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Edited by A. Dold, B. Eckmann and F. Takens

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Proceedings of the 11th International Seminar held in Sukhumi (Abkhazian Autonomous Republic) USSR, Sept. 25–Oct. 1, 1987

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INTRODUCTION

An International Seminar on Stability Problems for Stochastic Models was held in Sukhumi (Abkhasian Autonomic Republic) from 25 September till 1 October 1987. This seminar was the eleventh since the Steklov Mathematical Institute USSR Acad.Sci. (SMI) launched them in 1974*).

Traditionally other Institutes and Universities take part in organizing of the seminars too. Thus the Institute for Systems Studies (ISS) is a permanent co-organizer of these seminars. Essential help in the organizing and holding of the seminars (and, particularly in that of the Seminar-87) was received from International Research Institute for Management Sciences.

An active role in the organizing of the seminar in Sukhumi was played by the Abkhasian State University. The Rector of the University, Prof. Z.Avidzba, Vice-Rector Prof. O.Damenia and our colleagues from the University R.Absava, A.Gvaramia and L.Karba. All of them were members of Organizing Committee and we are grateful to them for their hospitality.

Participants of the seminar lived and worked on the Black Sea shore in the tourist hotel "XX s'ezd VLKCM". Remembering the good conditions which were created for us we especially thank the head of the Abkhasian tourist office, N.Akaba, and the director of the hotel, G.Meshveliani.

There were more than 100 participants at the seminar representing scientific centres and universities of 13 countries of Europe, Asia, Africa and America (both North and South). More than 60 reports were delivered during the 5 days.

The variety of topics of these reports can be explained by a

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*) See LN in Math., volumes 982, 1155, 1233

tradition: the principal aim of the seminar is to publicise ideas and methods used in stability theory of stochastic models and it does not imply a rigid topic selection for the reports.

The reports delivered made up the basis for two volumes of Proceedings. One of them traditionally is published by ISS Publishers in Russian*). The other is the present one.

The preparation of the manuscript of the Proceedings demanded a great deal of activity. Our sincere words of gratitude are addressed to active and permanent participants of the seminar L.B.Klebanov from Leningrad (for it was there that the final preparation of the manuscript took place) and I.A.Melamed from Tbilisi.

All the authors are indebted to Acad. L.D.Faddeev (adviser of the USSR Subseries of LNM) and Dr. A.P.Oskolkov for the possibility to meet again under the cover of a Lecture Notes volume.

V.M.Zolotarev

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*) All of these Proceedings are being translated into English in the "Journal of Soviet Mathematics", published by Plenum Publishers.

CONTENTS

Introduction

THE DENSITY FUNCTION'S ASYMPTOTIC REPRESENTATION IN THE CASE OF MULTIDIMENSIONAL STRICTLY STABLE DISTRIBUTIONS

S.V.Arkhipov

Introduction

During the last few years a continually increasing attention was payed to a peculiar - the so called strictly stable-class of multidimensional distributions, and the interest has not stopped growing up to the present. One of the reasons may be that in a certain sense this class forms the most part of the stable distributions.

To describe the strictly stable laws it will be advantageous for us to observe their characteristic functions (ch.f.) in a form not considered previously (except the case of $\alpha = 1$):

$$f(t) = \exp(g(|t|\tau), \quad t \in \mathbb{R}^n, \quad n \geq 2, \quad \tau = t/|t|,$$

$$g(t\tau) = \begin{cases} \Gamma(-\alpha)\, t^{\alpha} \int\limits_{S^{n-1}} (-i\tau, \xi)^{\alpha} \mathcal{M}(d\xi), & \alpha \in (0,1) \cup (1,2), \\ -\frac{\pi}{2}|t| \int\limits_{S^{n-1}} (\tau, \xi)\, \mathcal{M}(d\xi) + i(\tau, b), & \alpha = 1, \end{cases} \tag{1}$$

where $\mathcal{M}$ is a finite measure on the unit sphere $S^{n-1} = \{\xi : \xi = 1, \xi \in \mathbb{R}^n\}$, having for $\alpha = 1$ the supplementary property:

$$\int\limits_{S^{n-1}} \xi\, \mathcal{M}(d\xi) = 0.$$

The power of a complex number in (1) is interpreted with the aid of the principal value

$$(-iy)^{\alpha} = |y|^{\alpha} \exp\left(-\frac{\pi\alpha i}{2} \operatorname{sign} y\right), \quad y \in \mathbb{R}.$$

Except some particular cases there does not exist any clear expression for the density function of a stable law, consequently the investigation has to deal with different exact ot asymptotic representation of the density. The material, presented here, may be regarded as a further development of results, published in 12 . There was discussed the case $0<\alpha<2$, when the spectral density μ belonged to the special space $C^{\infty}(S^{n-1})$.

