McGRAW-HILL
ELECTRICAL AND
ELECTRONIC
ENGINEERING
SERIES

INTRODUCTION TO Nonlinear Analysis

W. J. Cunningham

PROFESSOR OF ELECTRICAL ENGINEERING
YALE UNIVERSITY

McGRAW-HILL BOOK COMPANY, Inc.
NEW YORK TORONTO LONDON
1958

INTRODUCTION TO NONLINEAR ANALYSIS

Library of Congress Catalog Card Number 58-7414

THE MAPLE PRESS COMPANY, YORK, PA.

INTRODUCTION TO NONLINEAR ANALYSIS

McGraw-Hill Electrical and Electronic Engineering Series

FREDERICK EMMONS TERMAN, *Consulting Editor*

W. W. HARMAN and J. G. TRUXAL, *Associate Consulting Editors*

Bailey and Gault · ALTERNATING-CURRENT MACHINERY
Beranek · ACOUSTICS
Bruns and Saunders · ANALYSIS OF FEEDBACK CONTROL SYSTEMS
Cage · THEORY AND APPLICATION OF INDUSTRIAL ELECTRONICS
Cauer · SYNTHESIS OF LINEAR COMMUNICATION NETWORKS, VOLS. I AND II
Cuccia · HARMONICS, SIDEBANDS, AND TRANSIENTS IN COMMUNICATION ENGINEERING
Cunningham · INTRODUCTION TO NONLINEAR ANALYSIS
Eastman · FUNDAMENTALS OF VACUUM TUBES
Evans · CONTROL-SYSTEM DYNAMICS
Feinstein · FOUNDATIONS OF INFORMATION THEORY
Fitzgerald and Higginbotham · BASIC ELECTRICAL ENGINEERING
Fitzgerald and Kingsley · ELECTRIC MACHINERY
Geppert · BASIC ELECTRON TUBES
Glasford · FUNDAMENTALS OF TELEVISION ENGINEERING
Happell and Hesselberth · ENGINEERING ELECTRONICS
Harman · FUNDAMENTALS OF ELECTRONIC MOTION
Harrington · INTRODUCTION TO ELECTROMAGNETIC ENGINEERING
Hayt · ENGINEERING ELECTROMAGNETICS
Hessler and Carey · FUNDAMENTALS OF ELECTRICAL ENGINEERING
Hill · ELECTRONICS IN ENGINEERING
Johnson · TRANSMISSION LINES AND NETWORKS
Kraus · ANTENNAS
Kraus · ELECTROMAGNETICS
LePage · ANALYSIS OF ALTERNATING-CURRENT CIRCUITS
LePage and Seely · GENERAL NETWORK ANALYSIS
Millman and Seely · ELECTRONICS
Millman and Taub · PULSE AND DIGITAL CIRCUITS
Rodgers · INTRODUCTION TO ELECTRIC FIELDS
Rüdenberg · TRANSIENT PERFORMANCE OF ELECTRIC POWER SYSTEMS
Ryder · ENGINEERING ELECTRONICS
Seely · ELECTRON-TUBE CIRCUITS
Seely · ELECTRONIC ENGINEERING
Seely · INTRODUCTION TO ELECTROMAGNETIC FIELDS
Seely · RADIO ELECTRONICS
Siskind · DIRECT-CURRENT MACHINERY
Skilling · ELECTRIC TRANSMISSION LINES
Skilling · TRANSIENT ELECTRIC CURRENTS
Spangenberg · FUNDAMENTALS OF ELECTRON DEVICES
Spangenberg · VACUUM TUBES
Stevenson · ELEMENTS OF POWER SYSTEM ANALYSIS
Storer · PASSIVE NETWORK SYNTHESIS
Terman · ELECTRONIC AND RADIO ENGINEERING
Terman and Pettit · ELECTRONIC MEASUREMENTS
Thaler · ELEMENTS OF SERVOMECHANISM THEORY
Thaler and Brown · SERVOMECHANISM ANALYSIS
Thompson · ALTERNATING-CURRENT AND TRANSIENT CIRCUIT ANALYSIS
Truxal · AUTOMATIC FEEDBACK CONTROL SYSTEM SYNTHESIS

