

INTRODUCTORY GRAPH THEORY

BÉLA ANDRÁSFAI

ADAM HILGER
BRISTOL

INTRODUCTORY GRAPH THEORY



INTRODUCTORY GRAPH THEORY

BÉLA ANDRÁSFAI

ADAM HILGER
BRISTOL

The original Hungarian
ISMERKEDÉS A GRÁFELMÉLETTEL
was published by Tankönyvkiadó, Budapest

Translated by
ANDRÁS RECSKI

© Akadémiai Kiadó, Budapest 1977

All rights reserved. No part of this publication may be reproduced,
stored in a retrieval system, or transmitted in any form or by any
means, electronic, mechanical, photo-copying, recording or otherwise,
without the prior permission of Akadémiai Kiadó

ISBN 0 85274 249 5

Published in Great Britain
by
Adam Hilger Ltd.,
Techno House, Redcliffe Way, BRISTOL, BS1 6NX
Adam Hilger is owned by The Institute of Physics
as a co-edition with
Pergamon Press Inc, Elmsford, New York
and
Akadémiai Kiadó, Budapest

Printed in Hungary at the Akadémiai Nyomda

TO THE MEMORY OF RÓZSA PÉTER

PREFACE

In giving an introductory lecture on mathematical topics one can usually start with the phrase: "It was the Greek civilization which first developed the elements of ...". All classical topics suggest this way of presentation, but even most of the "modern" ones were developed from classical nuclei, with a continually increasing level of abstraction. Although the first significant paper — Euler's — was written in 1736, graph theory is fairly new having been mentioned in mathematics only in about the last hundred years, following Kirchhoff's results on electrical networks. On the other hand, it has become an independent branch of mathematics in the last two decades owing to its rapid development. It should not be called modern, however, in this sense since it is still near to original thinking, still full of clarity, and its "charm" recalls the Greek civilization. It is a branch of combinatorics, it offers a great variety of natural problems and possible applications, it does not require advanced mathematical tools — but sometimes deep consideration. For this reason graph theory is an excellent research field for young people interested in mathematics. This has led to problems and results of graph theory becoming part of up-to-date school curricula. Extensive lecturing having made me aware of the interest at all levels, I wrote the Hungarian version of the present book in 1969, since no Hungarian book on the subject existed. (The first scientific monograph on graph theory was, in fact, written by the Hungarian professor Dénes König in 1936, but it was in German.) However, a large number of different books on graph theory have been published in the last 15 years (see References).

I have tried to give the exact proof of almost all the statements in the book — the easy theorems as well as the more difficult ones. The results are presented *in statu nascendi*, following the procedure of discovery, solution of sub-statements, definition of new concepts which prove to be useful, and determination of the possibilities of generalization from the solution of practical problems. Exercises, problems and their solutions are given throughout, with suggestions of new problems, simplification of complicated statements, and, above all, stimulation of readers.

The exercises are easy: a little drawing and calculation leads the reader to the solution of problems. A number of problems preceded by an asterisk (*) appear at the end of each chapter. The purpose of these is to develop the reader's ability to solve problems on topics treated in the chapter itself. Their solutions are presented in Chapter 7, but individual work on them is strongly recommended. This kind of activity is suggested throughout with the reader drawing graphs to illustrate the statements in the text. Even the wealth of illustrations in the book will not replace individual drawing, as this alone enables one to see the evolution of the figure. In this way, becoming familiar with the content of the book leads to the reader's own discovery of the results.

To avoid confusion the source of the statements is not indicated in the text but in the source-index. Chapters are not formally divided into sections, but the grouping of exercises and/or problems usually indicates a division; subjects covered are summarized in the Contents. In addition, to emphasize the various important methods of graph theory, they are not only mentioned in the subject index, but set in bold-faced type in the text. All the exercises, problems and statements are numbered as a whole, within each chapter, including those at the end of the chapter. Numbering recommences at the beginning of each chapter.

The first chapter deals with basic concepts; the following five consider five areas of graph theory, including some of the more modern results. Certain topics of interest are not discussed here, e.g. the relation of graphs to surfaces, matrices, and probability theory; map-colouring problems, detailed topological description of electrical networks and the solution of transportation problems. Most of these problems will be contained in the second volume of this book, which is in preparation. The reference list contains suggestions for further reading.

