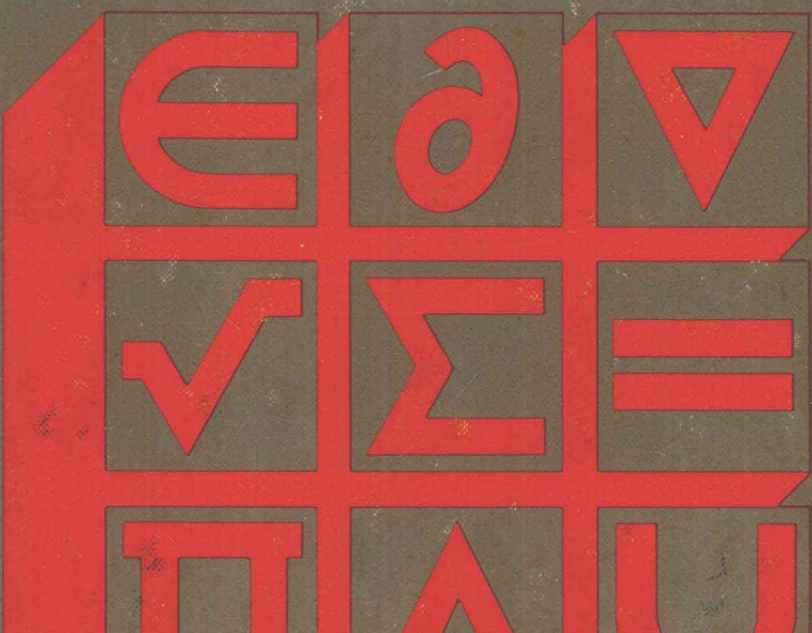
COLLEGE ALGEBRA



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preface

When writing this book, my main objectives were (a) to write a textbook that the student could read and understand, and (b) to *stress problem solving* as a means of obtaining an understanding of the topics that traditionally comprise college algebra courses. To accomplish these goals, I have done the following:

1. Worked out many examples in detail to emphasize both the theoretical and the computational aspects of the topics covered in the text.
2. Following many of the examples (particularly those in the earlier chapters), I have given other problems, similar in nature to the examples, as exercises for the student to work so that he can be sure that he understands the concepts discussed in the examples.
3. Given an extensive selection of problems at the end of each section so that the student can solidify his understanding of the concepts considered in that section.
4. Provided chapter review problems at the end of each chapter so that the student can further solidify his understanding of the topics considered in that chapter.

For the great majority of colleges, this textbook includes more material than can be covered in a one-semester course that meets a total of three classroom hours per week. This extra material was included for two reasons: to permit the instructor to choose only topics that are relevant to his students; and to provide enough material so that a second course in college algebra can be offered by those colleges that wish to.

The problems at the end of each section are arranged so that the student can obtain a balanced coverage of the material in that section by working every third problem. For the convenience of the instructor and the student, the answers to problems 1, 4, 7, 10, . . . and 2, 5, 8, 11, . . . are given immediately after the problem set. The answers to problems 3, 6, 9, 12, . . . are given in a separate answer book, which is available to the instructor upon request.

Canyon, Texas W. G. A.

contents

1

introductory topics

1.1 Real Numbers

Since numbers and their properties are of fundamental importance in the study of algebra, we begin with a discussion of various kinds of numbers.

Natural Numbers and Integers

The first kind of numbers with which you became acquainted were the **counting numbers** (also known as the **natural numbers**) 1, 2, 3, ... Next you were introduced to the number 0 and the negatives, −1, −2, −3, . . . , of the natural numbers. The natural numbers and their negatives, together with the number zero, are known as the **integers.** An integer is said to be an **even integer** if it can be expressed in the form $2k$, where k is an integer. An integer is said to be an **odd integer** if it can be expressed in the form $2k + 1$, where k is an integer. Note that every integer is either even or odd.

EXAMPLE 1

The integers −6, 0 and 10 are examples of even integers, since

$$-6 = 2(-3),$$
$$0 = 2(0),$$

and

$$10 = 2(5).$$

On the other hand, the integers −13, 1, and 11 are examples of odd integers, since

$$-13 = 2(-7) + 1,$$
$$1 = 2(0) + 1,$$

and

$$11 = 2(5) + 1.$$

If a, b, and c are integers with $a \cdot b = c$, then a (and also b) is said to be a **factor** (or **divisor**) of c, and c is said to be a **multiple** of a (and also of b).

EXAMPLE 2

Since $2 \cdot 3 = 6$, 2 and 3 are factors (divisors) of 6, and 6 is a multiple of 2 and of 3.