PREFACE

This book is intended to provide the engineer and scientist with information about some of the basic techniques for finding solutions for nonlinear differential equations having a single independent variable. Physical systems of many types and with very real practical interest tend increasingly to require the use of nonlinear equations in their mathematical description, in place of the much simpler linear equations which have often sufficed in the past. Unfortunately, nonlinear equations generally cannot be solved exactly in terms of known tabulated mathematical functions. The investigator usually must be satisfied with an approximate solution, and often several methods of approach must be used to gain adequate information. Methods which can be used for attacking a variety of problems are described here.

Most of the material presented in this book has been given for several years in a course offered to graduate students in electrical engineering at Yale University. Emphasis in this course has been on the use of mathematical techniques as a tool for solving engineering problems. There has been relatively little time devoted to the niceties of the mathematics as such. This viewpoint is evident in the present volume. Students enrolled in the course have had backgrounds in electrical circuits and mechanical vibrations and a degree of familiarity with linear differential equations. Their special field of interest has often been that of control systems. A knowledge of these general areas is assumed on the part of the reader of this volume.

Nonlinear problems can be approached in either of two very different ways. One approach is based on the use of a minimum of physical equipment, making use only of pencil, paper, a slide rule, and perhaps a desk calculator. The degree of complexity of problems which can be successfully handled in a reasonable time with such facilities is admittedly not very great. A second approach is based on the use of what may be exceedingly complicated computing machinery of either the analog or the digital variety. Such machinery allows much more information to be dealt with and makes feasible the consideration of problems of complexity far greater than can be handled in the first way. The efficient use of computing machinery is a topic too large to be considered in the present discussion. Only techniques useful with the modest facilities available to almost everyone are considered here.

A set of problems related directly to the subject matter of each chapter is added at the end of the text.

For a variety of reasons, it has seemed desirable to concentrate the bibliographic reference material in a special section also at the end of the text, rather than have it scattered throughout the book. In this way, a few comments can be offered about each reference and an effort made to guide the reader through what sometimes appears to be a maze of literature.

As is the case in most textbooks, by far the greater part of the material discussed here has developed through the continuing efforts of many workers, and but little is original with the present author. He would like, therefore, to acknowledge at this time his indebtedness to all those who have led the way through the thorny regions of nonlinearities.

W. J. Cunningham

CONTENTS

Preface *v*

1. Introduction 1

2. Numerical Methods 8

2.1 Introduction 2.2 Use of Taylor's Series 2.3 Modified Euler Method 2.4 Adams Method 2.5 Checking Procedures: *a. Checking by Plotting, b. Checking by Successive Differences, c. Checking by Numerical Integration* 2.6 Equations of Order Higher than First 2.7 Summary

3. Graphical Methods 28

3.1 Introduction 3.2 Isocline Method, First-order Equation 3.3 Isocline Method, Second-order Equation. Phase-plane Diagram 3.4 Time Scale on the Phase Plane 3.5 Delta Method, Second-order Equation 3.6 Lienard Method 3.7 Change of Variable 3.8 Graphical Integration 3.9 Preisman Method 3.10 Summary

4. Equations with Known Exact Solutions 55

4.1 Introduction 4.2 Specific Types of Equations: *a. Equation Directly Integrable, b. First-order Linear Equation, c. Higher-order Linear Equation with Constant Coëfficients, d. Equation with Variables Separable, e. Equation from an Exact Differential, f. Equation with Only Second Derivative, g. Second-order Equation without Dependent Variable Alone, h. Second-order Equation without Independent Variable Alone, i. Bernoulli's Equation, j. Riccati's Equation, k. Euler-Cauchy Equation, l. Linear Equation with Varying Coefficients* 4.3 Variation of Parameters 4.4 Equations Linear in Segments 4.5 Equations Leading to Elliptic Functions 4.6 Properties of Elliptic Functions 4.7 Summary

