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Approximation and Optimization

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PREFACE

This volume contains the proceedings of the Seminar on Approximation and Optimization, which took place in January 12-16, 1987 at the University of Havana, Havana, Cuba. The seminar was jointly organized by the University of Havana, the Cuban Academy of Science and the Cuban Mathematical Society to promote scientific contacts between specialists of two very closely related branches of mathematics, namely approximation theory and optimization theory.

We wish to thank the International Mathematical Union and the International Council of Scientific Unions for sponsoring the seminar: their financial support was decisive in obtaining a considerable participation from mathematicians of Western Europe, North America and Latin America. The Third World Academy of Sciences also made a financial support.

The contributions to this volume include original research papers as well as a few survey articles. All these papers were refereed. We have divided the contents into three sections: the first one contains the papers submitted by some of the invited speakers; in the last two, the rest of the papers are classified according to their contents.

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Nonparametric Polynomial Density Estimation in the L^p Norm

Z. CIESIELSKI

Abstract. *A simple construction of polynomial estimators for densities and distributions on the unit interval is presented. For densities from certain Lipschitz classes the error for the mean L^p deviation is characterized. The Casteljeau algorithm for calculating the values of the estimators is applied.*

1. Introduction. The space of all real polynomials of degree not exceeding m is denoted by Π_m. In Π_m we have the Bernstein basis i.e.

$$\Pi_m = \operatorname{span}[N_{i,m}, i = 0, \ldots, m],$$

where

$$N_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \qquad i = 0, \ldots, m.$$

The Casteljeau algorithm is based on the identity

$$N_{i,m}(x) = (1-x)N_{i,m-1}(x) + xN_{i-1,m-1}(x). \tag{1.1}$$

For given $w \in \Pi_m$

$$w(x) = \sum_{i=0}^{m} w_i N_{i,m}(x), \tag{1.2}$$

where the coefficients w_i are unique. Using (1.1) we find that for $0 \le k \le m$

$$w(x) = \sum_{i=0}^{m-k} w_i^{(k)}(x) N_{i,m-k}(x), \tag{1.3}$$

where $w_i^{(k)} \in \Pi_k$, and for $0 \le k < m$ we have

$$w_i^{(k+1)}(x) = (1-x)w_i^{(k)}(x) + x w_{i+1}^{(k)}(x), \qquad i = 0, \ldots, m-k-1. \tag{1.4}$$

In particular, $w(x) = w_0^{(m)}(x) = const.$

Some more properties of the Bernstein polynomials will be needed. Our attention will be restricted to the interval $I = [0,1]$ and the following notation will be used

$$(f,g) = \int_I f(x)g(x)\,dx,$$

$$\|f\|_p = \left(\int_I |f|^p\right)^{\frac{1}{p}}.$$

It is convenient to use simultaneously with $N_{i,m}$ the polynomials

$$M_{i,m} = (m+1)N_{i,m}.$$

The following elementary properties of the polynomials $N_{i,m}$ and $M_{i,m}$ will be used:

1°. $N_{i,m}(x) \geq 0$ for $x \in I, i = 0,\ldots,m$.

2°.

$$\sum_{i=0}^{m} N_{i,m} = 1.$$

3°. $(M_{i,m}, 1) = 1$, for $i = 0,\ldots,m$.

4°. For w as in (1.2) we have

$$\begin{aligned} Dw &= m\sum_{i=0}^{m-1} \Delta w_i N_{i,m-1} \\ &= \sum_{i=0}^{m-1} \Delta w_i M_{i,m-1}, \end{aligned}$$

where $\Delta w_i = w_{i+1} - w_i$ and $Dw = dw/dx$.

5°. For $i = 0,\ldots,m$

$$DN_{i,m} = M_{i-1,m-1} - M_{i,m-1}$$

with $M_{j,m} = 0$ whenever $j < 0$ or $j > m$.

