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STRUCTURE OF RINGS

BY

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PREFACE

Since the appearance of the author's *Theory of Rings* and Artin, Nesbitt and Thrall's *Rings with Minimum Condition*, a number of important developments have taken place in the theory of (non-commutative) rings. These are: the structure theory of rings without finiteness assumptions, cohomology of algebras, and structure and representation theory of non-semi-simple rings (Frobenius algebras, quasi-Frobenius rings). The main purpose of the present volume is to give an account of the first of these developments. The tools which have been devised for the study of general rings yield improved proofs of the older structure results on rings with minimum condition and of finite dimensional algebras. We have therefore considered the specialization of the general results and methods to these classical cases. Thus the present volume includes virtually all the results on semi-simple rings which can be found in the two books cited before. For example, the theory of centralizers of finite dimensional simple subalgebras of simple rings with minimum condition appears as a special case of the Galois theory of the complete ring of linear transformations of a vector space over a division ring. We believe that the passage to the more general case gives a better insight into these results.

The general structure theory is applicable also to a number of important new classes of rings. Of particular interest are the primitive rings with minimal ideals, algebraic algebras and algebras with a polynomial identity. The first class includes the rings of bounded operators in Banach spaces. Some of the results (e.g. the isomorphism theorem) were first obtained for this special case (Eidelheit's theorem). The study of algebraic algebras presents a number of interesting problems, one of the most interesting being Kurosch's analogue of Burnside's problem on periodic groups: Is every finitely generated algebraic algebra finite dimensional? In striking contrast with the situation in the group case, important positive results have been obtained for algebraic algebras. In particular, the analogue of the restricted Burnside problem has an affirmative answer for algebraic algebras. This fact is a corollary of a more general result on PI-algebras (algebras satisfying a polynomial identity).

The starting point in our considerations is the definition of a radical for an arbitrary ring. This is an ideal which measures the departure of a ring from semi-simplicity. A semi-simple ring is one which has enough irreducible representations to distinguish elements. A ring which has a faithful irreducible representation is called primitive. Chapter I is devoted to the basic properties of the radical, semi-simplicity and primitivity for rings and algebras. The considerations of Chapter II center around a density theorem for primitive rings. This is a special case of a more general result involving mappings of one vector space into a second one. The generalization and a lemma used in its proof are used in Chapter IV to derive all the elementary results on dual vector spaces. An extension of the density theorem for primitive rings to completely reducible modules is given in Chapter VI. Chapter III is concerned with rings satisfying the minimum condition for right ideals. In the first part we consider the theory of semi-simple rings with minimum condition. Next we collect a number of formal results on idempotents and matrix units. Finally we consider the notions of semi-primary and primary rings and we obtain structure theorems for these. Chapter IV is devoted to the structure theory of primitive rings with minimal ideals. We determine the isomorphisms, anti-isomorphisms and derivations for such rings. In Chapter V we define Kronecker products of modules and algebras and we reduce the problems of determining the structure of Kronecker products of simple algebras to the case of division algebras and fields. The notions of multiplication algebra and centroid play an important role in these considerations. Chapter VI is concerned with completely reducible modules and their centralizers. The last part of this chapter deals with the Galois theory of the complete ring of linear transformations of a vector space over a division ring. Chapter VII lays the foundations for the study of division rings which may be infinite dimensional over their centers. We consider the Galois theory of automorphisms for division rings, the structure of Kronecker products of division rings, and commutativity theorems (e.g. Wedderburn's theorem on finite division rings). In Chapter VIII we consider several types of nil radicals. One of these is the lower nil radical of Baer which coincides with the intersection of the prime ideals of a ring. We consider also nil subsystems of rings with maximum or minimum condition for right ideals. In Chapter IX we define a topology of the set of primitive ideals of a ring and we use this to obtain representations of rings as rings of continuous functions on topological spaces. The earliest result of this type is Stone's representation theorem for Boolean algebras. In Chapter X the structure theory is applied to commutativity theorems for general rings and to the study of PI-algebras and algebraic algebras. The main results on Kurosch's problem are derived here.

We have tried to make our presentation self-contained and to give complete proofs, particularly in the basic results. The only knowledge assumed is that of the rudiments of ring and module theory such as is found in any of the introductory texts to abstract algebra. Occasionally we have left proofs as exercises, but this has been done only in secondary results.

