INTRODUCTORY GRAPH THEORY

BÉLA ANDRÁSFAI

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BRISTOL

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TO THE MEMORY OF RÓZSA PÉTER

PREFACE

In giving an introductory lecture on mathematical topics one can usually start with the phrase: "It was the Greek civilization which first developed the elements of ...". All classical topics suggest this way of presentation, but even most of the "modern" ones were developed from classical nuclei, with a continually increasing level of abstraction. Although the first significant paper – Euler's – was written in 1736, graph theory is fairly new having been mentioned in mathematics only in about the last hundred years, following Kirchhoff's results on electrical networks. On the other hand, it has become an independent branch of mathematics in the last two decades owing to its rapid development. It should not be called modern, however, in this sense since it is still near to original thinking, still full of clarity, and its "charm" recalls the Greek civilization. It is a branch of combinatorics, it offers a great variety of natural problems and possible applications, it does not require advanced mathematical tools – but sometimes deep consideration. For this reason graph theory is an excellent research field for young people interested in mathematics. This has let to problems and results of graph theory becoming part of up-to-date school curricula. Extensive lecturing having made me aware of the interest at all levels, I wrote the Hungarian version of the present book in 1969, since no Hungarian book on the subject existed. (The first scientific monograph on graph theory was, in fact, written by the Hungarian professor Dénes Kőnig in 1936, but it was in German.) However, a large number of different books on graph theory have been published in the last 15 years (see References).

I have tried to give the exact proof of almost all the statements in the book – the easy theorems as well as the more difficult ones. The results are presented *in statu nascendi*, following the procedure of discovery, solution of sub-statements, definition of new concepts which prove to be useful, and determination of the possibilities of generalization from the solution of practical problems. Exercises, problems and their solutions are given throughout, with suggestions of new problems, simplification of complicated statements, and, above all, stimulation of readers.

The exercises are easy: a little drawing and calculation leads the reader to the solution of problems. A number of problems preceded by an asterisk (*) appear at the end of each chapter. The purpose of these is to develop the reader's ability to solve problems on topics treated in the chapter itself. Their solutions are presented in Chapter 7, but individual work on them is strongly recommended. This kind of activity is suggested throughout with the reader drawing graphs to illustrate the statements in the text. Even the wealth of illustrations in the book will not replace individual drawing, as this alone enables one to see the evolution of the figure. In this way, becoming familiar with the content of the book leads to the reader's own discovery of the results.

To avoid confusion the source of the statements is not indicated in the text but in the source-index. Chapters are not formally divided into sections, but the grouping of exercises and/or problems usually indicates a division; subjects covered are summarized in the Contents. In addition, to emphasize the various important methods of graph theory, they are not only mentioned in the subject index, but set in bold-faced type in the text. All the exercises, problems and statements are numbered as a whole, within each chapter, including those at the end of the chapter. Numbering recommences at the beginning of each chapter.

The first chapter deals with basic concepts; the following five consider five areas of graph theory, including some of the more modern results. Certain topics of interest are not discussed here, e.g. the relation of graphs to surfaces, matrices, and probability theory; map-colouring problems, detailed topological description of electrical networks and the solution of transportation problems. Most of these problems will be contained in the second volume of this book, which is in preparation. The reference list contains suggestions for further reading.

Those involved in mathematics at any level should find this book most useful; in fact, this is true for all whose work involves problem-solving since the development of the ability to solve problems by thinking in terms of graph theory is of benefit in any field.

I am especially grateful to the referees of the Hungarian version of the book, Professors R. Péter and T. Gallai, for their criticism, valuable suggestions and their comments on the mathematics as well as the presentation. Particular thanks are due to Professor P. Erdős, who suggested and fostered the idea of the English edition. I should like to thank A. Recski for his translation and constructive remarks and the staff of Akadémiai Kiadó for their helpfulness.

Béla Andrásfai

CONTENTS

Chapter 3

Chapter 4

CHAPTER 1

INTRODUCTION TO GRAPHS

A tournament between several teams is first considered. A certain number of matches has already been played and a clear picture is required to show which these are. This could take the form of a figure in which each team corresponds to a point; since all matches are between two teams, any match can be represented by a line joining the two corresponding points. Let us indicate all the completed matches in this way. A symbol corresponding to the particular team, should be written beside each point, otherwise some misunderstanding might arise because lines corresponding to matches could intersect each other and the points of intersection might appear to represent teams as well. Therefore, instead of points, small circles are used to denote the teams. Figure 1 illustrates the following situation: There are five teams: *a*, *b*, *c*, *d*, *e*, and the following matches have already been played:

$$a-d, \quad a-e, \quad b-c,$$
$$b-d, \quad c-d, \quad c-e,$$

Figure 1 is called the *graph* of the situation. (The word originated from the possibility of graphical demonstration.) The circles and the lines are called the *vertices* and *edges* of the graph, respectively.

