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Edited by A. Dold and B. Eckmann

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T. Sunada (Ed.)

Geometry and Analysis on Manifolds

Proceedings, Katata-Kyoto 1987

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Proceedings of the 21st International Taniguchi Symposium held at Katata, Japan, Aug. 23–29 and the Conference held at Kyoto, Aug. 31 – Sept. 2, 1987

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PREFACE

The twentyfirst Taniguchi International Symposium was held at Katata in Shiga prefecture, Japan from August 23rd through 29th, 1987 under the title

Geometry and Analysis on Manifolds.

The symposium was followed by a conference held at the Institute for Mathematical Science in Kyoto University from August 31st till September 2nd under the same title.

The symposium and conference were focused on various aspects of geometric analysis, including spectral analysis of the Laplacian on compact and noncompact Riemannian manifolds, harmonic analysis on manifolds, complex analysis and isospectral problems. The present volume contains expanded versions of most of the invited lectures in Katata and Kyoto.

We, the organizers and all the participants, would like to express our hearty thanks to Mr. Toyosaburo Taniguchi for his support. Thanks are due to Professor Shingo Murakami who, as the coordinator of the Taniguchi International Symposia, guided the organizing committee to the success of the symposium and conference.

Toshikazu Sunada

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PROGRAM OF SYMPOSIUM (KATATA)

Monday, 24. 8.:

9:30 S. Ozawa : Surveys and open problems concerning eigenvalues of the Laplacian on a wildly perturbed domain.

14:00 S. Bando : Ricci flat Kähler metrics on non-compact Kähler manifolds.

15:30 I. Enoki : On compact Kähler manifolds with nonpositive Ricci curvature.

Tuesday, 25. 8.:

9:30 M.T. Anderson : Topology of complete manifolds of non-negative Ricci curvature.

11:00 M.T. Anderson : Space of positive Einstein metrics on compact manifolds.

14:00 G. Besson : On the multiplicity of eigenvalues of the Laplacian.

15:30 T. Mabuchi : Einstein Kähler metrics on toric varieties.

Thursday, 27. 8.:

9:30 H. Donnelly : Decay of eigenfunctions on Riemannian manifolds.

14:00 J. Dodziuk : Examples of Riemann surfaces of large genus with large λ_1.

15:30 T. Sunada : Fundamental groups and spectrum.

Friday, 28. 8.:

9:30 M. Kanai : Rough isometries between open manifolds.

11:00 M. Kanai : Geodesic flows of negatively curved manifolds with smooth stable foliations.

14:00 J.-P. Demailly : Characterization of affine algebraic manifolds by volume and curvature estimates.

15:30 W. Müller : Manifolds with corners and eta-invariants.

Saturday, 29. 8.:

9:30 P. Buser : An upper bound for the number of pairwise isospectral Riemann surfaces.

11:00 P. Buser : A finiteness theorem for the spectrum of Riemann surfaces.

PROGRAM OF CONFERENCE (KYOTO)

Monday, 31. 8.:

10:00 M.T. Anderson : Compactification of complete minimal submanifolds in R^n by Gauss map.

11:10 A. Kasue : Harmonic functions of finite growth on a manifold with asymptotically non-negative curvature.

13:30 J. Dodziuk : Lower bounds for the bottom of the spectrum of negatively curved manifolds.

14:40 A. Katsuda : Density theorem for closed geodesics.

15:50 H. Kawakami : On a construction of complete simply-connected Riemannian manifolds with negative curvature.

Tuesday, 1. 9.:

9:30 G. Besson : Number of bounded states and estimates on some geometric invariants.

10:30 R. Kobayashi : Kähler-Einstein metrics on algebraic varieties of general type.

11:30 J.-P. Demailly : Vanishing theorems and Morse inequalities for complex vector bundles.

14:30 H. Donnelly : Decay of eigenfunctions on Riemannian manifolds.

15:40 K. Sugiyama : Spectrum and a vanishing theorem.