2. Polynomial operators. A linear operator in a function space with range contained in Π_m for some m is called a *polynomial operator* . The space of all real functions of bounded variation on I which are left continuous is denoted by $BV(I)$ and it is equipped with the norm

$$\|F\|_{BV(I)} = |F(0)| + var(F).$$

Moreover, define

$$\mathbf{D}(I) = \{F \in BV(I) : F \textit{ is nondecreasing on } I,\ F(0) = 0,\ F(1) = 1\}$$

The polynomial operator T_m is now defined for $F \in BV(I)$ by the formula

$$T_mF(x) = \sum_{i=0}^{m} \int_I M_{i,m}\,dF \int_0^x N_{i,m}(y)\,dy. \tag{2.1}$$

It then follows that

$$T_m : BV(I) \to \Pi_{m+1}, \tag{2.2}$$

and

$$T_m : \mathbf{D}(I) \to \Pi_{m+1} \cap \mathbf{D}(I). \tag{2.3}$$

The polynomial operators corresponding to the densities are going to be defined naturally by means of the kernel

$$R_m(x,y) = \sum_{i=0}^{m} M_{i,m}(x) N_{i,m}(y). \tag{2.4}$$

It follows by the definitions and properties of $M_{i,m}$ and $N_{i,m}$ that

$$R_m(x,y) = R_m(y,x), \quad 0 \le R_m(x,y) \le m+1 \quad for \quad x, y \in I. \tag{2.5}$$

Define

$$R_m f(x) = \int_I R_m(x,y) f(y)\, dy$$

Clearly, $R_m : L^p \to \Pi_m$ and since by (2.4)

$$R_m f = \sum_{i=0}^{m} (M_{i,m}, f) N_{i,n}, \tag{2.6}$$

it takes by 2° and 3° densities into densities.

It is worth to notice that for F being absolutely continuous (2.1) gives

$$DT_m F = R_m DF. \tag{2.7}$$

PROPOSITION 2.8. *For F in $BV(I)$ we have*

$$T_m F(x) = F(0)(1-x)^{m+1} + F(1)x^{m+1} + \sum_{i=1}^{m} (F, M_{i-1,m-1}) N_{i,m+1}(x).$$

PROOF: Direct computation gives

$$\int_I M_{i,m}\, dF = (m+1)\{\delta_{i,m} F(1) - \delta_{i,0} F(0) + (F, M_{i,m-1}) - (F, M_{i-1,m-1})\},$$

and therefore by 5°

$$\begin{aligned}T_mF(x) &= F(0) + \sum_{i=0}^{m} \int_I M_{i,m}\,dF \int_0^x N_{i,m}(x)\,dx \\ &= F(0)(1-x)^{m+1} + F(1+)x^{m+1} \\ &+ \sum_{i=0}^{m-1} (F, M_{i,m-1}) \int_0^x (m+1)\big(N_{i,m}(y) - N_{i+1,m}(y)\big)\,dy \\ &= F(0)(1-x)^{m+1} + F(1+)x^{m+1} \\ &+ \sum_{i=0}^{m-1} (F, M_{i,m-1}) \int_0^x DN_{i+1,m+1}(y)\,dy.\end{aligned}$$

3. Approximation properties of the polynomial operators. In this section we state the necessary results on approximation by the operators T_m and R_m. The following is a consequence of Proposition 2.8

COROLLARY 3.1. *For $m = 0, 1, \dots,$ and $F \in BV(I)$ we have*

$$\|T_mF\|_\infty \le 3\|F\|_\infty, \tag{3.2}$$

and for $F, G \in \mathbf{D}(I)$

$$\|T_mF - T_mG\|_\infty \le \|F - G\|_\infty. \tag{3.3}$$

PROPOSITION 3.4. *For $f \in L^p(I)$ we have*

$$\|R_mf\|_p \le \|f\|_p, \quad m = 0, 1, \dots, \tag{3.5}$$

and if $f \in L^p(I)$, then

$$\|f - R_mf\|_p \to 0 \quad as \quad m \to \infty. \tag{3.6}$$

For the proof we refer to [1].

PROPOSITION 3.7. *Let $F \in C(I)$. Then $\|F - T_mF\|_\infty \to 0$ as $m \to \infty$.*

PROOF: Since (3.2) takes place it is sufficient to check the statement for absolutely continuous F. However, in this case (2.6) implies for $f = DF$

$$|F(x) - T_mF(x)| \le \|DF - DT_mF\|_1 = \|f - R_mf\|_1,$$

and the last term by (3.6) tends to 0 as $m \to \infty$.

In order to define the proper Lipschitz classes following [6] we need the step-weight function

$$\phi(x) = \sqrt{x(1-x)}, \; x \in I$$

and the symmetric difference of the second order

$$\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h).$$

Now, the modulus of smoothness with step-weight ϕ is given as follows

$$\omega_{2,\phi,p}(f;\delta) = \sup_{0<h<\delta} \|\Delta^2_{h\phi(x)} f(x)\|_2,$$

where $\Delta^2_{h\phi(x)}$ is zero whenever either $x + h\phi(x)$ or $x - h\phi(x)$ is not in I. Now, we can formulate the important for us auxiliary result (see [5], Theorm 3.4).