5. Analysis of Singular Points 85

5.1 Introduction 5.2 Type of Equation under

Investigation 5.3 Linear Transformations 5.4 Equations in Normal Form 5.5 Types of Singularities: *a. Both Roots Real and Positive, b. Both Roots Real and Negative, c. Both Roots Real and Opposite in Sign, d. Roots Pure Imaginaries, e. Roots Complex Conjugates* 5.6 Sketching of Solution Curves 5.7 Forcing Functions as Steps or Ramps 5.8 Analysis Combining Singularities and Linear Segments 5.9 Potential Energy of Conservative Systems 5.10 Summary

6. Analytical Methods 121

6.1 Introduction 6.2 Perturbation Method 6.3 Reversion Method 6.4 Variation of Parameters 6.5 Use of Several Methods 6.6 Averaging Methods Based on Residuals: *a. Galerkin's Method, b. Ritz Method* 6.7 Principle of Harmonic Balance 6.8 Summary

7. Forced Oscillating Systems 171

7.1 Introduction 7.2 Principle of Harmonic Balance 7.3 Iteration Procedure 7.4 Perturbation Method 7.5 Extension of Concepts from Linear Systems: *a. The Impedance Concept, b. The Transfer-function Concept* 7.6 Forced Oscillation in a Self-oscillatory System 7.7 Summary

8. Systems Described by Differential-difference Equations 221

8.1 Introduction 8.2 Linear Difference Equations 8.3 Linear Differential-difference Equation 8.4 Nonlinear Difference Equation 8.5 Nonlinear Differential-difference Equations 8.6 Graphical Solutions 8.7 Summary

9. Linear Differential Equations with Varying Coefficients 245

9.1 Introduction 9.2 First-order Equation 9.3 Second-order Equation: *a. Removal of First-derivative Term, b. Determination of Particular Integral, c. The WKBJ Approximation, d. Approximation through Variation of Parameters* 9.4 Mathieu Equation: *a. Basic Theory, b. Approximate Solutions by Variation of Parameters, c. Location of Stability Boundaries by Perturbation Method, d. Applications* 9.5 Summary

10. Stability of Nonlinear Systems 281

10.1 Introduction 10.2 Structural Stability 10.3 Dynamical Stability 10.4 Test for Stability of System with Nonoscillatory Steady State 10.5 Test for Stability of Oscillating Systems 10.6 Stability of Feedback Systems 10.7 Orbital Stability of Self-oscillating Systems 10.8 Orbital Stability with Two Components 10.9 Summary

Problems 321

Bibliography 337

Index 345

CHAPTER 1

INTRODUCTION

Whenever a quantitative study of a physical system is undertaken, it is necessary to describe the system in mathematical terms. This description requires that certain quantities be defined, in such a way and in such number that the physical situation existing in the system at any instant is determined completely if the numerical values of these quantities are known. In an electrical circuit, for example, the quantities may be the currents existing in the several meshes of the circuit. Alternatively, the quantities may be the voltages existing between some reference point and the several nodes of the circuit. Since the quantities, the currents and voltages of the example, change with time, time itself becomes a quantity needed to describe the system.

For a physical system, time increases continuously, independent of other occurrences in the system, and therefore is an independent quantity or an independent variable. If the system is made up of lumped elements, such as the impedance elements of the electrical circuit, time may be the only independent variable. If the system is distributed, as an electrical transmission line, additional independent variables such as the position along the line must be considered. The independent variables may change in any way without producing an effect upon one another.

Also in a physical system are certain parameters which must be specified but which do not change further or at least which change only in a specified way. Among these parameters are the resistance, inductance, and capacitance of an electrical circuit and the driving voltage applied to the circuit from a generator of known characteristics.