Wednesday, 2. 9.:

9:30 P. Buser : Cayley graphs and planer isospectral domains.

10:30 K. Fukaya : Collapsing of Riemannian manifolds and eigenvalues of the Laplace operator.

11:30 W. Müller : On the generalized Hirzebruch conjecture.

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L^2 HARMONIC FORMS ON COMPLETE RIEMANNIAN MANIFOLDS

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In this paper, we briefly survey selected recent developments and present some new results in the area of L^2 harmonic forms on complete Riemannian manifolds. In light of studies of L^2 cohomology relating to singular varieties, discrete series representations of Lie groups, arithmetic quotients of symmetric spaces among others, our discussion will be rather limited, focussing only on aspects of L^2 harmonic forms in global Riemannian geometry. This paper is partly intended as a completion of the announcement [1]. One may refer to [13] for a previous survey in this area.

Throughout the paper, all manifolds will be connected, complete, oriented Riemannian manifolds, of dimension n.

§1. L^2 cohomology and L^2 harmonic forms.

[1.1] Let Δ_M denote the Laplace-Beltrami operator acting on C^∞ p-forms $C^\infty(\Lambda^p(M))$ on the manifold M. The space of L^2 harmonic p-forms $\mathcal{H}^p_{(2)}(M)$ consists of those forms $\omega \in C^\infty(\Lambda^p(M))$ such that $\Delta_M\omega = 0$ and $\omega \in L^2$, i.e. $||\omega||^2 = \int_M \omega\wedge *\omega < \infty$, where $*: \Lambda^p(M) \to \Lambda^{n-p}(M)$ is the Hodge $*$ operator. The regularity theory for elliptic operators implies that $\mathcal{H}^p_{(2)}(M)$ is a Hilbert space with L^2 inner product.

If M is compact, the Hodge theorem implies that $\mathcal{H}^p_{(2)}(M) \cong H^p(M,\mathbb{R})$ so that these spaces are topological invariants of M. One guiding problem is to understand to what extent this remains true for non-compact manifolds. Note that since Δ and $*$ commute, $*$ induces on isomorphism $\mathcal{H}^p_{(2)}(M) \cong \mathcal{H}^{n-p}_{(2)}(M)$ (representing Poincaré duality for M compact). In particular, $*$ induces an automorphism of $\mathcal{H}^{n/2}_{(2)}(M)$ with $*^2 = (-1)^{n/2}$. Since $*$ is a conformal invariant on n/2-forms, one obtains the important fact that the Hilbert space structure on $\mathcal{H}^{n/2}_{(2)}(M)$ depends only on the conformal structure of M.

A well-known result of Andreotti-Vesentini [10] implies that

$$\mathcal{H}^p_{(2)}(M) = \{\omega\in C^\infty(\Lambda^p(M))\cap L^2 : d\omega=0 \text{ and } \delta w=0\},$$

where d is the exterior derivative and δ its formal adjoint. Thus one has a natural map

$$\mathcal{H}^p_{(2)}(M) \to H^p_{de\mathbb{R}}(M).$$

We now relate the space of L^2 harmonic forms on M to its L^2 cohomology. The simplest definition of L^2 cohomology is

*Partially supported by NSF Grant DMS-8701137

$$H^p_{(2)}(M) = \frac{\ker d_p}{\mathrm{Im}\, d_{p-1}}, \tag{1.1}$$

where $\ker d_p = \{\omega \in C^\infty(\Lambda^p M)\cap L^2 : d\omega = 0\}$, $\mathrm{Im}\, d_{p-1} = \{\eta \in C^\infty(\Lambda^p M)\cap L^2 : d\alpha = \eta$, for some $\alpha \in \mathrm{dom}_{p-1}\}$ with $\mathrm{dom}\, d_{p-1} = \{\alpha \in C^\infty(\Lambda^{p-1}M)\cap L^2 : d\alpha \in L^2\}$. Clearly there is a natural map

$$i_\#: \mathcal{H}^p_{(2)}(M) \to H^p_{(2)}(M)$$

and one says the Strong Hodge Theorem holds if $i_\#$ is an isomorphism. Cheeger [6] has shown that $i_\#$ is always an injection (since M is assumed complete). However, in many cases $i_\#$ is not surjective. For example, it is easily calculated that $\mathcal{H}^1_{(2)}(\mathbb{R}) = \{0\}$, but $H^1_{(2)}(\mathbb{R})$ is infinite dimensional(c.f. [6]).

