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Quantum Independent Increment Processes I

From Classical Probability to Quantum Stochastic Calculus

Lecture Notes in Mathematics

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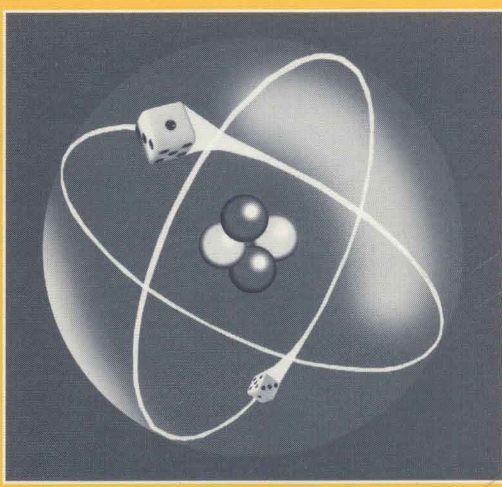
Uwe Franz



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Dedicated to the memory of Paul-André Meyer

Preface

This volume is the first of two volumes containing the lectures given at the School "Quantum Independent Increment Processes: Structure and Applications to Physics". This school was held at the Alfried Krupp Wissenschafts}-kolleg in Greifswald during the period March 9 – 22, 2003. We thank the lecturers for all the hard work they accomplished. Their lectures give an introduction to current research in their domains that is accessible to Ph. D. students. We hope that the two volumes will help to bring researchers from the areas of classical and quantum probability, operator algebras and mathematical physics together and contribute to developing the subject of quantum independent increment processes.

We are greatly indebted to the Volkswagen Foundation for their financial support, without which the school would not have been possible.

Special thanks go to Mrs. Zeidler who helped with the preparation and organisation of the school and who took care of the logistics.

Finally, we would like to thank the students for coming to Greifswald and helping to make the school a success.

Greifswald,
February 2005

Michael Schürmann
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Structure of Quantum Lévy Processes, Classical Probability and Physics

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Introduction

Random variables and stochastic processes are used to describe the behaviour of systems in a vast range of areas including statistics, finance, actuarial mathematics and computer science, as well as engineering, biology and physics. Due to an unavoidable lack of information about the state of the system concerned at a given moment in time, it is often impossible to predict these fluctuations with certainty — think of meteorology, for example. The unpredictable behaviour may be due to more fundamental reasons, as is the case in quantum mechanics. Here Heisenberg uncertainty limits the accuracy of simultaneous predictions of so-called complementary observables such as the position and momentum of a particle.

If the random fluctuations do not depend on time or position, then they should be described by stochastic processes which are homogeneous in space and time. In Euclidean space this leads to the important class of stochastic processes called Lévy processes, which have independent and stationary increments ([Lév65]). These processes have been attracting increasing interest over the last decade or so (see [Sko91], [Ber96], [Sat99], [BNMR01] and [App04]).

In quantum mechanics complete knowledge of the state is still insufficient to predict with certainty the outcomes of all possible measurements. Therefore its statistical interpretation has to be an essential part of the theory. Quantum probability starts from von Neumann's formulation of quantum mechanics ([vN96]) and studies quantum theory from a probabilistic point of view. Two key papers in the field are [AFL82] and [HP84].

A typical situation where quantum noise plays a role is in the description of a 'small' quantum system interacting with its 'large' environment. The state of the environment, also called heat bath or reservoir, cannot be measured or controlled completely. However it is reasonable to assume, at least as a first approximation, that it is homogeneous in time and space, and that the influence of the system on the heat bath can be neglected.

In concrete models the heat bath is generally described by a Fock space. The Hilbert space for the joint 'system plus heat bath' is then the tensor product of the Hilbert space representing the system with this Fock space. The separate time evolutions of the heat bath and system are coupled through their interaction to yield a unitary evolution of the system plus heat bath which is a cocycle with respect to the free evolution of the heat bath. Thus, through interaction (in other words, considered as an open system), the evolution of the system becomes non-unitary. In the Heisenberg picture this is given by a quantum dynamical semigroup, that is a one-parameter semigroup of completely positive maps (rather than *-automorphisms) on the system observables, see *Quantum Markov processes and applications to physics*, by Burkhard Kümmerer, in volume two of these notes. In the physics literature the dual Schrödinger picture is usually preferred; this is adopted in the influential monograph [Dav76].

Fock spaces arose in quantum field theory and in representation theory as continuous tensor products. The close connection between independent increment processes on the one hand, and current representations and Fock space on the other, was realised in the late sixties and early seventies ([Ara70], [PS72] and [Gui72], see also the survey article [Str00]). The development of a quantum stochastic calculus was a natural sequel to this discovery. This calculus involves the integration of operator 'processes', that is time-indexed families of operators adapted to a Fock-space filtration, with respect to the so-called creation, preservation and annihilation processes. It is modelled on the Itô integral, but in fact may be based on the nonadapted stochastic calculus of Hitsuda and Skorohod, see part three of this volume, *Quantum stochastic analysis — an introduction*, by Martin Lindsay. The relationship between classical and quantum stochastic calculus is also the subject of the final lecture of part one, *Lévy processes in Euclidean spaces and groups*, by David Applebaum.

Part four of this volume, *Dilations, cocycles and product systems* by Rajarama Bhat, concerns the relation between the unitary evolution of the *closed* system plus heat bath and the quantum dynamical semigroup which is the evolution of the *open* system itself. It addresses the question of which unitary evolutions correspond to a given quantum dynamical semigroup.