Further we change this requirement to a less restricting one supposing μ to be chosen from the famous Liouville space $L_2^{r}(S^{n-1})$. The spectral density μ belongs to $L_2^{r}(S^{n-1})$, $r>0$, when $\mu \in L_2(S^{n-1})$ and in in addition it has r derivative (r can be a fraction).

A fundamental problem is to get an asymptotical expression for the density $p(x)$ when $|x| \to \infty$. Taking the case of $\mu \in C^{\infty}(S^{n-1})$, when the density function $p(x)$ has a complete asymptotical decomposition, the situation considered now has the peculiarity that depending on the smoothness one can get only a finite order asymptotical decomposition.

The embedding theorems enable us to transfer our results to spectral densities chosen from Hölder spaces.

The method we suggest constructing the asymptotical form of the density function gives the analogous result for the derivatives of the density without any difficulties.

2. The fundamental result

Let $\{Y_{l_j}(\xi)\}$ be an orthonormal system of spherical harmonics (sph.h.) in $L_2(S^{n-1})$. The linearly independent harmonics of order l are indexed by j, $j=1,\ldots,d(l)$, where

$$d(l) = \frac{(2l+n-2)(l+n-3)!}{(n-2)!\; l!}$$

A survey of the theory of sph.h. can be found e.g. in [9] .

THEOREM. Let us suppose that the density μ of the spectral measure belongs to $L_2^r(S^{n-1})$, $\alpha \le r < \infty$. The following asymptotical representation holds true $x \neq 0$:

$$p(|x|\xi) = \frac{\mu(\xi)}{|x|^{\alpha+n}} + \sum_{k=2}^{m-1} \frac{2^{\alpha k}}{\pi^{n/2}\,\Gamma(k+1)} \cdot \frac{\mu_k(\xi)}{|x|^{\alpha k+n}} + R_m(x), \qquad (2)$$

where $\xi = x/x$ and the functions $\mu_k(\xi)$ are given on S^{n-1} by the following series, converging in the mean square sense

$$\mu_k(\xi) = \sum_{l=0}^{\infty} \sum_{j=1}^{d(l)} (-i)^l \frac{\Gamma((l+\alpha k+n)/2)}{\Gamma((l-\alpha k)/2)} g_{l_j}^k \mathcal{J}_{l_j}(\xi) \qquad \text{for } \alpha \neq 1,$$

$$\mu_k(\xi) = \sum_{l=L}^{\infty} \sum_{j=1}^{d(2l)} (-1)^l \frac{\Gamma(l+\frac{k+n}{2})}{\Gamma(l-\frac{k}{2})} g_{2l,j}^k \mathcal{J}_{2l,j}(\xi) \qquad \text{for } \alpha = 1.$$

$$L = \begin{cases} \frac{k}{2}+1 & \text{, when } k \text{ is even} \\ 0 & \text{, when } k \text{ is odd} \end{cases}$$

and where

$$g_{l_j}^k = \int_{S^{n-1}} g^k(\tau)\, \mathcal{J}_{l_j}(\tau)\, d\tau .$$

The remainder $R_m(x)$ can be estimated in the following way

$$\|R_m(x)\|_1 = \Big\{ \int_{S^{n-1}} (R_m(|x|\xi))^2 d\xi \Big\}^{1/2} \le \frac{C_1(m)}{|x|^{\alpha m+n}}; \qquad (3)$$

where $C_1(m)$ is a constant, depending on m, α and $\mu(\xi)$, furthermore on satisfies the condition

$$2 \leq m \leq [r/\alpha] + 1. \tag{4}$$

The upper bound for m shows the maximal possible number of the numbers in the representation (2).