Define the reduced L^2 cohomology by

$$\overline{H}^p_{(2)}(M) = \frac{\ker \overline{d}_p}{\overline{\mathrm{Im}\, d_{p-1}}} \tag{1.2}$$

where the closure is taken in L^2 and $\overline{d}_p$ is the strong closure of d_p in L^2,, i.e., $\overline{d}_p\alpha = \beta$ if $\exists \alpha_i \in \mathrm{dom}\, d_p$ such that $\alpha_i \to \alpha$ and $d\alpha_i \to \beta$ in L^2. There is a natural surjection $H^p_{(2)}(M) \to \overline{H}^p_{(2)}(M)$ and we have the basic fact [6] that

$$\mathcal{H}^p_{(2)}(M) \cong \overline{H}^p_{(2)}(M) \tag{1.3}$$

for any complete Riemannian manifold M. (1.3) may be viewed as a non-compact Hodge theorem: the reduced L^2 de Rham cohomology of M is isomorphic to the space of L^2 harmonic forms. For a de Rham-type theorem relating $\overline{H}^p_{(2)}(M)$ to the simplicial L^2 cohomology of M, c.f. [12].

An immediate consequence of (1.3) is that $\mathcal{H}^p_{(2)}(M)$ (up to equivalence) depends only on the quasi-isometry class of the metric on M, since it is easily verified that the topology on $\overline{H}^p_{(2)}(M)$ is a quasi-isometry invariant. In particular, if M is a (non-compact) regular cover of a compact manifold N with metric lifted from N, then $\mathcal{H}^p_{(2)}(M)$ does not depend on the metric. One is led to expect in this case that $\mathcal{H}^p_{(2)}(M)$ is a topological invariant of the pair $(N=M/\Gamma,\Gamma)$ in this case. In fact, Dodziuk [11] has shown that the action of Γ on $\mathcal{H}^p_{(2)}(M)$ is a homotopy invarant of (N,Γ) (up to equivalence). In particular, the L^2 Betti numbers $b^p_\Gamma(N) = \dim_\Gamma \mathcal{H}^p_{(2)}(M)$, c.f. §2, are homotopy invariants of N.

In general, one is interested in understanding relations between the topology and geometry of M and the spaces $\mathcal{H}^p_{(2)}(M)$. However, in many cases $\dim \mathcal{H}^p_{(2)}(M)$ has been difficult to estimate, even whether it vanishes or not. Some examples and discussion follow.

[1.2] (i) $\mathcal{H}^0_{(2)}(M) = \mathcal{H}^n_{(2)}(M) = \begin{cases} 0 \text{ if } \mathrm{vol}\, M = \infty \\ \mathbb{R} \text{ if } \mathrm{vol}\, M < \infty \end{cases}$. Further, if M is simply connected, then $\mathcal{H}^1_{(2)}(M)$ is naturally identified with the space of harmonic functions $u: M \to \mathbb{R}$ with finite energy or Dirichlet integral $\int_M |du|^2 < \infty$.

(ii) $\mathcal{H}^p_{(2)}(\mathbb{R}^n, \text{flat}) = 0$, for $0 \leq p \leq n$.

(iii) Let $H^n(-1)$ be the hyperbolic space of constant curvature -1. Then

$$\dim \mathcal{H}^p_{(2)}(H^n(-1)) = \begin{cases} 0 & p \neq n/2. \\ \infty & p = n/2. \end{cases}$$

In fact, the same result holds for any irreducible symmetric space of non-compact type, c.f. [4].