EXERCISE 1

List all factors of 6.

Answer The factors of 6 are 1, −1, 2, −2, 3, −3, 6, and −6, since $6 = (6)(1) = (-6)(-1) = (3)(2) = (-3)(-2)$.

EXERCISE 2

Is −9 a multiple of 3?

Answer Yes, since $3(-3) = -9$.

A natural number greater than 1 is said to be a **prime number** if it has no natural numbers as divisors except itself and 1. A natural number greater than 1 is said to be a **composite number** if it is not a prime number.

EXAMPLE 3

The natural number 2 is a prime number, since 2 (itself) and 1 are the only divisors of 2 that are natural numbers. Other prime numbers are 3, 5, 7, 11, 13, 17, 19, and so on. The natural number 6 is a composite number, since 6 and 1 are not the only divisors of 6 that are natural numbers. Other such divisors are 2 and 3. Some other composite numbers are 4, 8, 9, 10, 12, 14, 15, 16, 18, and so on.

Every composite number can be expressed as the product of primes in a way that is *unique* except for the order of the factors. For example,

$$12 = 4 \cdot 3 = (2 \cdot 2) \cdot 3 = 2 \cdot 2 \cdot 3$$

and

$$12 = 6 \cdot 2 = (2 \cdot 3) \cdot 2 = 2 \cdot 3 \cdot 2.$$

In each case, the factors are the same. However, the order of the factors is different.

Two integers are said to be **relatively prime** if they have no prime factors in common.

EXAMPLE 4

The prime factors of 10 are 2, 5, and 10, and the prime factors of 21 are 3, 7, and 21. Thus 10 and 21 are relatively prime, since none of the prime factors of 10 and of 21 are the same.

EXERCISE 3

Are 10 and 35 relatively prime?

Answer No. They have a common prime factor, namely 5.

Rational Numbers and Irrational Numbers

A **rational number** is a number that can be expressed in the form p/q, where p and q are integers and $q \neq 0$. For example, $\frac{3}{5}$, $-\frac{2}{7}$, and 2 are rational numbers. Note that 2 is a rational number, since $2 = \frac{2}{1}$.

By using ordinary division, each rational number can be expressed as a **periodic decimal** (i.e., a decimal in which a block of one or more digits in the decimal repeats itself during the division process). Examples are:

$$\frac{4}{5} = .8000\ldots = .8\overline{0},$$
$$\frac{1}{3} = .333\ldots = .\overline{3},$$
$$\frac{2}{11} = .181818\ldots = .\overline{18}.$$

In each of the previous examples, the bar above the decimal is used to indicate the block of numbers that repeats itself.

It is also true that each periodic decimal can be expressed as the quotient of two integers. That is, each periodic decimal represents a rational number.

EXAMPLE 5

Express $3.\overline{24}$ as the quotient of two integers.

Solution Let $x = 3.2424\ldots$. Then $100x = 324.2424\ldots$. Subtracting the sides of the first equation from the corresponding sides of the second equation, we get

$$\begin{array}{rl} 100x = 324.2424\ldots & \\ x = 3.2424\ldots & \text{(subtract)} \\ \hline 99x = 321 & \end{array}$$

from which it follows that

$$x = \frac{321}{99} = \frac{107}{33}.$$

∎

EXERCISE 4

Express each of the numbers (a) $2.\overline{171}$ and (b) $4.3\overline{95}$ as the quotient of two integers.

Answer (a) $\frac{2169}{999}$ (b) $\frac{4352}{990}$

Since each rational number can be represented as a periodic decimal, and vice versa, it follows that each decimal that is not periodic must represent some kind of number other than a rational number. We call this new kind of number an irrational number. Thus an **irrational number** is a number whose decimal representation is not periodic. For example, the number $.1010010001\ldots$, which is formed by adding one additional zero between each two successive 1's, is an irrational number, since it is not a periodic decimal. Other examples of irrational numbers are π, $\sqrt{2}$, and $\sqrt{3}$. To prove that π is irrational requires more mathematics than is available to us in this course. However, we shall show in Section 1.3 that each of the numbers $\sqrt{2}$ and $\sqrt{3}$ is an irrational number. Additional examples of irrational numbers are $\pi/5$, $2 - \sqrt{2}$, $\sqrt[3]{2}$, $\sqrt[5]{3}$, and $\sqrt{5} + \sqrt{3}$.

Each irrational number can be approximated to any desired degree of accuracy by some rational number. For example, the rational number 3.14159 is a five-decimal approximation of the irrational number π. Four-decimal approximations of $\sqrt{2}$ and $\sqrt{3}$ are 1.4142 and 1.7321, respectively.