The words *point*, *node*, *junction* are also used instead of vertex. The edge corresponding to the match $a-d$ is sometimes denoted by $\{a, d\}$. Obviously $\{a, d\}$ and $\{d, a\}$ denote the same edge. Similarly $\{a, e\}$ is the same as $\{e, a\}$, etc. The vertices *a* and *d* are also called the *endpoints* of the edge $\{a, d\}$; the edge $\{a, d\}$ *joins* the vertices *a* and *d*, or it is *incident* to *a* and *d*, *a* is a *neighbour* of *d*, *a* is *adjacent* to *d*, or *a* and *d* are *adjacent vertices*. Figure 1 is the graph G_1, containing the vertices *a*, *b*, *c*, *d* and *e* and the edges $\{a, d\}$, $\{a, e\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$ and $\{c, e\}$.

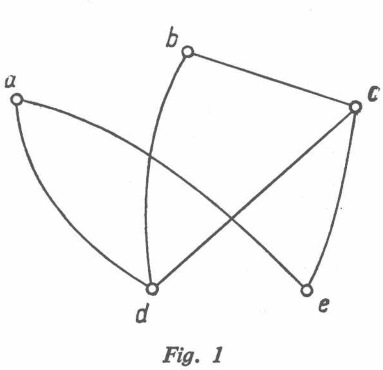

Fig. 1

It is possible that a given team has not yet played a match; and

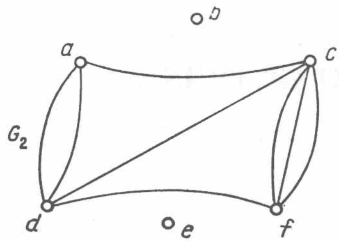

Fig. 2

in certain competitions there may be several matches between the same two teams. The graph G_2 of Fig. 2 shows situations corresponding to both these cases. The vertices without incident edges are called *isolated vertices*. If two or more edges join the same pair of vertices, the graph is said to contain *multiple edges*. b and e are therefore isolated vertices of G_2. The edges connecting c and f can be distinguished by subscripts, for example $\{c,f\}_1$, $\{c,f\}_2$, $\{c,f\}_3$. Similarly the edges corresponding to matches between a and d are $\{a, d\}_1$ and $\{a, d\}_2$.

Acquaintance between certain people can be represented by similar figures if the acquaintance is assumed to be mutual. Each person corresponds to a vertex and an edge joins two vertices if the two people know each other. The fact that a person a knows himself can be illustrated by an edge incident to vertex a only (see Fig. 3). Edges like this are usually called *loops*.

In what follows a figure is called a *graph* if it contains points and lines (vertices and edges), and each line joins two – not necessarily distinct – vertices. The number of the ends of the edges incident to the vertex p is called the *degree* or the *valency* of p and is denoted by $\varphi(p)$. A vertex of degree n is sometimes called *n-valent*.

Figure 3 contains a single vertex, a single edge, and $\varphi(a) = 2$. The graph G_3 of Fig. 4 contains 8 vertices and 10 edges, two of the edges being loops; in this graph $\varphi(a_8) = 0$, $\varphi(a_4) = \varphi(a_5) = 1$, $\varphi(a_6) = 2$, $\varphi(a_7) = 3$, $\varphi(a_1) = \varphi(a_2) = 4$ and $\varphi(a_3) = 5$.

Fig. 3

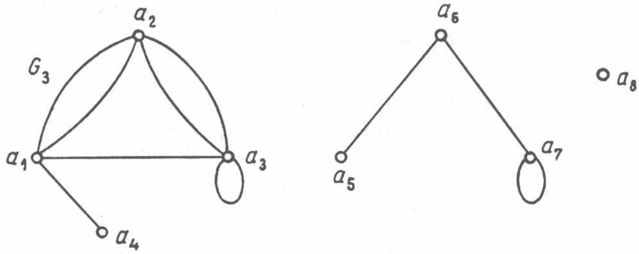

Fig. 4