PROPOSITION 3.8. *Let α, and f be given such that $0 < \alpha < 1$, $f \in L^p(I)$, $1 \le p \le \infty$. Then*

$$\|f - R_m f\|_p = O\left(\frac{1}{m^\alpha}\right) \quad as \quad m \to \infty \quad \Longleftrightarrow \quad \omega_{2,\phi,p}(f;\delta) = O\left(\delta^{2\alpha}\right) \quad as \quad \delta \to 0_+.$$

4. The estimators. Let us start with a simple sample of sise $n : X_1, \dots X_n$. It is assumed that the common distribution function F of these i.i.d. random variables has its support in I. For the given sample let us introduce

$$f_{m,n}(x) = \frac{1}{n} \sum_{j=0}^{n} R_m(X_j, x), \quad x \in I. \tag{4.1}$$

Clearly $f_{m,n}$ is a polynomial of degree m which, by (2.6), is a density on I. Let now F_n be the empirical distribution i.e. $F_n = |\{i : X_i < x\}|/n$ and let

$$F_{m,n} = T_m F_n. \tag{4.2}$$

It follows by (2.1) that

$$DF_{m,n} = f_{m,n}. \tag{4.3}$$

PROPOSITION 4.4. *Let F and $X_1, X_2, \dots$ be given as above. Then*

$$P\{\ F_{m,n} \Longrightarrow F \ \ as \ \ m,n \to \infty\} = 1,$$

where $\Longrightarrow$ means the weak convergence of probability distribution functions.

PROOF: Let us start with following identity

$$F - F_{m,n} = (F - T_m F) + T_m(F - F_n). \tag{4.5}$$

It will be shown at first that $T_m F$ converges weakly to F as $m \to \infty$ for each $F \in \mathbf{D}(I)$. For ϕ continuous on $(-\infty, \infty)$ and with compact support according to (2.1) and (3.6) we obtain

$$\int_{-\infty}^{\infty} \phi\, dT_m F = \int_0^1 R_m(\phi|_I)\, dF \to \int_{-\infty}^{\infty} \phi\, dF, \quad as \quad m \to \infty.$$

For the second part of (4.5) we obtain by (3.3) that

$$\|T_m(F - F_n)\|_\infty \le \|F - F_n\|_\infty,$$

but by Glivenko's theorem (see [8])

$$P\{\|F - F_n\|_\infty \to 0 \quad as \quad n \to \infty\} = 1.$$

Thus, with probability one $T_m(F - F_n)$ tends uniformly on I to 0 as $m, n \to \infty$. Since $F(x) - T_m F(x)$ tends to zero as $m \to \infty$ at each continuity point of F it follows by (4.5) that with probability one $F_{m,n}(x) \to F(x)$ with at all such points.

PROPOSITION 4.6. *Let $F \in \mathbf{D}(I) \cap C(I)$. Then*

$$P\{\|F - F_{m,n}\|_\infty \to 0 \text{ as } m, n \to \infty\} = 1.$$

This follows from the proof of Proposition 4.4 and by Proposition 3.7. We need the following inequality from [7].

LEMMA 4.7. *For continuous probability distribution F on $(-\infty, \infty)$ there are constants C, γ such that $0 < \gamma \le 2$, $0 < C < \infty$, and*

$$Pr\{\|F - F_n\|_\infty > \frac{\lambda}{\sqrt{n}}\} \le Ce^{-\gamma\lambda^2} \qquad for \qquad \lambda > 0,\ n = 1, 2, \ldots$$

For later convenience let us introduce the set of all densities on I

$$\mathbf{P}(I) = \{f \in L^1(I) : \int_I f = 1,\ f \ge 0\}.$$

LEMMA 4.8. *Let $f \in L^p(I) \cap \mathbf{P}(I)$ for some p, $1 \le p \le \infty$. Then*

$$|\, \|f - f_{m,n}\|_p - \|f - R_m f\|_p| \le \|R_m f - f_{m,n}\|_p \le 2(m+1)\|F - F_n\|_\infty$$

Moreover, for each finite p there is finite C such that

$$\left(\mathcal{E}\|R_m f - f_{m,n}\|_p^p\right)^{\frac{1}{p}} \le C\,\frac{m+1}{\sqrt{n}} \qquad for \quad n, m+1 = 1, 2, \ldots$$