Real Numbers

The rational numbers together with the irrational numbers are called the **real numbers.** That is, a real number is any number that has a decimal representation. Even though there are other kinds of numbers, when we speak of numbers in this book we shall mean real numbers unless otherwise specified.

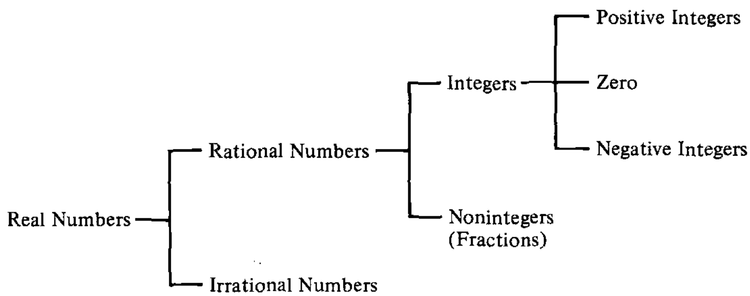

Figure 1.1

The various kinds of (real) numbers considered in the previous discussion and their relationships to one another are summarized in Figure 1.1.

Problems 1.1

1. Which of the numbers -4, $\frac{2}{3}$, 0, $\pi/2$, $\sqrt[3]{2}$, 7, $\sqrt{5}$, -1, .02, and $.\overline{02}$ are natural numbers?
2. Which of the numbers in problem 1 are integers?
3. Which of the numbers in problem 1 are rational numbers?
4. Which of the numbers in problem 1 are irrational numbers?
5. Explain why 50 is an even integer.
6. Explain why -37 is an odd integer.
7. List the factors of 12 that are natural numbers.
8. List the factors of 99 that are natural numbers.
9. List five multiples of 7.
10. List five multiples of 5.
11. List the prime numbers between 20 and 30.
12. List the prime numbers between 30 and 40.
13. Express 98 as the product of primes.
14. Express 323 as the product of primes.
15. Are 77 and 117 relatively prime? Explain.
16. Are 35 and 99 relatively prime? Explain.
17. Every rational number can be expressed as a __?__ decimal.
18. Every irrational number can be expressed as a __?__ decimal.

In problems 19–24, determine the decimal expansion of the given rational number.

19. $\frac{2}{7}$	20. $\frac{4}{9}$	21. $\frac{6}{13}$
22. $\frac{23}{99}$	23. $\frac{4}{37}$	24. $\frac{23}{7}$

In problems 25–36, express the given decimal as the quotient of two integers.

25. .444 . . .	26. .3838 . . .	27. .5353 . . .
28. 1.369369 . . .	29. 2.765765 . . .	30. 1.88 . . .
31. $.\overline{35}$	32. $.\overline{7}$	33. $.\overline{123}$

34. $.021\overline{21}$ **35.** $1.3\overline{7}$ **36.** $2.7\overline{3}$

37. The rational number $\frac{22}{7}$ is frequently used as an approximation of the irrational number π. To how many decimal places are the decimal expansions of $\frac{22}{7}$ and π the same? Obtain a rational number that represents π correct to four decimal places.

38. Express $.239\overline{9}$ as the quotient of two integers. Express $.240\overline{0}$ as the quotient of two integers. Can a rational number have two different decimal expansions?

Answers 1.1

1. 7	**2.** $-4, 0, 7, -1$	**4.** $\frac{\pi}{2}, \sqrt[3]{2}, \sqrt{5}$	**5.** 2(25)
7. 1, 2, 3, 4, 6, 12	**8.** 1, 3, 9, 11, 33, 99	**10.** $-5, 0, 5, 10, 15$	**11.** 23, 29
13. $2 \cdot 7 \cdot 7$	**14.** $17 \cdot 19$	**16.** yes	**17.** periodic
19. $\overline{.285714}$	**20.** $.\overline{4}$	**22.** $.\overline{23}$	**23.** $.\overline{108}$
25. $\frac{4}{9}$	**26.** $\frac{38}{99}$	**28.** $\frac{152}{111}$	**29.** $\frac{307}{111}$
31. $\frac{35}{99}$	**32.** $\frac{7}{9}$	**34.** $\frac{7}{330}$	**35.** $\frac{62}{45}$
37. 2, $\frac{31413}{9999}$		**38.** yes	

1.2 Sets

The main objective of this section is to present an introduction to sets, set terminology, and set symbolism. Our primary interest in set terminology and symbolism is as a means of abbreviating and clarifying various concepts that appear in later sections of this book. The idea of a set is such a basic idea in mathematics that we will not attempt to give a formal definition of a set. However, a **set** should be thought of intuitively as a collection of objects. Each object in this collection of objects is called an **element** of the set. A set must be *well defined.* That is, there must be some criterion for deciding whether a particular object *is* or *is not* an element of the set. The following are examples of sets:

> The set of all letters in the word "TARANTULA."
> The set of all points of a given line.
> The set of all persons in your class.