COROLLARY 1. Assume that $\mu(\xi) \in L_2^r(S^{n-1})$, $r \geq \alpha + n/2$. So the functions $\mu_k(\xi)$ can be regarded as elements from the Hölder space of the sphere. More accurately $\mu_k(\xi) \in C_*^{r-\alpha(k-1)-n/2}(S^{n-1})$, if the exponent of smoothness is an integer, and $\mu_k(\xi) \in C^{r-\alpha(k-2)-n/2}(S^{n-1})$ otherwise. The representation (2) remains true, but the estimation (3) must be changed as follows:

$$\|R_m(x)\|_2 = max\{ |R_m(|x|\xi)| : \xi \in S^{n-1} \} \leq \frac{C_2(m)}{x^{\alpha m + n}},$$

where

$$2 \leq m \leq [\frac{r - n/2}{\alpha}] + 1.$$

COROLLARY 2. Let X be a strictly stable random vector in $\mathbb{R}^n$, and G a cone with its vertex at zero such that $g \in \operatorname{supp} \mu$, where $g \in S^{n-1} \cap G$. Denoting by

$$G(u(\xi)) = \{ x = |x|\xi : \ |x| \geq |u(\xi)|, \ \xi \in g \}$$

we have

$$P\{X \in G(u(\xi))\} = (\alpha + n - 1)^{-1} \int_g \mu(\xi)(|u(\xi)|)^{-(\alpha+n-1)} d\xi +$$

$$+ \sum_{k=2}^{m-1} 2^{\alpha k} (\alpha k + n - 1)^{-1} \pi^{-n/2} (\Gamma(k+1))^{-1} \int_g \mu_k(\xi)(|u(\xi)|)^{-(\alpha k + n - 1)} d\xi +$$

$$+ \int_g R_m(|u(\xi)|\xi) \, d\xi.$$

This representation is asymptotical in the sense that the remainder tends to zero when

$$\min\{|u(\xi)| : \xi \in g\} \longrightarrow \infty .$$

3. Some preliminary results from the theory of functional spaces on the sphere

The system $\{\mathcal{Y}_{l_j}(\xi)\}$ of sph.h. is complete in $L_2(S^{n-1})$ and every function $\theta(\xi) \in L_2(S^{n-1})$ can be expanded into series converging the the mean square sense in the following way (see [5], § 31):

$$\theta(\xi) = \sum_{l,j} \theta_{l_j} \mathcal{Y}_{l_j}(\xi),$$

where

$$\theta_{l_j} = \int_{S^{n-1}} \theta(\xi) \mathcal{Y}_{l_j}(\xi)\, d\xi.$$

DEFINITION 1. The operator determined by the equation

$$T\theta(\xi) = \sum_{l,j} t_l \theta_{l_j} \mathcal{Y}_{l_j}(\xi)$$

is called the multiplier operator, and its spectrum $\{t_l\}$ the multiplier by spherical harmonics.

Now let us turn to the multiplier operator $(E+\delta)^{r/2}$ with the multiplier $\{(1+l(l+n-2))^{r/2}\}$, where E is the unit operator and and δ is the spherical part of the Laplacian, or the Laplace-Beltram operator (cf [5], § 31).

DEFINITION 2. For $0<r<\infty$ we call the space $L_2^r(S^{n-1})$ the sphere S^{n-1}, satisfying that

$$(E+\delta)^{r/2}\theta(\xi) \in L_2(S^{n-1}).$$

We note immediately that the spaces $L_2^r(S^{n-1})$ determined just now coincide in the the sets of their elements with the well-known Sobolev-Slo detzki spaces $W_2^r(S^{n-1})$. For the sake of better comparability of the he results we recall the definition used in the theory of singular integrals ([5] , § 31), and which differs from the one given regularly in the theory of partial differential equation.

Let θ be extended by constant (i.e. $\tilde{\theta}(x) = \theta(x/|x|)$) to the spherical segment Ω: $0 < \varrho_1 \leq \varrho \leq \varrho_2 < \infty$. The space denoted by $W_2^r(S^{n-1})$ consists of those functions extended to Ω as mentioned above, they belong to the usual Sobolev-Slobodetzki space $W_2^r(\Omega)$ (cf. [10]).

REMARK 1. Usually in the theory of multivariate differentiable functions theorems are completely proved only for the classes of functions defined on the whole $\mathbb{R}^n$. With the aid of multiplication of the function $\theta(\xi) \in W_2^r(\Omega)$ by the function $\theta_0(\varrho) \in C^\infty(\mathbb{R})$: $\theta_0(\varrho) = 1$ on $[\varrho_1, \varrho_2]$ and $\theta_0(\varrho) = 0$ outside the segment $[\varrho_1/2, 2\varrho_2]$ we get the needed extension to $\mathbb{R}^n$ saving the class. This gives the possibility to extend the theorems proved for $\mathbb{R}^n$ to arbitrary domain $\Omega \subset \mathbb{R}^n$ (cf. [6]).