Those involved in mathematics at any level should find this book most useful; in fact, this is true for all whose work involves problem-solving since the development of the ability to solve problems by thinking in terms of graph theory is of benefit in any field.

I am especially grateful to the referees of the Hungarian version of the book, Professors R. Péter and T. Gallai, for their criticism, valuable suggestions and their comments on the mathematics as well as the presentation. Particular thanks are due to Professor P. Erdős, who suggested and fostered the idea of the English edition. I should like to thank A. Recski for his translation and constructive remarks and the staff of Akadémiai Kiadó for their helpfulness.

Béla Andrásfai

CONTENTS

Preface	5
Chapter 1	
<i>Introduction to graphs</i>	15
Basic concepts	15
Connections between degrees and the number of vertices and edges: 1—13	17
The pigeonhole principle	21
The number of edges in a complete graph with n vertices: 11	21
A problem concerning complementary graphs: 16 i.e. 14	23
Connections between degrees and the number of vertices and edges in connected graphs: 18—22	27
Simple problems concerning paths and circuits: 23 and 24	29
The method of the longest path	30
Two properties of connected graphs: 25 and 26	31
Exercises, Problems	31
Chapter 2	
<i>Trees and forests</i>	33
Connections between the number of vertices and edges in a tree: 5 and 6 (1—4 are preparatory to this)	34
Applications in chemistry: 7 and 8	35
Paths in trees: 9	36
Forest (10 is preparatory to this)	39
Characterization of spanning trees: 11	39
Characterization of fundamental circuits, fundamental system of circuits: 17	40
Spanning forest of graphs	42
Rank and nullity of graphs: 18 (13—15 are preparatory to this)	42
Economical way of building a network without a circuit; three methods	43

Finding of spanning trees with minimal and with maximal value respectively	49
The use of spanning trees in computing electrical networks	49
The two Kirchhoff laws	50
Exercises, Problems	54

Chapter 3

<i>Routes following the edges of a graph</i>	57
The Königsberg Bridge problem, 4	57
Open and closed edge-trains	61
Exact conditions for the existence of an open and a closed Eulerian line respectively: 6 and 7 (5 is preparatory to this)	63
Basic concepts concerning directed graphs	63
Directed paths, circuits and edge-trains	64
Formulation of traffic problems with the aid of directed graphs	66
Traffic-condition, strongly connected graph	67
Connection between bridge and circuit: 12 and 13	68
Connected graphs without bridge can be oriented so as to be strongly connected: 18 (10 and 14 are preparatory to this)	69
The methods of starting from the maximum and the minimum	69
Exact condition for the existence of a closed Eulerian line in directed graphs: 19 (15 is preparatory to this)	71
An application to non-oriented graphs: 20	71
Note on infinite graphs	73
In the maze	74
Two labyrinth rules	76
Following the corridors of an exhibition	78
The structure of randomly Eulerian graphs: 23 and 24 (21 and 22 are preparatory to this)	80
Exercises, Problems	82

Chapter 4

<i>Routes covering the vertices of a graph</i>	86
The dodecahedron game, 1	86
Hamiltonian circuit, Hamiltonian path	87
Condition implying the non-existence of a Hamiltonian circuit and path respectively: 3, cut-vertex	88
Application: a knight's move on a chessboard: 4 and 5 (Fig. 99)	88
The final analysis of the dodecahedron game (6 and 7 are preparatory to this)	92

Degree-condition implying the existence of a circuit of length greater than a prescribed number: 13 i.e. 8	98
Degree-conditions implying the existence of a Hamiltonian circuit and a Hamiltonian path respectively: 14 (9 is preparatory to this), 15 (10—12 are preparatory to this), and 16	99
Hamiltonian circuit in polyhedrons bounded by triangles	104
Directed Hamiltonian circuits and paths	106
Tournamens possessing a Hamiltonian path: 18 (17 is preparatory to this)	106
Conditions implying the existence of a directed Hamiltonian circuit and path respectively: 19—22	107
Notes on Hamiltonian paths of infinite graphs	108
Exercises, Problems	109