[1.3] Let $(C^\infty_0(M), d)$ denote the de Rham complex of forms of compact support on M and let $H^p_0(M)$ be the corresponding cohomology with compact supports. There is a natural map

$$\Pi: H^p_0(M) \to \mathcal{H}^p_{(2)}(M) \tag{1.4}$$

given by $\Pi[\omega] = P(\omega)$, where P denotes orthogonal projection (in L^2) onto $\mathcal{H}^p_{(2)}(M)$. The weak Hodge theorem of Kodaira [20]

$$L^2 \cong \overline{\text{Im } \delta_{p+1,o}} \oplus \overline{\text{Im } d_{p-1,o}} \oplus \mathcal{H}^p_{(2)}(M) \tag{1.5}$$

implies that Π is well defined on cohomology. A result of Gaffney [16] that $\int_c \omega = \int_c P(\omega)$ for any compact p-cycle c in M and closed p-form $\omega \in C^\infty_0(\Lambda^p M)$ implies that $\ker \Pi \subset Z \equiv \{\omega \in C^\infty_0(\Lambda^p M): d\omega = 0, \int_c \omega = 0$ for all compactly supported p-cycles c$\}$.

For example, suppose $E \to X$ is a vector bundle over the compact n-manifold X with fibre $\mathbb{R}^m$. The Thom class $U \in H^m_{deR}(E)$ of E is represented by a closed m-form in $C^\infty_0(\Lambda^m E)$ and is non-zero (in $H^m_{deR}(E)$) if for instance the Euler class $e(E) \in H^m_{deR}(X)$ is non-zero, since $e(E) = z^* U$, where z is the 0-section. It follows in this case that $P[U] \in \mathcal{H}^m_{(2)}(E)$ represents a non-trivial L^2 harmonic n-form on E, for any complete metric on E.

[1.4] As an application of [1.3] we mention the following. Suppose M is a complete non-compact Riemannian manifold which admits a complete metric with non-negative curvature operator $R: \Lambda^2(TM) \to \Lambda^2(TM)$. The well-known Bochner-Weitzenbock formula implies that for any harmonic p-form ω on M

$$\tfrac{1}{2}\Delta|\omega|^2 = |\nabla\omega|^2 + \langle F_p\omega, \omega\rangle. \tag{1.6}$$

It is shown in [17] that $R \geq 0 \Rightarrow \langle F\omega, \omega\rangle \geq 0$, for any p. If $\omega \in L^2$, then one may integrate (1.6) by parts, and using a standard cutoff function argument it follows that

$$\int_M |\nabla\omega|^2 + \langle F_p\omega, \omega\rangle = 0.$$

Thus, $|\nabla\omega|^2 \equiv 0$, i.e. ω is a parallel p-form on M. It follows that $|\omega|^2 = \text{const}$ and since $\omega \in L^2$ and $\text{vol}(M) = \infty$, we see $\omega = 0$.

This shows that for any complete metric on M with $R \geq 0$, there are no L^2 harmonic p-forms, for any p. Since F_1 is basically the Ricci curvature Ric_M of M, we also see that if $\text{Ric}_M \geq 0$ then there are no non-trivial harmonic 1-forms on M.

Corollary. Let X be a compact n-manifold and $E \to X$ a rank m vector bundle with non-zero Thom class $U \in H^m_{deR}(E)$. Then E admits no complete metric with non-negative curvature operator R.

Proof. By [1.3] above, $\Pi[U]$ is a non-trivial L^2 harmonic m-form on E. If E has a complete metric with $R \geq 0$, we contradict the discussion above.

For example, TS^n admits no complete metric with $R \geqslant 0$ for n even, even though S^n admits metrics with $R > 0$. Note that TS^n admits complete metrics with sectional curvature $K \geq 0$. An open question of Gromoll asks whether every vector bundle over a compact manifold of non-negative sectional curvature admits a complete metric of non-negative sectional curvature.