The the results of [1] it follows that

i) $L_2^r(S^{n-1}) = W_2^r(S^{n-1})$, $r > 0$,

ii) for $r > 0$ the space $L_2^r(S^{n-1})$ consists of distributions.

LEMMA 1. If $\theta_1(\xi)$, $\theta_2(\xi) \in L_2^r(S^{n-1})$, $r > n/2$ then $\theta_1(\xi) \cdot \theta_2(\xi) \in L_2^r(S^{n-1})$.

PROOF. The statement of the theorem is the obvious consequence of the remark 1, the results of [11] on multipliers in $\mathbb{R}^n$ and i).

Now we define the Hölder spaces of functions on the sphere.

DEFINITION 3. We say that $\theta(\xi) \in C^\lambda(S^{n-1})$, $\lambda > 0$, when the function $\theta(x/|x|) \in C^{[\lambda]}(\mathbb{R}^n \setminus \{0\})$ ($[\lambda]$ is the integer part of λ), and in addition if $\lambda \neq [\lambda]$ then the derivatives of order $[\lambda]$: $u_k(x) = (D^k\theta)x$, $|k| = [\lambda]$ satisfy the following condition on the sphere:

$$|u_k(\xi_1) - u_k(\xi_2)| \leq C\,|\xi_1 - \xi_2|^{\lambda - [\lambda]}, \qquad \xi_1, \xi_2 \in S^{n-1}.$$

DEFINITION 4. We say that $\theta(\xi) \in C_*^{\lambda}(S^{n-1})$, $\lambda = 1,2,\ldots$, when $\theta(x/|x|) \in C^{\lambda-1}(\mathbb{R}^n \setminus \{0\})$, and the derivatives of order $\lambda - 1$: $u_k(x) = (D^k\theta)(x)$, $|k| = \lambda - 1$ satisfy the following condition on the sphere:

$$|u(\xi_1) - u(\xi_2)| \le C|\xi_1 - \xi_2| \ln 2(|\xi_1 - \xi_2|)^{-1}, \quad \xi_1, \xi_2 \in S^{n-1}.$$

LEMMA 2. If the function $\theta(\xi)$ belongs to the space $L_2^r(S^{n-1})$, $r > n/2$, then

$$\theta(\xi) \in C_*^{r-n/2}(S^{n-1}) \quad \text{, when } r - n/2 \text{ is integer}$$

$$\theta(\xi) \in C^{r-n/2}(S^{n-1}) \quad \text{otherwise.}$$

The proof is based on the embedding theorems of the spaces $W_2^r(\mathbb{R}^n)$ into $H_\infty^{r-n/2}(\mathbb{R}^n)$ (see [6], p.229, [7], p.67), furthermore on the remark 1 and i).

LEMMA 3. Let $\{t_l\}$ be the spectrum of the multiplier operator T. T is a continuous operator from $L_2^r(S^{n-1})$ to $L_2^{r+u}(S^{n-1})$ iff $t_l = O(l^{-u})$.

PROOF. The lemmas statement directly follows from the proposition 6.1 of [1] and from ii).

The following lemma explains the connection between the order of smoothness of the functions $g(r)$ and $\mu(\xi)$ from (1).

LEMMA 4. If $\mu(\xi) \in L_2^r(S^{n-1})$, $r > 0$ then $g(r) \in L_2^{r+\frac{n}{2}+\alpha}(S^{n-1})$, $0 < \alpha < 2$ and we have the summation formulae:

$$g(r) = \pi^{n/2} 2^{-\alpha} \sum_{l,j} i^l \Gamma((l-\alpha)/2)(\Gamma((l+n+\alpha)/2))^{-1} \mu_{l_j} \mathcal{I}_{l_j}(r), \quad \alpha \ne 1, \tag{5}$$

$$g(r) = \frac{1}{2}\pi^{n/2} \sum_{l=0}^{\infty} \sum_{j=1}^{d(2l)} (-1)^l \Gamma(l - \tfrac{1}{2})(\Gamma(l - (n+1)/2))^{-1} \mu_{2l,j} \mathcal{I}_{2l,j}(r), \quad \alpha = 1. \tag{6}$$