Chapter 5

<i>Matching problems, factors</i>	111
Organizing a tournament	111
The complete graph as the product of 1-factors: 1 (organizing a tournament is preparatory to this)	112
k -factors, regular graphs	113
Set of independent edges, maximal set of independent edges	113
Regular graph of even degree is the product of 2-factors: 13 (3, 5, 10—12 are preparatory to this)	117
The complete graph as the product of Hamiltonian circuits (Fig. 135)	117
Bipartite graph (4, 6 and 7 are preparatory to this)	118
Characterization of bipartite graphs: 14 and 15	118
Regular bipartite graph as the product of 1-factors: 18 (8, 9, 16 and 17 are preparatory to this)	120
Edges covering a set of vertices. The marriage problem: 19 (4, 6 and 17 are preparatory to this)	121
The method of alternating paths	122
Algorithm for finding a maximal set of independent edges in a bipartite graph (Hungarian method) 20 (an application of 19 is preparatory to this)	125
Set of covering vertices, minimal set of covering vertices	126
$ic_{\max} = cv_{\min}$ for bipartite graphs: 22	127
Set of independent vertices, maximal set of independent vertices	130
Set of covering edges, minimal set of covering edges	130
$iv_{\max} = ce_{\min}$ for bipartite graphs without an isolated vertex: 30	132

Degree-condition implying the existence of more than a prescribed number of independent edges: 31 (25 is preparatory to this)	132
Degree-conditions implying the existence of a Hamiltonian circuit in bipartite graphs: 32 and 33 (26 is preparatory to this)	133
Exact condition for the existence of a 1-factor of a bipartite graph: 34 (27 is preparatory to this)	135
Exact condition for the existence of a 1-factor of an arbitrary graph: 35	136
Application to 3-regular graphs without a bridge: 36—41	136
Regular graphs which cannot be decomposed into the product of factors: 42 (Figs 149 and 154)	140
Exercises, Problems	140

Chapter 6

<i>Extremal values, extremal graphs</i>	144
Some types of extremal-value problems	144
Some elementary combinatorial theorems: 4—8 (1—3 are preparatory to this)	146
Three ways of defining the $n(m, k)$ Ramsey numbers	149
A special case of the Ramsey theorem: 22; estimation of Ramsey numbers and some exact values: 10, 12, 15, 16, 18, 19, 23 and 24 (11, 13, 14, 17, 19, 20 and 21 are preparatory to this)	156
More general Ramsey numbers	158
The solution of a Ramsey-type extremal-value problem with the aid of the structure of graphs without a directed circuit. An application in number theory: 25 and 28 and Note 2	158
Some special cases of a Ramsey-type further problem: 26, 27, 29 and 30	159
Conditions for degrees and number of edges implying the existence of a triangle: 38—40 (17 and 31—35 are preparatory to this)	168
Conditions for degrees and number of edges implying the existence of a complete subgraph with k vertices: 43 and 44 respectively (36—42 are preparatory to this)	172
A geometrical application of 43: 49 (45—48 are preparatory to this)	176

Connection between cv_{\min} , the number of edges and vertices: 53 and 54 (50—52 are preparatory to this)	179
Conditions for degrees and number of edges implying the existence of a triangle (or a circuit of odd length less than a given number) while iv_{\max} is fixed or bounded: 55, 62—66 (56—61 are preparatory to this)	182
The concept of the blocks of a graph (67 is preparatory to this)	193
Degree-condition implying the existence of a path longer than a prescribed number: 70 (68 is preparatory to this)	195
Conditions for number of edges implying the existence of a path or circuit longer than a prescribed number: 71 and 72 respec- tively (69 and 70 are preparatory to this)	196
Condition for number of edges implying the existence of vertex- disjoint circuits: 80 (73, 75 and 76 are preparatory to this)	200
Condition for number of edges implying the existence of edge- disjoint circuits: 81 (74 and 77—79 are preparatory to this)	203
Exercises, Problems	204

Chapter 7

<i>Solutions to the exercises and problems</i>	209
--	-----

Chapter 1	209
Chapter 2	216
Chapter 3	222
Chapter 4	226
Chapter 5	236
Chapter 6	244

Source Index	263
--------------	-----

References	265
------------	-----

Subject Index	267
---------------	-----



INTRODUCTION TO GRAPHS

A tournament between several teams is first considered. A certain number of matches has already been played and a clear picture is required to show which these are. This could take the form of a figure in which each team corresponds to a point; since all matches are between two teams, any match can be represented by a line joining the two corresponding points. Let us indicate all the completed matches in this way. A symbol corresponding to the particular team, should be written beside each point, otherwise some misunderstanding might arise because lines corresponding to matches could intersect each other and the points of intersection might appear to represent teams as well. Therefore, instead of points, small circles are used to denote the teams. Figure 1 illustrates the following situation: There are five teams: a, b, c, d, e , and the following matches have already been played:

$$\begin{array}{lll} a-d, & a-e, & b-c, \\ b-d, & c-d, & c-e, \end{array}$$

Figure 1 is called the *graph* of the situation. (The word originated from the possibility of graphical demonstration.) The circles and the lines are called the *vertices* and *edges* of the graph, respectively.