Formally, quantum groups arise from groups in a similar way to how quantum probability arises from classical probability, and to how C^*-algebra theory is now commonly viewed as noncommutative topology. Namely, one casts the axioms for a group (or probability space, or topological space) in terms of the appropriate class of functions on the group (respectively, probability, or topological space). This yields a commutative algebra with extra structure, and the quantum object is then defined by dropping the commutativity axiom. This procedure has been successfully applied to differential geometry ([Con94]).

For example taking the algebra of representative functions on a group (i.e. those functions which can be written as matrix elements of a finite-dimensional representation of the group), one obtains the axioms of a commutative Hopf algebra ([Swe69]). Dropping commutativity, one arrives at one definition of a Hopf algebra. At least in finite dimension, the Hopf algebra axioms give a satisfactory definition of a (finite) quantum group.

Similarly the essentially bounded measurable functions on a probability space, with functions equal almost everywhere identified, form a commutative von Neumann algebra on which the expectation functional yields a state which is faithful and normal. Conversely, every commutative von Neumann algebra with faithful normal state is isomorphic to such an algebra of (measure equivalence classes of) random variables on a probability space with state given by the expectation functional.

Thus the axioms depend on the choice of functions. For example *all* functions on a group form a Hopf algebra only if the group is finite. The guiding principle for finding the 'right' set of axioms is that it should yield a rich theory which incorporates a good measure of the classical theory. In the case of quantum probability there is a straightforward choice. A unital *-algebra with a state is called an algebraic noncommutative probability space, and simply a noncommutative probability space when the algebra is a von Neumann algebra and the state is normal. In the latter case the state is often, but not always, assumed to be faithful. In fact recent progress in the understanding of noncommutative stochastic independence has benefitted from a loosening of the axioms to allow noninvolutive algebras, see *Lévy processes on quantum groups and dual groups*, by Uwe Franz in volume two of these notes.

In what is now known as topological quantum group theory, the search for the 'right' foundations has a long history. Only recently have Kustermans and Vaes obtained a relatively simple set of axioms that is both rich enough to contain all the examples one would want to consider as quantum groups whilst still having a satisfactory duality theory, see part two of this volume, *Locally compact quantum groups*, by Johan Kustermans.

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Lévy Processes in Euclidean Spaces and Groups

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1 Introduction

"Probability theory has always generated its problems by its contact with other areas. There are very few problems that are generated by its own internal structure. This is partly because, once stripped of everything else, a probability space is essentially the unit interval with Lebesgue measure."

S.R.S. Varadhan, AMS Bulletin January (2003)

One of the most beautiful and fruitful ideas in probability theory is that of *infinite divisibility*. For a random variable to be infinitely divisible, we require that it can be decomposed as the sum of n independent, identically distributed random variables, for any natural number n. Many distributions of importance for both pure and applied probability have been shown to be infinitely divisible and some of the best known in a very long list are the normal law, the Poisson and compound Poisson laws, the t-distribution, the χ^2 distribution, the log-normal distribution, the stable laws, the normal inverse Gaussian and the hyperbolic distributions. The basic ideas of infinite divisibility chrystallised during the heroic age of classical probability in the 1920s and 1930s - the key result is the beautiful Lévy-Khintchine formula which gives the general form of the characteristic function for an infinitely divisible probability distribution. Another important discovery from this era is that such distributions are precisely those which arise as limit laws for row sums of asymptotically negligible triangular arrays of independent random variables. Gnedenko and Kolmogorov [40] is a classic text for these results - for a more modern viewpoint, see Jacod and Shiryaev [51].

When we pass from single random variables to stochastic processes, the analogue of infinite divisibility is the requirement that the process has stationary and independent increments. Such processes were first investigated systematically by Paul Lévy (see e.g. Chapter 5 of [56]) and now bear his name in honour of his groundbreaking contributions.

Many important stochastic processes are Lévy processes - these include Brownian motion, Poisson and compound Poisson processes, stable processes and subordinators. Note that any infinitely divisible probability distribution can be embedded as the law of $X(1)$ in some Lévy process $(X(t), t \geq 0)$. A key structural result, which gives great insight into sample path behaviour, is the Lévy-Itô decomposition which asserts that any Lévy process can be decomposed as the sum of four terms - a deterministic (drift) which increases linearly with time, a diffusion term which is controlled by Brownian motion, a compensated sum of small jumps and a (finite) sum of large jumps. In particular, this shows that Lévy processes are a natural subclass of semimartingales with jumps (see e.g. [66], [51]).

Lévy processes are also Markov (in fact Feller) processes and their infinitesimal generators are represented as integral perturbations of a second order elliptic differential operator, in a structure which mirrors the Lévy-Khintchine form. Alternatively, the generator is represented as a pseudo-differential operator with a symbol determined by the Lévy-Khintchine formula. This latter structure is paradigmatic of a wide class of Feller processes, wherein the symbol has the same form but an additional spatial dependence. This is a major theme of Niels Jacob's books ([48, 49, 50]).

The last decade has seen Lévy processes come to the forefront of activity in probability theory and there have been several major developments from both theoretical and applied perspectives. These include fluctuation theory ([19]), codification of the genealogical structure of continuous branching processes ([55]), investigations of turbulence via Burger's equation ([20]), the study of stochastic differential equations with jumps and associated stochastic flows [54], construction of Euclidean random fields [2], properties of linearly viscoelastic materials [23], new examples of times series [24] and a host of applications to option pricing in "incomplete" financial markets (see e.g. [75], chapter 5 of [13], and references therein). In addition, two important monographs have appeared which are devoted to the subject ([19], [74]) and a third is to appear shortly ([13]). Since 1998, conferences to review and discuss new developments have taken place on an annual basis - the proceedings of the first of these are collected in [15].

The first four sections of these notes aim to give an overview of the key structural properties of Lévy processes taking values in Euclidean space, and of the associated stochastic calculus. They are based very closely on parts of