[1.5] By [1.3], one may produce L^2 harmonic forms on M if M has compactly supported cohomology and homology. In fact, it is easily verified that $H_0(M)$ and Z are topological invariants of M so that $Y \equiv H_0(M)/Z \subset \text{Im}\,\Pi \subset \mathcal{H}_{(2)}(M)$ depends only on the topology of M.

Note further that one may use the * operator to produce harmonic forms not in Im Π. For instance, if $n = 4k+2$, then $*^2 = -1$ on $2k+1$ forms so that $\mathcal{H}^{2k+1}_{(2)}(M)$ is even dimensional. If $E \to X$ is a rank $2k+1$ bundle over X^{2k+1} with non-zero Thom class, then $\text{Im}\,\Pi_{2k+1} = \langle \Pi[U] \rangle$ so that $*\Pi(U) \notin \text{Im}\,\Pi$.

The space $*Y$ is also seen to be a topological invariant so if we set $\mathcal{H}^p_c = \overline{Y_p \cup *Y_{n-p}}$ we may form $\mathcal{H}^p_\infty \equiv \mathcal{H}^p_{(2)}(M)/\mathcal{H}^p_c$. A basic problem is to determine whether, and if so how, $\mathcal{H}^p_\infty$ is characterized by the geometry of M at infinity (up to quasi-isometry).

Based on examples, one expects to a certain extent that if M is "large" at infinity, e.g. strictly negative curvature, exponential volume growth, etc., then M should possess L^2 harmonic p-forms in $\mathcal{H}^p_\infty$, for some p. If M is "small" at infinity, e.g. non-negative curvature, polynomial volume growth, etc., then M may possess no L^2 harmonic p-forms in $\mathcal{H}^p_\infty$.

In this respect, we raise the following questions.

1) If M is a complete, non-compact manifold of positive sectional curvature, is it true that M admits no non-trivial L^2 harmonic p-forms? Of course, the Gromoll-Meyer theorem [19] implies that M is diffeomorphic to $\mathbb{R}^n$ so there is no compactly supported homology. By the remarks above, the question is true if M has non-negative curvature operator or M is quasi-isometric to $\mathbb{R}^n$.

One may ask if the curvature condition above can be weakened. For example, if $H_1(M) = H_{n-1}(M) = 0$ and $M^n, n \geq 3$, is of polynomial volume growth, i.e. $\mathrm{vol}B(r) \leq c \cdot r^k$ for $r \geq 1$, some k, where B(r) is the geodesic r-ball in M, is every L^2 harmonic 1-form on M zero? Similarly, if $H_p(M) = H_{n-p}(M) = 0$, is every L^2 harmonic p-form on M^n, $p \neq \frac{n}{2}$ zero? One must exclude the middle dimensions because of conformal invariance. However, we will see in [3.7] below that these questions are false in every dimension.

2) If M^{2n} is simply connected with curvature $K_M \leq -1$, does M admit a non-trivial L^2 harmonic p-form? One expects the answer should be yes and that one can weaken the hypothesis to $K_M \leq -1$ outside a compact set.

§2. L^2 Betti numbers.

[2.1] The L^2 Betti numbers $b^k_{(2)}(M)$, introduced by Atiyah in [2], are homotopy invariants attached to a compact manifold M defined as follows. Let $\tilde{M}$ be the universal cover of M equipped with a lifted metric so that $\Gamma = \pi_1(M)$ acts by isometries. The Hilbert space $\mathcal{H}^k_{(2)}(\tilde{M})$ thus becomes a Γ-module. Let $P: L^2 \to \mathcal{H}^k_{(2)}(\tilde{M})$ be the orthogonal projection with associated smooth kernel $p(x,y) \in \mathrm{Hom}(\Lambda^k_y(\tilde{M}), \Lambda^k_x(\tilde{M}))$. Note that Γ commutes with Δ and preserves inner products so that it commutes with P. In particular,