The words *point*, *node*, *junction* are also used instead of vertex. The edge corresponding to the match $a-d$ is sometimes denoted by $\{a, d\}$. Obviously $\{a, d\}$ and $\{d, a\}$ denote the same edge. Similarly $\{a, e\}$ is the same as $\{e, a\}$, etc. The vertices a and d are also called the *endpoints* of the edge $\{a, d\}$; the edge $\{a, d\}$ joins the vertices a and d , or it is incident to a and d , a is a *neighbour* of d , a is *adjacent* to d , or a and d are *adjacent vertices*. Figure 1 is the graph G_1 , containing the vertices a, b, c, d and e and the edges $\{a, d\}$, $\{a, e\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$ and $\{c, e\}$.

It is possible that a given team has not yet played a match; and

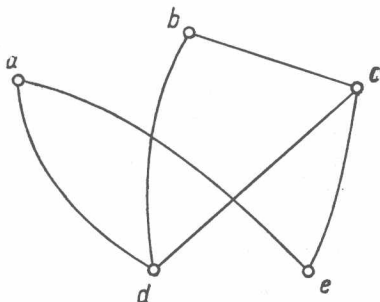


Fig. 1

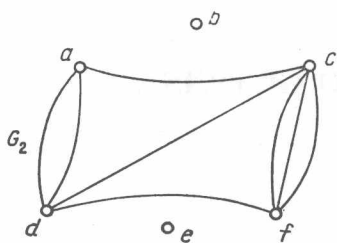


Fig. 2

in certain competitions there may be several matches between the same two teams. The graph G_2 of Fig. 2 shows situations corresponding to both these cases. The vertices without incident edges are called *isolated vertices*. If two or more edges join the same pair of vertices, the graph is said to contain *multiple edges*. b and e are therefore isolated vertices of G_2 . The edges connecting c and f can be distinguished

by subscripts, for example $\{c, f\}_1$, $\{c, f\}_2$, $\{c, f\}_3$. Similarly the edges corresponding to matches between a and d are $\{a, d\}_1$ and $\{a, d\}_2$.

Acquaintance between certain people can be represented by similar figures if the acquaintance is assumed to be mutual. Each person corresponds to a vertex and an edge joins two vertices if the two people know each other. The fact that a person a knows himself can be illustrated by an edge incident to vertex a only (see Fig. 3). Edges like this are usually called *loops*.

In what follows a figure is called a *graph* if it contains points and lines (vertices and edges), and each line joins two — not necessarily distinct — vertices. The number of the ends of the edges incident to the vertex p is called the *degree* or the *valency* of p and is denoted by $\varphi(p)$. A vertex of degree n is sometimes called *n -valent*.

Figure 3 contains a single vertex, a single edge, and $\varphi(a) = 2$. The graph G_3 of Fig. 4 contains 8 vertices and 10 edges, two of the edges being loops; in this graph $\varphi(a_8) = 0$, $\varphi(a_4) = \varphi(a_5) = 1$, $\varphi(a_6) = 2$, $\varphi(a_7) = 3$, $\varphi(a_1) = \varphi(a_2) = 4$ and $\varphi(a_3) = 5$.



Fig. 3

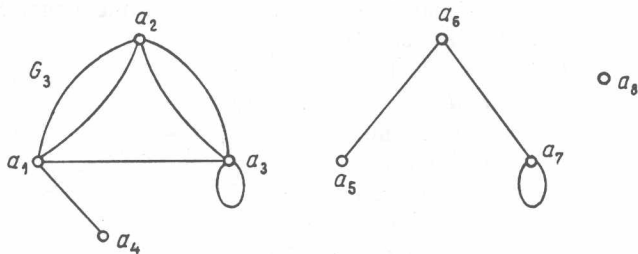


Fig. 4