The principal contributors to the structure theory of rings without finiteness conditions have been Amitsur, Azumaya, Baer, Chevalley, Dieudonné, Kaplansky, Kurosch, Levitzki, McCoy, Nakayama and the present author. We had planned originally to write a series of notes indicating individual contributions. However, we had to abandon this project since it would have delayed still further the publication of this book which has been in process for several years. Instead we have substituted brief textual references to sources from time to time and we have listed the basic papers bearing on the subject of each chapter at the end of the chapter. We have also added a few references to papers which give further results on the topics considered. The bibliography at the end of the book is fairly complete for papers appearing since about 1943. For earlier references we refer to the bibliography of our *Theory of Rings* (Mathematical Surveys, No. 2, 1943).

We are greatly indebted to a number of friends for assistance in preparing this manuscript. The first version of this book was based on our lecture notes, which were prepared by M. Weisfeld. Later versions were read by Dieudonné, A. Rosenberg and Zelinsky who made a number of important suggestions for improvements. We are indebted also to Amitsur and to the late Professor Levitzki for communicating to us results prior to publication. Finally, we wish to express our hearty thanks to C. W. Curtis, F. Quigley, A. Rosenberg, G. Seligman and F. D. Jacobson for valuable help with the proofs.

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TABLE OF CONTENTS

PAGE

CHAPTER I

The Radical and Semi-simplicity 1

CHAPTER II

Irreducible Modules and Primitive Rings 24

CHAPTER III

Rings with Minimum Condition 38

CHAPTER IV

Primitive Rings Having Minimal One-sided Ideals 60

CHAPTER V

Kronecker Products 95

CHAPTER VI

Completely Reducible Modules. Galois Theory for the Ring of Linear Transformations 124

CHAPTER VII

Division Rings 157

CHAPTER VIII

Nil Ideals and Prime Ideals 193

CHAPTER IX

Structure Spaces 203

CHAPTER X

Applications 217

Bibliography 251

Index 261

CHAPTER I

THE RADICAL AND SEMI-SIMPLICITY

In this chapter we shall define the basic concepts of primitive ring, semi-simple ring and the radical. While these notions can be approached from a number of points of view, the most natural one seems to be that of representation theory. At any rate, we shall adopt this point of view here. In this chapter, as in most of the book, the main theme of our discussion will concern rings without operators. Nearly everything carries over easily to the more general case of algebras over a commutative ring. The extension to algebras will be indicated briefly from time to time. For the ideas of the present chapter this is done in §9, where we consider also the question of the dependence of the various concepts on the domain of operators. A number of examples illustrating the basic concepts are given in §11.

1. Representations and Modules

In this section we recall the fundamental notions of the theory of representations of rings. The reader is presumably familiar with these; hence we shall confine ourselves to a brief summary of the main definitions and results.

We recall first that a *representation* of a ring $\mathfrak{A}$ is a homomorphism of $\mathfrak{A}$ into the ring of endomorphisms of some commutative group $\mathfrak{M}$. An *anti-representation* is an anti-homomorphism of $\mathfrak{A}$ into a ring of endomorphisms. A representation (anti-representation) is said to be *faithful* if and only if it is 1-1, that is, an isomorphism (anti-isomorphism). Let $a \rightarrow \bar{a}$ be a representation acting in $\mathfrak{M}$. Then we can define a composition of the product set $\mathfrak{M} \times \mathfrak{A}$ to $\mathfrak{M}$ by setting $ma = m\bar{a}$, $m \in \mathfrak{M}$, $a \in \mathfrak{A}$. In this way one obtains a right module in the sense of

Definition 1. $\mathfrak{M}$ is called a *right $\mathfrak{A}$-module* if and only if

(i) A composition $+$ is defined in $\mathfrak{M}$ such that $(\mathfrak{M}, +)$ is a commutative group.

(ii) $\mathfrak{A}$ is a ring.

(iii) A law of composition on $\mathfrak{M} \times \mathfrak{A}$ into $\mathfrak{M}$ is defined, which, for $x, y \in \mathfrak{M}$ and $a, b \in \mathfrak{A}$, satisfies

(a) $(x + y)a = xa + ya$,

(b) $x(a + b) = xa + xb$,

(c) $x(ab) = (xa)b$.

If, moreover, $\mathfrak{A}$ has a multiplicative identity 1 and $x1 = x$ for all $x \in \mathfrak{M}$, then $\mathfrak{M}$ is called *unital*.[1]

[1] This is often called "unitary". However, this use of the term "unitary" is in conflict with its well-established meaning in geometry. This has led us to adopt the new term "unital" here.

We have noted that a representation determines a right module. The converse holds also. Thus if $\mathfrak{M}$ is a right $\mathfrak{A}$-module then we define the mapping a_R in $\mathfrak{M}$ by $ma_R = ma$. It is easy to check that a_R is an endomorphism of $(\mathfrak{M}, +)$ and that the correspondence $a \to a_R$ is a representation. Thus we see that the concepts of representation and module are essentially equivalent. In a similar fashion one sees that the concepts of anti-representation and left module are essentially the same. The latter is defined by replacing the product ma by a product am satisfying the relations obtained from (a), (b) and (c) by reversing the order of the symbols which occur in these. Henceforth the term "module" without modifier will always mean right module.

We recall now the fundamental concepts and elementary results of the theory of modules.

DEFINITION 2. $\mathfrak{N}$ is called an *$\mathfrak{A}$-submodule* of the module $\mathfrak{M}$ if and only if
(i) $(\mathfrak{N}, +)$ is a subgroup of $(\mathfrak{M}, +)$.
(ii) For all $a \in \mathfrak{A}$ and all $y \in \mathfrak{N}$, $ya \in \mathfrak{N}$.

DEFINITION 3. Let $\mathfrak{N}$ be an $\mathfrak{A}$-submodule of $\mathfrak{M}$. The *difference $\mathfrak{A}$-module* $\mathfrak{M} - \mathfrak{N}$ is defined as follows: Form the difference group $(\mathfrak{M}, +) - (\mathfrak{N}, +)$. Then regard the difference group as an $\mathfrak{A}$-module by defining $(\mathfrak{N} + x)a = \mathfrak{N} + xa$. It is easy to see that this definition is independent of the choice of x in its coset and that the rules for a module hold.

The ring $\mathfrak{A}$ itself is a module relative to right multiplication as module composition; the corresponding representation is called the *regular representation*. The submodules of this module are the right ideals of the ring. In this volume the word "ideal" without modifiers will always mean two-sided ideal. If $\mathfrak{B}$ is an ideal in a ring $\mathfrak{A}$ then the difference ring will be denoted by $\mathfrak{A}/\mathfrak{B}$ while, as above, the difference module will be denoted as $\mathfrak{A} - \mathfrak{B}$.

DEFINITION 4. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be $\mathfrak{A}$-modules. A mapping A of $\mathfrak{M}$ into $\mathfrak{M}'$ is called a ($\mathfrak{A}$-) *homomorphism* if and only if
(i) A is a group homomorphism of $(\mathfrak{M}, +)$ into $(\mathfrak{M}', +)$.
(ii) For all $x \in \mathfrak{M}$ and all $a \in \mathfrak{A}$, $(xa)A = (xA)a$.
If A is 1-1 it is called an ($\mathfrak{A}$-) *isomorphism*. If there exists a homomorphism of $\mathfrak{M}$ onto $\mathfrak{M}'$, then $\mathfrak{M}'$ is called a *homomorphic image* of $\mathfrak{M}$, and if there exists an isomorphism of $\mathfrak{M}$ onto $\mathfrak{M}'$, then $\mathfrak{M}$ and $\mathfrak{M}'$ are called *isomorphic* modules. Two representations of a ring $\mathfrak{A}$ are called *equivalent* if and only if the associated $\mathfrak{A}$-modules are isomorphic.