The remaining quantities that describe the system are dependent upon the values of the parameters and the independent variables. These remaining quantities are the dependent variables of the system. The system is described mathematically by writing equations in which the variables appear, either directly or as their derivatives or integrals. The constants in the equations are determined by the parameters of the system. A complicated system requires description by a number of simultaneous equations of this sort. The number of equations must be the same as the number of dependent variables which are the unknown quantities of the system. If a single independent variable is sufficient, only derivatives and integrals involving this one quantity appear. Furthermore, it is often possible to eliminate any integrals by suitable differentiation. The resulting equations having only derivatives with respect to one variable are called ordinary differential equations. Where there is more than one independent variable, partial differentials with respect to any or all such variables appear and the equations are partial differential equations. Only the case of ordinary differential equations is considered in this book.

The equations are said to be linear equations if the dependent variables or their derivatives appear only to the first power. If powers other than the first appear, or if the variables appear as products with one another or with their derivatives, the equations are nonlinear. Since a transcendental function, for example, can be expanded as a series with terms of progressively higher powers, an equation with a transcendental function of a dependent variable is nonlinear.

The following equation for a series electrical circuit is a linear equation:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = E\sin\omega t \tag{1.1}$$

In this equation, the time t is the independent variable and the current i is the dependent variable. The coefficients represent constant parameters. The equation describing approximately a circuit with an iron-core inductor which saturates magnetically is a nonlinear equation,

$$L(1 - ai^2)\frac{di}{dt} + Ri = E\sin\omega t \tag{1.2}$$

The term involving $i^2\,di/dt$ is a nonlinear term.

In the simplest case, the parameters of the physical system are constants and do not change under any condition that may exist in the system. For many cases, however, the parameters are not constants. If the parameters change in some fashion with a change in value of one or more of the dependent variables, the equation becomes nonlinear. This fact is evident since the parameter which is a function of the dependent variable appears as a coefficient of either that variable or another one. Thus, a term is introduced having a variable raised to a power other than unity, or appearing as a product with some other variable. Such a term appears in Eq. (1.2), where variable i is squared and its product with the derivative di/dt appears.

A second possibility is that the parameters change in some specified way as the independent variable changes. In this case, the equation remains a linear equation but has varying coefficients. The equation for the current in the electrical circuit containing a telephone transmitter subjected to a sinusoidal sound wave is linear but has a time-varying coefficient,

$$R_1(1 + a \sin \omega t)i + R_2 i = E \tag{1.3}$$

It is possible, of course, to have both nonlinear terms and terms with varying coefficients in the same equation. This would result for the example of a circuit having both a saturating inductor and a telephone transmitter. The equation would be a combination of Eqs. (1.2) and (1.3).

The solution for a set of differential equations is a set of functions of the independent variables in which there are no derivatives. When these functions are substituted into the original equations, identities result. It is generally quite difficult, and often impossible, to find a general solution for a differential equation. There are certain special kinds of equations, however, for which solutions are readily obtainable.

The most common type of differential equation for which solutions are easily found is a linear equation with constant coefficients. These equations can be solved by applying certain relatively simple rules. Operational methods are commonly employed for this purpose. If the number of dependent variables is large, or if the equations are otherwise complicated, application of the rules may be a fairly tedious process. In theory, at least, an exact solution is always obtainable.

A most important property of linear equations is that the principle of superposition applies to them. In essence, this principle allows a complicated solution for the equations to be built up as a linear sum of simpler solutions. For a homogeneous equation with no forcing function, the solution may be created from the several complementary functions, their number being determined by the order of the equation. An equation with a forcing function has as a solution the sum of complementary functions and a particular integral. A complicated forcing function may be broken down into simpler components and the particular integral found for each component. The complete particular integral is then the sum of all those found for the several components. This is the principle allowing a complicated periodic forcing function to be expressed as a Fourier series of simple harmonic components. Particular integrals for these components are easily found, and their sum represents the solution produced by the original forcing function. Superposition is not possible in a nonlinear system.