$$p(\gamma x, \gamma y) = p(x,y), \ \forall \gamma \in \Gamma. \tag{2.1}$$

If $\{\phi_n\}$ is an orthonormal basis of $\mathcal{H}^k_{(2)}(\tilde{M})$, then one has the sequence of partial sums

$$p_N(x,y) = \sum_1^N \phi_n(x) \otimes \phi^*_n(y),$$

where ϕ^*_n is the dual of ϕ_n defined by the metric in $(\Lambda^p(M))^*$. P_N defines a projection of finite rank on L^2 and the sequence P_N converges strongly in L^2 to P. In particular,

$$p(x,y) = \sum_1^\infty \phi_n(x) \otimes \phi^*_n(y),$$

where the convergence is uniform on compact sets in $\tilde{M}$. In particular,

$$\mathrm{tr}\, p(x,x) = \sum_1^\infty |\phi_n(x)|^2 \geq 0. \tag{2.2}$$

Since p(x,y) is Γ-invariant, tr p(x,x) is a Γ-invariant function on $\tilde{M}$ and thus defines a function on M. Define the L^2 Betti number $b^k_{(2)}(M)$ by

$$b^k_{(2)}(M) = \dim \mathcal{H}^k_{(2)}(\tilde{M}) = \int_M \mathrm{tr}\, p(x,x) dV_x. \tag{2.3}$$

Note that one may define the L^2 Betti numbers $b^k_2(M_1\Gamma)$ of a Galois cover $\bar{M} \to M$ with group Γ in exactly the same way. We list below several properties and results for L^2 Betti numbers. Many of these carry over to the case $b^k_{(2)}(M,\Gamma)$, but for simplicity we assume $\Gamma = \pi_1(M)$.

[2.2] It is clear that the L^2 Betti numbers are largely dependent on the structure of $\pi_1(M)$. Of course, if $\pi_1(M) = 0$, then $b^k_{(2)}(M) = \dim H^k(M,\mathbb{R})$ by the Hodge theorem. Thus, one is really only interested in the case $|\pi_1(M)| = \infty$ and we will always assume this below. The group $\pi_1(M)$ enters in two ways: (i) in determining the basic features of the geometry and topology of $\tilde{M}$ and thus of the space $\mathcal{H}^k_{(2)}(\tilde{M})$, and (ii) via the action of Γ on $\mathcal{H}^k_{(2)}(\tilde{M})$. Roughly speaking, (i) determines whether $b^k_{(2)}(\tilde{M})$ is zero or not, while (ii) leads to the exact value of $b^k_{(2)}(M)$ (assumed positive).

As an example of the π_1-dependence, note that if M_1 and M_2 are compact manifolds with $\tilde{M}_1$ isometric to $\tilde{M}_2$ (or quasi-isometric), then $b^k_{(2)}(M_1) > 0 \Longleftrightarrow b^k_{(2)}(M_2) > 0$. For example, L^2 Betti numbers of all flat manifolds are zero.

[2.3] Since we assume $|\pi_1(M)| = \infty$, the following statements are equivalent:

(i) $b^k_{(2)}(M) > 0$

(ii) $\dim \mathcal{H}^k_{(2)}(\tilde{M}) > 0$, i.e. there is a non-trivial L^2 harmonic k-form on $\tilde{M}$.

(iii) $\dim \mathcal{H}^k_{(2)}(\tilde{M}) = \infty$.

The L^2 Betti numbers behave multiplicatively under finite covers, i.e. if $\tilde{M} \to M$ is an ℓ-sheeted cover, then

$$b^k_{(2)}(\tilde{M}) = \ell \cdot b^k_{(2)}(M).$$

Also, the L^2 Betti numbers satisfy a Poincaré duality $b^k_{(2)}(M) = b^{n-k}_{(2)}(M)$.