PROOF: The (2.1), 2° and 5° of Section 1 give for fixed $x \in I$

$$|R_m f(x) - f_{m,n}(x)| = |DT_m(F - F_n)(x)|$$

$$= |\sum_{i=0}^{m} \int_I M_{i,m}\, d(F - F_n)\ N_{i,m}(x)| = |\sum_{i=0}^{m} \int_I (F - F_n)(y) DM_{i,m}(y)\, dy\ N_{i,m}(x)|$$

$$\le \|F - F_n\|_\infty \sum_{i=0}^{m} \|DM_{i,m}\|_1 N_{i,m}(x) \le 2(m+1)\|F - F_n\|_\infty,$$

whence $\|R_m f - f_{m,n}\|_p \le 2(m+1)\|F - F_n\|_\infty$, and this completes the first part of the proof, which in combination with Lemma 4.7 gives the second part.

THEOREM 4.9. *Let either $f \in L^p(I) \cap \mathbf{P}(I)$ for some p, $1 \le p < \infty$, or $f \in C(I) \cap \mathbf{P}(I)$, and let $m = [n^\beta]$ for some $\beta > 0$. Then, for $0 < \beta < \frac{1}{2}$*

$$Pr\{\|f - f_{m,n}\|_p = o(1) \quad as \quad n \to \infty\} = 1.$$

Moreover, if $0 < \alpha < 1$, $0 < \beta < \frac{1}{2}\frac{1}{1+\alpha}$, then the following conditions are equivalent:

(i)
$$\omega_{2,\phi,p}(f;\delta) = O(\delta^{2\alpha}) \qquad as \qquad \delta \to 0_+,$$

(ii)
$$Pr\{\|f - f_{m,n}\|_p = O(\frac{1}{n^{\alpha\beta}}) \quad as \quad n \to \infty\} = 1.$$

PROOF: We know from [1] that $\|f - R_m f\|_p = o(1)$. On the other hand Lemma 4.7 gives for $\epsilon > 0$

$$(4.10) \qquad Pr\{(m+1)\|F - F_n\|_\infty > \epsilon\} \le C\, e^{-\gamma\epsilon^2\frac{n}{m^2}} \le C\, e^{-\gamma\epsilon^2 n^{1-2\beta}}.$$

This implies
$$Pr\{(m+1)\|F - F_n\|_\infty = o(1) \quad as \quad n \to \infty\} = 1.$$

To complete the first part of the proof it is sufficient now to apply Lemma 4.8. Substituting in (4.10) $\epsilon = \frac{c}{n^{\alpha\beta}}$ we get

$$(4.11) \qquad Pr\{(m+1)\|F - f_n\|_\infty > \frac{c}{n^{\alpha\beta}}\} \le C\, e^{-\gamma c^2 n^{1-2\beta}},$$

which implies

$$Pr\{(m+1)\|F - F_n\|\infty = O(\frac{1}{n^{\alpha\beta}}) \quad as \quad n \to \infty\} = 1.$$

Now the equivalence of (i) and (ii) follows by Lemma 4.8.

Next Theorem concerns the order of the mean L^p deviations for the estimators $f_{m,n}$. To this end we need the following auxiliary inequalities. The first is elementary and it is well known.

PROPOSITION 4.12. *Let $J =< -a, a >$, $a > 0$, $R = (-\infty, \infty)$. Then,*

$$0 \le |x+h|^p + |x-h|^p - 2|x|^p \le p(p-1)a^{p-2}|h|^2 \quad for \quad p > 2,\ x+h, x-h \in J.$$

To formulate the second inequality we recall the definition of the customary second order modulus of smoothness i.e for $g \in C(J)$ define

$$(4.13) \quad \omega_{2,\infty}(g;\delta)_J = \sup_{x_1,x_2 \in J, |x_1 - x_2| \le 2\delta} |g(\frac{x_1 + x_2}{2}) - \frac{g(x_1) + g(x_2)}{2}|, \qquad 0 < \delta \le \frac{1}{2}.$$

The following useful estimate we find in [9].

PROPOSITIOPN 4.14. *Let X be a random variable with values in J, J being finite or infinite interval, $EX^2 < \infty$ and let $g \in C(J)$. Then*

$$|g(EX) - Eg(X)| \leq 15\, \omega_{2,\infty}\left(g; \frac{1}{2}\sqrt{E(X-EX)^2}\right)_J .$$

LEMMA 4.15. *Let $f \in L^p(I) \cap \mathbf{P}(I)$, $1 \leq p < \infty$. Then, for the sample $X_1, \dots, X_n$ corresponding to the density f we have*

$$\left(E\|R_m f - f_{m,n}\|_p^p\right)^{\frac{1}{p}} \leq C\, \frac{(m+1)^{\frac{1}{q\wedge 2}}}{n^{\frac{1}{p\vee 2}}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $a \vee b = max(a,b)$, $a \wedge b = min(a,b)$.