Much of the basic theory of the operation of physical systems rests on the assumption that they can be described adequately by linear equations with constant coefficients. This assumption is a valid one for many systems of physical importance. Most of the beautiful and complicated theory of electrical circuits, for example, is based on this assumption. It is probably no exaggeration to say, however, that all physical systems become nonlinear and require description by nonlinear equations under certain conditions of operation. Whenever currents or voltages in an electrical circuit become too large, nonlinear effects become evident. Iron cores saturate magnetically; the dielectric properties of insulating materials change; the temperature and resistance of conductors vary; rectification effects occur. Currents and voltages that are large enough to produce changes of this sort sometimes prove to be embarrassingly small. It is fortunate for the analyst of physical phenomena that his assumption of linearity with constant coefficients does apply to many systems of very real interest and importance. Nevertheless, he must remember that his assumption is likely to break down whenever the system is pushed toward its limits of performance.

The procedures for finding solutions for nonlinear equations, or equations with varying coefficients, generally are more difficult and less satisfying than are those for simpler equations. Only in a few cases can exact solutions in terms of known functions be found. Usually, only an approximate solution is possible, and this solution may apply with reasonable accuracy only within certain regions of operation. The analyst of a nonlinear system must use all the facilities at his disposal in order to predict the performance of the system. Often he must be guided by intuition based on a considerable insight into the physical operation of the system, and not be dependent upon a blind application of purely mathematical formulas. It is a truism to observe that it helps greatly to know the answer ahead of time. Experimental data concerning the operation of the system may be of great value in analyzing its performance.

If the amount of nonlinearity is not too large, or if the equations are of a few special types, analytical methods may be used to yield approximate solutions. Analytical methods give the solution in algebraic form without the necessity for inserting specific values for the numerical constants in the course of obtaining the solution. Once the solution is obtained, numerical values can be inserted and the effects of wide variations in these values explored rather easily.

If the amount of nonlinearity is larger, analytical methods may not be sufficient and solution may be possible only by numerical or graphical methods. These methods require that specific numerical values for the parameters of the equations and for the initial conditions of the variables be used in the course of obtaining the solution. Thus, any solution applies for only one particular set of conditions. Furthermore, the process of obtaining the solution is usually tedious, requiring considerable manipulation if good accuracy is to be obtained. Then, if a solution for some different set of values for the numerical parameters is required, the whole process of solution must be repeated. Because of the large amount of manipulation needed in solving equations by numerical methods, a suitable digital computer is almost a necessity if many equations of this sort are to be solved.

In summary, the situation facing the analyst can be described as follows: The equations describing the operation of many physical systems can be reduced to sets of simultaneous ordinary differential equations. When the equations are linear with constant coefficients, standard rules can be applied to yield a solution. If the number of dependent variables is large, or if the order of the differential equations is large, the application of the rules may be tedious and time-consuming. A solution is possible, at least in theory. When the differential equations are nonlinear, solution by analytical methods is possible only if the amount of nonlinearity is small and, even then, available solutions are usually only approximate. Equations with a considerable degree of nonlinearity can be solved only by numerical or graphical methods.

In many practical problems of interest, the situation is so complicated that an exact solution is not possible, or at least not possible in terms of available time and effort. In such a case, it becomes necessary to simplify the problem in some way by neglecting less important effects and concentrating upon the really significant features. Often suitable assumptions can be made which lead to simplified equations that can be solved. While these equations do not describe the situation exactly, they may describe important features of the phenomenon. The simplified equations can be studied by available techniques and certain information obtained. This process may be compared with setting up a simplified mathematical model analogous to the physical system of interest. The model may be studied in various ways so as to find how it behaves under a variety of conditions. While the information so obtained may not always be sufficient to allow a complete design of a physical device, the information may well be enough to provide useful criteria for the design. This kind of procedure has proved useful in many practical situations. An electronic oscillator, for example, is usually too complicated to be studied if all its details are taken into account. However, the classic van der Pol equation describes many of the operating features of the oscillator and is simple enough so that much information can be found about its solutions. A study of the van der Pol equation has led to much useful knowledge about the operation of self oscillators.