[2.4] The L^2 index theorem of Atiyah [2] implies that a number of topological invariants of M can be computed in terms of the L^2 Betti numbers. In particular, for the Euler characteristic $\chi(M)$ one has

$$\chi(M) = \sum_{k=1}^{n} (-1)^k b^k_{(2)}(M). \tag{2.4}$$

For the signature $\sigma(M)$, assuming $n \equiv 0(4)$, one has

$$\sigma(M) = \beta_+^{n/2} - \beta_-^{n/2}, \tag{2.5}$$

where $\beta_\pm^{n/2}$ denotes the Γ-dimensions of the ± 1 eigenspaces of the $*$ operator acting on $\mathcal{H}^{n/2}_{(2)}(\tilde{M})$.

[2.5] It follows easily from standard elliptic theory that one has an estimate of the form

$$\operatorname{tr} P(x,x) \leq c(\operatorname{geo}(\tilde{M})), \tag{2.6}$$

where c is a constant depending on $\sup|K_M|$ and $\inf \operatorname{Inj}(x,\tilde{M})$. Thus, if M collapses with bounded covering geometry, i.e. if M admits a sequence of metrics g_i such that $geo(\tilde{M},g_i) \leq c$ and $vol(M,g_i) \to 0$, then

$$b^k_{(2)}(M) = \int_M tr\ P_i(x,x)dv_i \leq c \cdot vol(M,g_i) \to 0.$$

This observation of Cheeger-Gromov [7] shows that the L^2 Betti numbers are obstructions to the collapse of M with bounded covering geometry. For instance, any manifold M of the form $M = N \times S^1$ collapses with bounded covering geometry. In general, of course, (2.6) leads to the upper bound

$$b^k_{(2)}(M) \leq c(n,k,geo(\tilde{M})) \cdot volM.$$

[2.6] Suppose $\pi_1(M)$ is an amenable group. This may be characterized geometrically by the condition $h_{Ch}(\tilde{M}) = 0$, where $h_{Ch}(\tilde{M}) = \inf_{U \subset\subset M} \frac{vol \partial U}{vol U}$ is the Cheeger isoperimetric constant on $\tilde{M}$ (c.f. [5]). Examples of amenable discrete groups include all nilpotent and solvable groups, as well as groups of subexponential growth with respect to the word metric. On the other hand, fundamental groups (and subgroups) of compact negatively curved manifolds are non-amenable (unless infinite cyclic).

Cheeger-Gromov [9] prove the interesting result that if $\pi_1(M)$ is amenable, then the natural map

$$\rho: H^p_{(2)}(\tilde{M}) \to H^p_{deR}(\tilde{M}) \qquad (2.7)$$

is injective, for any p. This has the following immediate consequence.

(i) $b^1_{(2)}(\tilde{M}) = 0$, i.e. M has no L^2 harmonic 1-forms. Using a result of Brooks [5], this implies that if X is any compact manifold, then $\tilde{X}$ has L^2 harmonic 1-forms only if $\lambda_0 > 0$, where λ_0 is the infinum of the L^2-spectrum of Δ on functions. Similarly, by a result of Lyons-Sullivan [21], $\tilde{X}$ has L^2 harmonic 1-forms only if $\tilde{X}$ carries a non-constant bounded harmonic function.

(ii) If M is a $K(\pi,1)$, then $\tilde{M}$ has no L^2 harmonic p-forms, for any p.

(iii) If $\tilde{M}$ has a non-zero L^2 harmonic p-form, then $\dim H^p_{deR}(M) = \infty$.

For general $\pi_1(M)$, the method of Cheeger-Gromov can easily be shown to imply that

$$b^p_{(2)}(M,\rho) \leq c(geo(M)) \cdot h_{ch}(\tilde{M}),$$

where $b^p_{(2)}(M,\rho) = \dim_\Gamma \ker \rho$. It would be interesting to bound c in terms of weaker invariants, e.g. $\inf Ric_M$ and $diam_M$.

Note that the converse of the result above is false, i.e., there exist compact manifolds with non-amenable $\pi_1(M)$ with all L^2 Betti numbers zero, c.f. [2.5] for example.

Also, (2.7) is not valid for general non-compact manifolds with say, polynomial volume growth, c.f. [3.7].