K(t)) = \int_{\partial K(t)} \varphi(x, K(t)) \cdot c(t,x)\, d\mathscr{H}^{N-1}x$$

with the coefficient function

$$\varphi(z,M) := \sum_{k=0}^{2} \partial_{k+1}\psi\Big(\mathscr{L}^N(M), \int_M G\,dx, \int_M G^2\,dx\Big) \cdot G(z)^k.$$

The basic idea of solving the segmentation problem is quite easy: The composition $\Phi \circ K(\cdot)$ is nonincreasing if the integrand of its (weak) derivative is nonpositive, i.e. $\varphi(x, K(t)) \cdot c(t,x) \leq 0$ for all $t \in [0,T[,\ x \in \partial K(t)$. As a consequence we get the following criterion of the construction of $c(\cdot,\cdot)$: for all $t \in [0,T[,\ x \in \partial K(t)$,

$$\varphi(x, K(t)) > 0 \implies c(t,x) = 0.$$

Roughly speaking, the sign of $\varphi(\cdot, K(t))$ ought to be locally "stable" because Reynolds Transport Theorem (in § A.6) supposes $c(\cdot,\cdot)$ to be continuous with respect to space (at least). In this context we benefit from the assumption $G \in C_c^0(\mathbb{R}^N)$ for the first time: