Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

749

V. Girault
P.-A. Raviart

Finite Element Approximation of the Navier-Stokes Equations

Springer-Verlag
Berlin Heidelberg New York

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

749

V. Girault
P.-A. Raviart

Finite Element Approximation of the Navier-Stokes Equations

Springer-Verlag
Berlin Heidelberg New York 1979

Authors

Vivette Girault
Pierre-Arnaud Raviart

Analyse Numérique
Tour 55–65, 5ème étage
Université Pierre et Marie Curie
4, Place Jussieu
F-75230 Paris Cedex 05

AMS Subject Classifications (1970): 35Q10, 65-06, 65MXX, 65M15, 65N30, 76-XX

ISBN 3-540-09557-8 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-09557-8 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data
Girault, Vivette, 1943-
Finite element approximation of the Navier-Stokes equations.
(Lecture notes in mathematics; 749)
Bibliography: p.
Includes index.
1. Viscous flow. 2. Navier-Stokes equations-- Numerical solutions. 3. Finite element method. I. Raviart, P. A., 1939-- joint author. II. Title. III. Series: Lecture notes in mathematics (Berlin); 749.
QA3.L28 no. 749 [QA929] 510'.8s [519.4] 79-21917
ISBN 0-387-09557-8

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2141/3140-543210

INTRODUCTION

The contents of this publication have been taught at the University Pierre & Marie Curie as a graduate course in numerical analysis during the academic year 1977-78.

In the last few years, many engineers and mathematicians have concentrated their efforts on the numerical solution of the Navier-Stokes equations by finite element methods. The purpose of this series of lectures is to provide a fairly comprehensive treatment of the most recent mathematical developments in that field. It is not intended to give an exhaustive treatment of all finite element methods available for solving the Navier-Stokes equations. But instead, it places a great emphasis on the finite element methods of mixed type which play a fundamental part nowadays in numerical hydrodynamics. Consequently, these lecture notes can also be viewed as an introduction to the mixed finite element theory.

We have tried as much as possible to make this text self-contained. In this respect, we have recalled a number of theoretical results on the pure mathematical aspect of the Navier-Stokes problem and we have frequently referred to the recent book by R. Temam [44]. The reader will find in this reference further mathematical material.

Besides R. Temam, the authors are gratefully indebted to M. Crouzeix for many helpful discussions and for providing original proofs of a number of theorems.

CONTENTS

CHAPTER I

MATHEMATICAL FOUNDATION OF THE STOKES PROBLEM

§ 1 - GENERALITIES ON SOME ELLIPTIC BOUNDARY VALUE PROBLEMS

In this paragraph we study briefly the Dirichlet's and Neumann's problems for the harmonic and biharmonic operators.

1.1. Basic concepts on Sobolev spaces

Our purpose here is to recall the main notions and results, concerning the classical Sobolev spaces, which we shall use later on. Most results are stated without proof. The reader will find more details in the references listed at the end of this text .

To simplify the discussion, we shall work from now on with real-valued functions, but of course every result stated here will carry on to complex-valued functions.

Let Ω denote an open subset of $\mathbb{R}^n$ with boundary Γ . We define $\mathcal{D}(\Omega)$ to be the linear space of functions infinitely differentiable and with compact support on Ω . Then, we set

$$\mathcal{D}(\overline{\Omega}) = \{\varphi_{|\Omega} \ ; \ \varphi \in \mathcal{D}(\mathbb{R}^n)\} ,$$

or equivalently, if $\mathcal{O}$ denotes any open subset of $\mathbb{R}^n$ such that $\overline{\Omega} \subset \mathcal{O}$,

$$\mathcal{D}(\overline{\Omega}) = \{\varphi_{|\Omega} \ ; \ \varphi \in \mathcal{D}(\mathcal{O})\} .$$

Now, let $\mathcal{D}'(\Omega)$ denote the dual space of $\mathcal{D}(\Omega)$, often called the space of distributions on Ω . We denote by $< . , . >$ the duality between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$ and we remark that when f is a locally integrable function, then f can be identified with a distribution by

$$< f,\varphi > = \int_\Omega f(x)\varphi(x)\,dx \qquad \forall\ \varphi \in \mathcal{D}(\Omega) .$$

Now, we can define the derivatives of distributions. Let $\alpha = (\alpha_1 ,\dots, \alpha_n) \in \mathbb{N}^n$

and $|\alpha| = \sum_{i=1}^{n} \alpha_i$. For u in $\mathcal{D}'(\Omega)$, we define $\partial^\alpha u$ in $\mathcal{D}'(\Omega)$ by :

$$< \partial^\alpha u, \varphi > = (-1)^{|\alpha|} < u, \partial^\alpha \varphi > \qquad \forall \varphi \in \mathcal{D}(\Omega) ;$$

i.e. if $u \in \mathcal{C}^{|\alpha|}(\bar{\Omega})$ then $\partial^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$.

For $m \in \mathbb{N}$ and $p \in \mathbb{R}$ with $1 \leqslant p \leqslant \infty$, we define the Sobolev space :

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega) \ ; \ \partial^\alpha v \in L^p(\Omega) \quad , \quad \forall \ |\alpha| \leqslant m\} ,$$

which is a Banach space for the norm

(1.1) $$\| u \|_{m,p,\Omega} = (\sum_{|\alpha| \leqslant m} \int_\Omega |\partial^\alpha u(x)|^p dx)^{1/p} \quad , \quad p < \infty$$

or

$$\| u \|_{m,\infty,\Omega} = \sup_{|\alpha| \leqslant m} (\sup_{x \in \Omega} \text{ess} \ |\partial^\alpha u(x)|) \quad , \quad p = \infty .$$

We also provide $W^{m,p}(\Omega)$ with the following seminorm

(1.2) $$|u|_{m,p,\Omega} = (\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha u(x)|^p dx)^{1/p} \quad ,$$

for $p < \infty$, and we make the above modification when $p = \infty$.

When $p = 2$, $W^{m,2}(\Omega)$ is usually denoted by $H^m(\Omega)$, and if there is no ambiguity, we drop the subscript $p = 2$ when refering to its norm and seminorm. $H^m(\Omega)$ is a Hilbert space for the scalar product :

(1.3) $$(u,v)_{m,\Omega} = \sum_{|\alpha| \leqslant m} \int_\Omega \partial^\alpha u(x) \partial^\alpha v(x) dx .$$

In particular, we write the scalar product of $L^2(\Omega)$ with no subscript at all.

As $\mathcal{D}(\Omega) \subset H^m(\Omega)$, we define

(1.4) $$H_o^m(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^m(\Omega)} ,$$

i.e. $H_o^m(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ for the norm $\| \cdot \|_{m,\Omega}$. We denote by $H^{-m}(\Omega)$ the dual space of $H_o^m(\Omega)$ normed by :

(1.5) $$\| f \|_{-m,\Omega} = \sup_{\substack{v \in H_o^m(\Omega) \\ v \neq 0}} \frac{|< f,v >|}{\| v \|_{m,\Omega}} .$$

The following lemma characterizes the functionals of $H^{-m}(\Omega)$.

LEMMA 1.1.

A distribution f belongs to $H^{-m}(\Omega)$ if and only if there exist functions f_α in $L^2(\Omega)$, for $|\alpha| \leqslant m$, such that

$$f = \sum_{|\alpha| \leqslant m} \partial^\alpha f_\alpha .$$

THEOREM 1.1. (Poincaré-Friedrichs' inequality)

If Ω is connected and *bounded at least in one direction*, then for each $m \in \mathbb{N}$, there exists a constant C_m such that

$$\|v\|_{m,\Omega} \leqslant C_m |v|_{m,\Omega} \qquad \forall\, v \in H_o^m(\Omega) . \tag{1.6}$$

Hence the mapping $v \longmapsto |v|_{m,\Omega}$ is a norm on $H_o^m(\Omega)$ equivalent to $\|v\|_{m,\Omega}$.

In order to study more closely the boundary values of functions of $H^m(\Omega)$, we assume that Γ , the boundary of Ω , is *bounded and Lipschitz continuous* - i.e. Γ can be represented parametrically by Lipschitz continuous functions. Let $d\sigma$ denote the surface measure on Γ and let $L^2(\Gamma)$ be the space of square integrable functions on Γ with respect to $d\sigma$, equipped with the norm

$$\|v\|_{o,\Gamma} = \{\int_\Gamma (v(\sigma))^2 d\sigma\}^{1/2} .$$

THEOREM 1.2.

1°) $\mathcal{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$.

2°) There exists a constant C such that

$$\|\gamma_o\varphi\|_{o,\Gamma} \leqslant C\, \|\varphi\|_{1,\Omega} \qquad \forall\, \varphi \in \mathcal{D}(\overline{\Omega}) , \tag{1.7}$$

where $\gamma_o\varphi$ denotes the value of φ on Γ .

It follows from Theorem 1.2. that the mapping γ_o defined on $\mathcal{D}(\overline{\Omega})$ can be extended by continuity to a mapping, still called γ_o , from $H^1(\Omega)$ into $L^2(\Gamma)$, i.e. $\gamma_o \in \mathcal{L}(H^1(\Omega) ; L^2(\Gamma))$. By extension, $\gamma_o\varphi$ is called the boundary value of φ on Γ ; to simplify notations, we drop the prefix γ_o when it is clearly implied.

THEOREM 1.3.

1°) $\mathrm{Ker}(\gamma_o) = H_o^1(\Omega)$.

2°) The range space of γ_o is a proper and dense subspace of $L^2(\Gamma)$, called $H^{1/2}(\Gamma)$.

For μ in $H^{1/2}(\Gamma)$, we define

(1.8)
$$\| \mu \|_{1/2,\Gamma} = \inf_{\substack{v \in H^1(\Omega) \\ \gamma_o v = \mu}} \| v \|_{1,\Omega} .$$

The mapping $\mu \longmapsto \| \mu \|_{1/2,\Gamma}$ is a norm on $H^{1/2}(\Gamma)$, and $H^{1/2}(\Gamma)$ is a Hilbert space for this norm. Let $H^{-1/2}(\Gamma)$ be the corresponding dual space of $H^{1/2}(\Gamma)$, normed by

(1.9)
$$\| \mu^\star \|_{-1/2,\Gamma} = \sup_{\substack{\mu \in H^{1/2}(\Gamma) \\ \mu \neq 0}} \frac{|<\mu^\star,\mu>|}{\| \mu \|_{1/2,\Gamma}} ,$$

where again $< .,. >$ denotes the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. We remark that $<.,.>$ is an extension of the scalar product of $L^2(\Gamma)$ in the sense that when $\mu^\star \in L^2(\Gamma)$, we can identify $< \mu^\star,\mu >$ with $\int_\Gamma \mu^\star(\sigma)\mu(\sigma)\, d\sigma$.

Let $\vec{\nu} = (\nu_1 ,\dots, \nu_n)$ be the unit outward normal to Γ which exists almost everywhere on Γ thanks to the hypothesis of Lipschitz continuity. If v is a function in $H^2(\Omega)$, we define its normal derivative by :

(1.10)
$$\frac{\partial v}{\partial \nu} = \sum_{i=1}^{n} \nu_i \gamma_o \left(\frac{\partial v}{\partial x_i}\right) .$$

It can be proved that the mapping $v \longmapsto \frac{\partial v}{\partial \nu} \in \mathcal{L}(H^2(\Omega) ; H^{1/2}(\Gamma))$. Moreover, we can characterize $H_o^2(\Omega)$ as follows :

THEOREM 1.4.

$$H_o^2(\Omega) = \{v \in H^2(\Omega) ;\ \gamma_o v = 0 \ \underline{\text{and}}\ \frac{\partial v}{\partial \nu} = 0\} .$$

When Γ is sufficiently smooth, the range space of γ_o can also be extended as follows. For $m \in \mathbb{N}$, $m \geqslant 1$, we define $H^{m-1/2}(\Gamma)$ as the image of $H^m(\Omega)$ by the transformation γ_o , equipped with the norm :

$$\| f \|_{m-1/2,\Gamma} = \inf_{\substack{v \in H^m(\Omega) \\ \gamma_o v = f}} \| v \|_{m,\Omega} .$$

Then, it can be checked that $\frac{\partial u}{\partial \nu} \in H^{m-3/2}(\Gamma)$ for u in $H^m(\Omega)$, and the following result holds :

THEOREM 1.5.

The mapping $u \longmapsto \{\gamma_o u, \frac{\partial u}{\partial \nu}\}$ *defined on* $H^m(\Omega)$ *is onto* $H^{m-1/2}(\Gamma) \times H^{m-3/2}(\Gamma)$.

We close this section with two useful applications of the Green's formula.

LEMMA 1.2.

1°) *Let* u *and* $v \in H^1(\Omega)$. *Then, for* $1 \leq i \leq n$,

(1.11) $$\int_\Omega u \frac{\partial v}{\partial x_i} dx = - \int_\Omega \frac{\partial u}{\partial x_i} v \, dx + \int_\Gamma u v \nu_i \, d\sigma .$$

2°) *Moreover, if* $u \in H^2(\Omega)$, *then*

(1.12) $$\sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = - \sum_{i=1}^{n} \int_\Omega \frac{\partial^2 u}{\partial x_i^2} v \, dx + \sum_{i=1}^{n} \int_\Gamma \nu_i \frac{\partial u}{\partial x_i} v \, d\sigma .$$

Adopting the usual notations :

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} , \quad \overrightarrow{\text{grad}}\, u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}) ,$$

(1.12) becomes :

(1.13) $$(\overrightarrow{\text{grad}}\, u, \overrightarrow{\text{grad}}\, v) = - (\Delta u, v) + \int_\Gamma \frac{\partial u}{\partial \nu} v \, d\sigma .$$

1.2. Abstract elliptic theory

This section gives a brief account of a fundamental tool used in studying linear partial differential equations of elliptic type.

Let V be a real Hilbert space with norm denoted by $\| . \|_V$; let V' be its dual space and let $<.,.>$ denote the duality between V' and V .

Let $(u,v) \longmapsto a(u,v)$ be a real bilinear form on $V \times V$, ℓ an element of V' and consider the following problem :

(P) $$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = < \ell, v > \quad \forall v \in V . \end{cases}$$

The following theorem is due to Lax and Milgram [35] .

THEOREM 1.6.

We assume that a is continuous and elliptic on V , i.e. there exist two constants M and $\alpha > 0$ such that

(1.14) $$|a(u,v)| \leq M \|u\|_V \|v\|_V \qquad \forall\, u,v \in V$$

and

(1.15) $$a(v,v) \geq \alpha \|v\|_V^2 \qquad \forall\, v \in V .$$

Then problem (P) has one and only one solution u in V . Moreover, the mapping $\ell \mapsto u$ is an *isomorphism* from V' *onto* V .

COROLLARY 1.1.

When a is symmetric - i.e. $a(u,v) = a(v,u)$ $\forall\, u, v \in V$ - then the solution u of (P) is also the only element of V that minimizes the following quadratic functional (also called energy functional) on V :

(1.16) $$J(v) = \frac{1}{2} a(v,v) - \langle \ell, v \rangle .$$

1.3. Example 1 : Dirichlet's harmonic problem

In all the examples, we assume that Ω is bounded and Γ Lipschitz continuous.

Consider the following non-homogeneous Dirichlet's problem :

(D) Given f in $H^{-1}(\Omega)$ and g in $H^{1/2}(\Gamma)$, find a function u that satisfies

(1.17) $$-\Delta u = f \quad \text{in } \Omega$$

(1.18) $$u = g \quad \text{on } \Gamma .$$

Let us formulate this problem in terms of problem (P). We set $V = H_o^1(\Omega)$ and

$$a(u,v) = (\overrightarrow{\text{grad}}\, u, \overrightarrow{\text{grad}}\, v).$$

It is clear that a is continuous on $[H_o^1(\Omega)]^2$, and owing to Theorem 1.1,

$$a(v,v) = \|\overrightarrow{\text{grad}}\, v\|_{o,\Omega}^2 = |v|_{1,\Omega}^2 \geq C_1 \|v\|_{1,\Omega}^2 .$$

Besides that since $H^{1/2}(\Gamma)$ is the range space of γ_o , let u_o in $H^1(\Omega)$ satisfy $\gamma_o u_o = g$, and examine the following problem :

$$(D')\begin{cases}\text{Find } u \text{ in } H^1(\Omega) \text{ such that} \\ (1.19)\quad u - u_o \in H^1_o(\Omega) \\ (1.20)\quad a(u-u_o,v) = < f,v > - a(u_o,v) \quad \forall\, v \in H^1_o(\Omega) .\end{cases}$$

Since a is continuous, the mapping $v \overset{\ell}{\longmapsto} < f,v > - a(u_o,v)$ belongs to $H^{-1}(\Omega)$. Therefore, thanks to the Lax-Milgram theorem, problem (D') has one and only one solution u in $H^1(\Omega)$.