In the chapters which follow, methods are considered for the solution of ordinary differential equations that either are nonlinear or have varying coefficients. The first methods described are based on numerical or graphical techniques. While only relatively simple cases are considered here, these methods can be extended to apply to systems with considerable complexity. Usually, more information about the system can be obtained more quickly if an analytical solution is possible. Some of the simpler analytical procedures that can be used are considered in later chapters. Much of the work that has been done with nonlinear equations is limited to equations of first and second order. For this reason, the emphasis here is on equations of these orders. The methods of solution can be extended to equations of higher order, but complications increase rapidly when this is done.

It is assumed throughout the discussion that the reader is familiar with methods of solving the simpler kinds of differential equations and, in particular, differential equations that are linear with constant coefficients. The emphasis is on the use of mathematical techniques as a tool for studying physical systems, and no attempt has been made to provide elaborate justification for the techniques that are discussed. It is assumed that the reader has a reasonable degree of familiarity with analysis of electrical circuits and mechanical systems. A single method of attacking a nonlinear problem often gives only a partial solution, and other methods must be used to obtain still more information. For this reason, the same examples appear at a number of different places in the book, where different methods of analysis are applied to them. An attempt is made, however, to keep each chapter of the book as nearly self-contained as possible.

It is, perhaps, not out of place to remark that some of the procedures commonly used by mathematicians in studying a particular problem seem at first to be unessential and, in fact, confusing. In considering a single differential equation of high order, for example, it is common practice to replace it with a set of simultaneous first-order equations. This process is usually necessary when a numerical method of solution is being employed, since most numerical methods apply only to first-order equations. For other methods of analysis, separation into first-order equations is largely a convenience in that it allows better organization of the succeeding steps in the solution. In still other cases, it is actually necessary to recombine the equations into a single higher-order equation at certain points in the solution.

Another device commonly employed is that of changing the form of certain variables in the equation. Sometimes, a change of variable replaces an original equation which could not be solved with an alternate equation which can be solved. A few nonlinear equations can be made linear by appropriate changes of variable. At other times, a change of variable is made for purposes of convenience. Often, in problems concerning physical systems, simplification is possible by combining into single factors certain of the dimensional parameters that appear. By proper choice of factors, the resulting equation can be made dimensionless. The variables in the equation are then said to be normalized. A normalized equation and its solution often present information in a more compact and more easily analyzed form than does the equation before normalization. A normalized equation is free of extraneous constants, and the constants that do remain are essential to the analysis.

It should be recognized that this book, as well as most others of its type, represents the distillation of the work done by a great many investigators over a long period of time. The discussions presented here show the effects of continued revision and retain what are thought to be the most useful and essential principles. The examples that are chosen for discussion are admittedly ones which illustrate a particular principle and are chosen because a solution for them is possible. Changes of variable and definitions of new quantities are introduced at many points in the discussion. The fact that analysis is, after all, a sort of experimental science should not be overlooked. Often, the changes that are made are found only after a great many such modifications have been tried. The one which successfully simplifies the equation is the one, of course, which appears in the discussion. An analyst faced with a new and unknown problem may have to try many approaches before he finds one that gives the information he needs.

One of the fundamental problems in the analysis of any physical system is the question of its stability. For this reason, discussions relating to stability arise in many places in the following chapters. Because of the kinds of phenomena which occur in a nonlinear system, it is not possible to use a single definition for stability that is meaningful in all cases. Several definitions are necessary, and the appropriate one must be chosen in any given situation. Since the question of stability is fundamental to almost every problem, it is a topic which ought, perhaps, to be the first considered. On the other hand, the intricacies of the question can be appreciated only after background has been acquired. Therefore, points relating to stability appear at many places throughout the book. Then, in the last chapter, these points are brought together and presented in a more complete manner. A certain amount of repetition is necessarily involved in this kind of presentation, but it does provide a closing chapter in which many basic ideas are tied together.