PROOF: The case $1 \leq p \leq 2$ is easy. As in [2] we have

$$\left(E\|R_m f - f_{m,n}\|_p^p\right)^{\frac{1}{p}} \leq \left(E\|R_m f - f_{m,n}\|_2^2\right)^{\frac{1}{2}} \leq \left(\frac{m+1}{n}\right)^{\frac{1}{2}}.$$

Let now $p > 2$ and let for $i = 0, \dots, m$,

$$X^{(i)} = \frac{1}{n}\sum_{j=1}^{n}\left(M_{i,m}(X_j) - EM_{i,m}(X_j)\right).$$

It follows that the values of $X^{(i)}$ are in $J = (-m-1, m+1)$. Since $EX^{(i)} = 0$, Proposition 4.14 applied to $X^{(i)}$ and to $g(x) = |x|^p$ gives

$$|Eg(X^{(i)})| \leq 15\, \omega_{2,\infty}\left(g; \frac{1}{2}\sqrt{\frac{1}{n}\int_I M_{i,m}^2 f}\right)_J,$$

whence by Proposition 4.12

$$E|X^{(i)}|^p \leq C^p (m+1)^{p-2}\frac{1}{n}\int_I M_{i,m}^2 f \leq C^p \frac{(m+1)^p}{n}\int_I N_{i,m} f.$$

Now, by Jensen's inequality

$$E\|R_m f - f_{m,n}\|_p^p = \int_I E\left|\sum_{i=0}^{m} X^{(i)} N_{i,m}(y)\right|^p dy$$

$$\leq \int_I \sum_{i=0}^{m} E|X^{(i)}|^p N_{i,m}(y)\, dy = \frac{1}{m+1}\sum_{i=0}^{m} E|X^{(i)}|^p \leq C^p \frac{(m+1)^{p-1}}{n}.$$

To formulate the last result we introduce for $0 < \alpha < 1$ and $1 \leq p < \infty$

$$\psi(\alpha,p) = \begin{cases} \frac{1}{2\alpha+1}, & \text{if } 1 \leq p < 2; \\ \frac{1}{p\alpha+p-1}, & \text{if } 2 \leq p < 2+\frac{1}{\alpha+1}; \\ \frac{1}{2\alpha+2}, & \text{if } p \geq 2+\frac{1}{\alpha+1}. \end{cases}$$

THEOREM 4.16. *Let $1 \leq p < \infty$ and let $f \in L^p(I) \cap \mathbf{P}(I)$. Let α, $0 < \alpha < 1$, and β, $0 < \beta \leq \psi(\alpha,p)$ be given. Moreover, let $m = [n^\beta]$. Then the following conditions are equivalent:*

(i)
$$\omega_{2,\phi,p}(f;\delta) = O(\delta^{2\alpha}) \quad as \quad \delta \to 0_+,$$

(ii)
$$(E\|f - f_{m,n}\|_p^p)^{\frac{1}{p}} = O(\frac{1}{n^{\alpha\beta}}) \quad as \quad n \to \infty.$$

For the proof we apply Lemmas 4.15, 4.8 and Proposition 3.8.

COROLLARY 4.17. *Under the assumptions of Theorem 4.16 the best choice of β with respect to (ii) is given by formula $\beta = \psi(\alpha,p)$.*

EXAMPLE: Using the examples on page 228 of [6] we find that for $1 \leq p < 2$ the arcsin law density i.e. for

$$f_0(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}, \qquad x \in <0,1>$$

we have

$$\omega_{2,\phi,p}(f_0;\delta) \sim \delta^{\frac{2}{p}-1} \quad as \quad \delta \to 0_+.$$

Thus, for this density $\alpha = \frac{1}{p} - \frac{1}{2}$ and the optimal choice for β is $\beta = \frac{p}{2}$.