It remains only to prove that u may be characterized as the unique solution of problem (D). Taking $v \in \mathcal{D}(\Omega)$ in (1.20) gives :

$$a(u,v) = - < \Delta u,v > = < f,v > \qquad \forall\, v \in \mathcal{D}(\Omega) .$$

Hence u satisfies

$$(D_1)\begin{cases}(1.19)\quad u-u_o \in H^1_o(\Omega) , \\ (1.17)\quad -\Delta u = f \text{ in } H^{-1}(\Omega) .\end{cases}$$

Conversely, every solution of (D_1) is a solution of (D') by the density of $\mathcal{D}(\Omega)$ in $H^1_o(\Omega)$. But

$$u - u_o \in H^1_o(\Omega) \text{ iff } \gamma_o u = g ,$$

therefore problems (D_1) and (D) are the same.

As far as the regularity of u is concerned, we know, from the Lax-Milgram's theorem that the mapping $\ell \longmapsto u - u_o$ is an isomorphism from $H^{-1}(\Omega)$ onto $H^1(\Omega)$. Therefore,

$$\| u-u_o \|_{1,\Omega} \leqslant C_2 \| \ell \|_{-1,\Omega} .$$

Clearly,

$$\| \ell \|_{-1,\Omega} \leqslant \| f \|_{-1,\Omega} + \| u_o \|_{1,\Omega} .$$

Hence

$$\| u \|_{1,\Omega} \leqslant C_3 \{\| f \|_{-1,\Omega} + \| u_o \|_{1,\Omega} \}$$

$\forall\, u_o \in H^1(\Omega)$ such that $\gamma_o u_o = g$. From definition (1.8) this implies that

$$\| u \|_{1,\Omega} \leqslant C_3 \{\| f \|_{-1,\Omega} + \| g \|_{1/2,\Gamma}\} .$$

Thus, we have proved the following proposition :

PROPOSITION 1.1.

Problem (D) has one and only one solution u in $H^1(\Omega)$ and

(1.21) $$\|u\|_{1,\Omega} \leqslant C\{\|f\|_{-1,\Omega} + \|g\|_{1/2,\Gamma}\} ,$$

i.e. u depends continuously upon the data of (D).

Remarks 1.1.

1°) Let $m \in \mathbb{N}$, $m \geqslant 1$. When Γ is sufficiently smooth, it can be shown that if $f \in H^{m-2}(\Omega)$ and $g \in H^{m-1/2}(\Gamma)$, then $u \in H^m(\Omega)$ and

(1.22) $$\|u\|_{m,\Omega} \leqslant C(\|f\|_{m-2,\Omega} + \|g\|_{m-1/2,\Gamma}) .$$

2°) When Γ is only Lipschitz continuous, the same result is still valid for $m = 2$, provided Ω is convex. ∎

1.4. Example 2 : Neumann's harmonic problem

Here, we assume in addition that Ω is connected and we deal with the non-homogeneous Neumann's problem :

(N) Find u such that :

(1.23) $$-\Delta u = f \text{ in } \Omega ,$$

(1.24) $$\frac{\partial u}{\partial \nu} = g \text{ on } \Gamma ,$$

where $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\Gamma)$ satisfy the relation :

(1.25) $$\int_\Omega f\,dx + \langle g, 1\rangle_\Gamma = 0 .$$

Since problem (N) only involves the derivatives of u , it is clear that its solution is never unique. We turn the difficulty by seeking u in the quotient space $H^1(\Omega)/\mathbb{R}$ equipped with the quotient norm :

(1.26) $$\|\dot{v}\|_{H^1(\Omega)/\mathbb{R}} = \inf_{v \in \dot{v}} \|v\|_{1,\Omega} .$$

The theorem below states an important property of this space ; its proof can be found in Nečas [39] .

THEOREM 1.7.

The space $H^1(\Omega)/\mathbb{R}$ is a Hilbert space for the quotient norm (1.26). Moreover, on this space the seminorm $\dot{v} \longmapsto |v|_{1,\Omega}$ is a norm, equivalent to (1.26).