Capital letters, $A, B, C, \ldots$, are generally used to represent sets, while lowercase letters, $a, b, c, \ldots$, are used to represent the elements of sets. If the object a belongs to the set A, we write "$a \in A$" to mean that "a is an element of the set A." Similarly, we write "$a \notin A$" to mean that "a is *not* an element of the set A." Thus, if A is the set of all countries in Europe, we write

$$\text{France} \in A \qquad \text{and} \qquad \text{Canada} \notin A.$$

Sets are usually described by one of two methods. In either method, the set description is included inside braces { }. A set is said to be described by the **roster method** if each of the elements of the set is actually listed inside the braces. For example, the set consisting of the last four letters of the English alphabet is denoted by

$$\{\text{w, x, y, z}\}.$$

Another example of a set described by the roster method is the set

$$\{\text{New Mexico, Oklahoma, Arkansas, Louisiana}\},$$

whose elements are the names of those states which share a border in common with Texas.

A set is said to be described by the **rule method** if a common property that describes the elements of the set, and only the elements of the set, is enclosed inside the braces. In such situations, the set is generally expressed in the form

$$\{x|x \text{ has property } P\},$$

which is read "the set of all objects x such that x has the property P." For example, the set $\{1, 2, 3, 4, 5\}$ can be described as

$$\{x|x \text{ is a counting number less than } 6\}.$$

As another example, the set {California, Colorado, Connecticut} can be described as

$$\{x|x \text{ is a state of the United States whose name begins with C}\}.$$

As a matter of future convenience, we shall reserve the capital letters N, I, Q, and R to represent certain special sets of numbers. These sets can now be described in set notation by the rule method as follows:

$$\begin{aligned} N &= \{x|x \text{ is a natural number}\}, \\ I &= \{x|x \text{ is an integer}\}, \\ Q &= \{x|x \text{ is a rational number}\}, \\ R &= \{x|x \text{ is a real number}\}. \end{aligned}$$

The letter x used in describing a set by the rule method is an example of a variable. In general, a symbol, such as x, which represents an unspecified element of a given set, is called a **variable.** On the other hand, a **constant** is a symbol, such as 2 or π, which is used to represent exactly one object.

Two sets of special interest are the empty set and the universal set. The **empty set** (sometimes called the **null set**), denoted by $\varnothing$, is defined as the set that contains no elements. As an example, the set of all men who are over 10 feet tall is the empty set. Can you think of another set that is the empty set? In many instances, it is convenient to decide upon a totality of objects that are to be considered as elements of sets in a particular discussion. In such cases we define the **universal set,** denoted by U, as the set that contains all the objects that are to be considered as elements of sets. The choice of the universal set depends upon the particular problem being discussed.

Related Sets

The set A is said to be a **subset** of the set B, denoted by $A \subseteq B$, if and only if each element of A is also an element of B. If A is not a subset of B, we write $A \nsubseteq B$.

EXAMPLE 1

(a) $\{2, 7\} \subseteq \{2, 5, 7, 9\}$ (b) $\{2, 7\} \subseteq \{2, 7\}$
(c) $\varnothing \subseteq \{2, 7\}$ (d) $\{2, 7\} \nsubseteq \{2, 3, 6, 8\}$

Observe that the empty set is a subset of every set. Also observe that every set is a subset of itself.

If $C = \{2, 3, 7\}$, then $\{2\} \subseteq C$ and $2 \in C$. It is incorrect to write $2 \subseteq C$, since 2 is not a set. It is also incorrect to write $\{2\} \in C$, since $\{2\}$ is a set rather than an element.

EXERCISE 1

If $B = \{0, 1, \pi\}$, which of the following are true?
(a) $\varnothing \subseteq B$ (b) $0 \subseteq B$ (c) $\pi \in B$
(d) $2 \notin B$ (e) $\{2, \pi\} \subseteq B$ (f) $\{\pi\} \nsubseteq B$

Answer (a), (c), and (d) are true.

Two sets A and B are **equal** to one another, denoted by $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$. That is, two sets are equal if and only if they have exactly the same elements.

EXAMPLE 2

The sets $\{a, b, c\}$ and $\{c, a, b\}$ are equal, since each set is a subset of the other set.