5. Algorithm for computing the density and distribution estimators. Let $X_1, \ldots, X_n$ be given as in the previous section. Since

$$f_{m,n}(x) = \frac{1}{n}\sum_{j=1}^{n} R_m(X_j,x),$$

to compute $f_{n,m}(x)$ for fixed x we need to compute $R_m(X_j,x)$ for $j = 1,\ldots,n$. However,

$$R_m(X_j,x) = \sum_{i=0}^{m} M_{i,m}(X_j)N_{i,m}(x)$$

and therefore we use the Casteljeau algorithm for the first time to compute $M_{i,m}(X_j)$ and for the second time to calculate $R_m(X_j,x)$. Now, the density $f_{m,n}$ has also the following representation

$$f_{m,n}(x) = \sum_{i=0}^{m} a_i N_{i,m}(x),$$

where

$$a_i = \frac{1}{n}\sum_{j=1}^{n} M_{i,m}(X_j), \qquad i = 0,\dots,m.$$

Thus, at almost no cost the following coefficients

$$b_0 = 0,\ b_1 = 1,\ b_j = \frac{a_0 + \dots + a_{j-1}}{m+1}, \qquad j = 1,\dots,m+1$$

can be computed. To compute $F_{m,n}(x)$ one applies once more the Casteljeau algorithm to the following formula

$$F_{m,n}(x) = \int_0^x f_{m,n}(y)\,dy = \sum_{j=1}^{m+1} b_j M_{j,m+1}(x).$$

6. Comments. This note is related to [3] and [2] but the tools used here are different. This made it possible to extend the results from [2]. The author is indebted to G. Krzykowski who has brought to our attention Lemma 4.7.

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Local Spline Interpolation Schemes in One and Several Variables

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Abstract: In the first part of this paper we briefly review some recent results pertaining to the construction of compactly supported fundamental functions for univariate Lagrange interpolation by splines. In the second part of the paper we discuss several possible extensions of these results to a multivariate setting.*

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1. Introduction

The importance of splines for the numerical solution of interpolation problems is a well-established fact. Nevertheless, computationally spline interpolation requires the solution of large sparse linear systems whose order is roughly equal to the number of data being interpolated. This is reflected in the fact that generally the interpolant at any point depends on all the data. Equivalently, if we let $\{L_i\}_{i\in\mathbb{Z}}$ be the fundamental Lagrange splines for interpolation on the sequence $\{x_i\}_{i\in\mathbb{Z}}$, that is,

$$L_i(x_j) = \delta_{ij}, \; i,j \in \mathbb{Z} \tag{1.1}$$

then each L_i is supported on all of $\mathbb{R}$. It seems desirable to have fundamental functions of *compact support*. This can prove useful when updating of the interpolant is desired as new data are available or when solving several smaller linear systems to determine the Lagrange splines is preferred over solving one large set of equations. The use of compactly supported fundamental functions has already proved useful in numerical grid generation [7] as well as in computer aided design, [1].

An efficient method for constructing Lagrange splines of compact support is the addition of knots beyond those chosen at the data locations. The problem then is how to use these degrees of freedom in such a way that either shape control and/or high accuracy is achieved. Various such questions have been systematically analyzed in [6]. Some of these results are briefly reviewed in Section 2. In Section 3 we propose several extensions of these results to multivariate interpolation problems. Due to the wider variety of possibilities in the multivariate case, these results only provide an initial investigation into a problem that has important applications for practical data fitting in several variables.

2. Univariate Compactly Supported Fundamental Functions

Let $X = \{x_i\}_{i\in\mathbb{Z}}$ be a strictly increasing sequence of real numbers. As usual, the B-splines of order k on X are defined by

$$N_{i,k,X}(x) = (x_{i+k} - x_i)\,[x_i, \dots, x_{i+k}](\bullet - x)_+^{k-1}$$

where $[x_i, \dots, x_{i+k}]f$ denotes the k-th order divided difference of f and

$$x_+^{\ell} = \begin{cases} x^{\ell}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

For any fixed integer $1 \leq q \leq k-1$ the function

$$L_{i+q}(x) = \left(\prod_{\substack{j=1 \\ j\neq q}}^{k-1} \frac{x - x_{i+j}}{x_{i+q} - x_{i+j}} \right) N_{i,k,X}(x)/N_{i,k,X}(x_{i+q}) \tag{2.1}$$

is a piecewise polynomial of degree $2k-3$ having $k-2$ continuous derivatives. Moreover, as is clearly apparent

$$L_i(x_j) = \delta_{ij}, \; i,j \in \mathbb{Z}, \tag{2.2}$$

and

$$\operatorname{supp} L_i = [x_{i-q}, \; x_{i-q+k}] = \operatorname{supp} N_{i-q,k,X}.$$