With this space, we can put problem (N) in the abstract setting of problem (P). Let $V = H^1(\Omega)/\mathbb{R}$,

$$a(\dot{u},\dot{v}) = (\overrightarrow{\text{grad}}\, u, \overrightarrow{\text{grad}}\, v) \qquad \forall\, u \in \dot{u}\ ,\ v \in \dot{v}$$

and

$$\ell : \dot{v} \longmapsto \int_\Omega fv\, dx + \langle g,v \rangle_\Gamma \qquad \forall\, v \in \dot{v}\ . \tag{1.27}$$

Note that the right-hand side of (1.27) is independent of the particular $v \in \dot{v}$ thanks to the compatibility condition (1.25). Furthermore, $\ell \in V'$ because, owing to (1.8), we have :

$$\left|\int_\Omega fv\, dx + \langle g,v \rangle_\Gamma\right| \leqslant (\| f \|_{o,\Omega} + \| g \|_{-1/2,\Gamma}) \inf_{v \in \dot{v}} \| v \|_{1,\Omega}\ .$$

Thus

$$\| \ell \|_{V'} \leqslant \| f \|_{o,\Omega} + \| g \|_{-1/2,\Gamma}\ . \tag{1.28}$$

Obviously, $a(\dot{u},\dot{v})$ is continuous on $V \times V$, and by virtue of Theorem 1.7,

$$a(\dot{v},\dot{v}) = |v|^2_{1,\Omega} \geqslant C_1 \| \dot{v} \|^2_{H^1(\Omega)/\mathbb{R}}\ .$$

Hence, by the Lax-Milgram's theorem, the following problem

$$(\text{N}')\ \begin{cases} \text{Find } \dot{u} \text{ in } H^1(\Omega)/\mathbb{R} \text{ satisfying} \\ (1.29)\qquad a(\dot{u},\dot{v}) = \langle \ell,\dot{v} \rangle \qquad \forall\, \dot{v} \in H^1(\Omega)/\mathbb{R}\ , \end{cases}$$

has a unique solution $\dot{u} \in H^1(\Omega)/\mathbb{R}$.

Let us interpret problem (N'). When v is restricted to $\mathcal{D}(\Omega)$, (1.29) yields :

$$-\Delta u = f \quad \text{in} \quad L^2(\Omega) \qquad \forall\, u \in \dot{u}\ . \tag{1.30}$$

Next, by taking the scalar product of (1.30) with v and comparing with (1.29), we find :

$$(\overrightarrow{\text{grad}}\, u, \overrightarrow{\text{grad}}\, v) = (-\Delta u,v) + \langle g,v \rangle_\Gamma \qquad \forall\, v \in H^1(\Omega)\ . \tag{1.31}$$

Therefore, problem (N') is equivalent to :

find u in $H^1(\Omega)$ satisfying (1.30) and (1.31).

It remains to interpret (1.31) as a boundary condition. At the present stage this cannot be done without assuming that $u \in H^2(\Omega)$. Then Green's formula (1.13) yields :

$$\int_\Gamma \frac{\partial u}{\partial \nu} v \, d\sigma = \langle g,v \rangle_\Gamma \qquad \forall\, v \in H^1(\Omega) \ ,$$

i.e.

$$\frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \Gamma \ .$$

As u is supposed to belong to $H^2(\Omega)$, this implies in particular that $g \in H^{1/2}(\Gamma)$. In that case, problems (N) and (N') are equivalent. Of course this is not entirely satisfactory in that the existence of a solution of problem (N) is subjected to the regularity of the solution of (N'). Further on, with more powerful tools, we shall be able to eliminate this regularity hypothesis.

Now, let us examine the regularity of u . According to the Lax-Milgram' s theorem, (1.28) and the equivalence Theorem 1.7, we obtain :

$$|u|_{1,\Omega} \leqslant C_2(\|f\|_{o,\Omega} + \|g\|_{-1/2,\Gamma}) \ .$$

We have thus proved the following result.

PROPOSITION 1.2.

Let the solution $\dot{u}$ of problem (N') belong to $H^2(\Omega)/\mathbb{R}$. Then $\dot{u}$ is the only solution of problem (N) and each $u \in \dot{u}$ is continuous with respect to the data, i.e.

(1.32) $$|u|_{1,\Omega} \leqslant C(\|f\|_{o,\Omega} + \|g\|_{-1/2,\Gamma}) \ .$$

Remark 1.2.

As in the previous example, if Γ is very smooth and if $f \in H^{m-2}(\Omega)$ and $g \in H^{m-3/2}(\Gamma)$ with $m \geqslant 2$, then it can be shown that $\dot{u} \in H^m(\Omega)/\mathbb{R}$ and

(1.33) $$|u|_{m,\Omega} \leqslant C(\|f\|_{m-2,\Omega} + \|g\|_{m-3/2,\Gamma}) \quad \text{for every} \quad u \in \dot{u} \ .$$ ∎