We recall also that if $\mathfrak{N}$ is a submodule and $\bar{\mathfrak{M}} = \mathfrak{M} - \mathfrak{N}$ then the mapping $\nu: x \to \bar{x} = x + \mathfrak{N}$ is a homomorphism of $\mathfrak{M}$ onto $\bar{\mathfrak{M}}$. We call this the *natural homomorphism* of $\mathfrak{M}$ onto $\bar{\mathfrak{M}}$. If A is a homomorphism of $\mathfrak{M}$ into a second module $\mathfrak{N}$ then the *image* $\mathfrak{M}A = \{xA \mid x \in \mathfrak{M}\}$ is a submodule of $\mathfrak{N}$. Also the *kernel* $\mathfrak{K}$ of A, which is the set of elements y such that $yA = 0$, is a submodule of $\mathfrak{M}$. If $\mathfrak{L}$

is a submodule contained in the kernel $\mathfrak{K}$ then the mapping $\bar{A}: x + \mathfrak{L} \to xA$ is single-valued (i.e. independent of the choice of x in its coset). If follows that $\bar{A}$ is a homomorphism of $\mathfrak{M} - \mathfrak{L}$ into $\mathfrak{N}$. We call $\bar{A}$ the mapping *induced* by A and we note that $A = \nu\bar{A}$ where ν is the natural homomorphism of $\mathfrak{M}$ onto $\mathfrak{M} - \mathfrak{L}$. The kernel of $\bar{A}$ is $\mathfrak{K} - \mathfrak{L}$. Hence $\bar{A}$ is an isomorphism if and only if $\mathfrak{K} = \mathfrak{L}$. This result implies the "fundamental theorem of homomorphisms" that any homomorphic image $\mathfrak{M}A$ of $\mathfrak{M}$ is isomorphic to a difference module $\mathfrak{M} - \mathfrak{K}$, $\mathfrak{K}$ the kernel of A.

It is sometimes convenient to regard a given module $\mathfrak{M}$ as a module with respect to a second ring. Clearly if $\mathfrak{B}$ is a subring then any $\mathfrak{A}$-module $\mathfrak{M}$ can be considered as a $\mathfrak{B}$-module. Also if $\mathfrak{B}$ is an ideal in $\mathfrak{A}$ and $\mathfrak{M}$ is an $\mathfrak{A}/\mathfrak{B}$-module then the definition $xa = x(a + \mathfrak{B})$ for $x \in \mathfrak{M}$, $a \in \mathfrak{A}$ defines $\mathfrak{M}$ as $\mathfrak{A}$-module. This is clear from the representation point of view; for all that we are doing is defining a homomorphism for the ring $\mathfrak{A}$ as a resultant of the natural ring homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{B}$ and the homomorphism of $\mathfrak{A}/\mathfrak{B}$ into the ring of endomorphisms of $\mathfrak{M}$. It is clear also that the submodules of $\mathfrak{M}$ as $\mathfrak{A}$-module are the submodules of $\mathfrak{M}$ as $\mathfrak{A}/\mathfrak{B}$-module and conversely. If $\mathfrak{D}$ is the kernel of the representation of $\mathfrak{A}$ determined by $\mathfrak{M}$ then evidently $\mathfrak{D} \supseteq \mathfrak{B}$ and $\mathfrak{D}/\mathfrak{B}$ is the kernel of the representation of $\mathfrak{A}/\mathfrak{B}$. Conversely, suppose that we are given an $\mathfrak{A}$-module $\mathfrak{M}$ and an ideal $\mathfrak{B}$ contained in the kernel of the representation of $\mathfrak{A}$. Ther we can define $x(a + \mathfrak{B}) = xa$ and verify that this makes $\mathfrak{M}$ into an $\mathfrak{A}/\mathfrak{B}$-module. We summarize these results in the following

PROPOSITION 1. *Let $\mathfrak{A}$ be a ring, $\mathfrak{B}$ an ideal in $\mathfrak{A}$. If $\mathfrak{M}$ is an $\mathfrak{A}/\mathfrak{B}$-module then $\mathfrak{M}$ can be considered as an $\mathfrak{A}$-module. Conversely, if $\mathfrak{M}$ is an $\mathfrak{A}$-module and $\mathfrak{B}$ is contained in the kernel of the representation then $\mathfrak{M}$ can be regarded as an $\mathfrak{A}/\mathfrak{B}$-module. In either case the submodules of $\mathfrak{M}$ as $\mathfrak{A}$-module are the same as those of $\mathfrak{M}$ as $\mathfrak{A}/\mathfrak{B}$-module and if $\mathfrak{D}$ is the kernel for $\mathfrak{A}$ then $\mathfrak{D}/\mathfrak{B}$ is that for $\mathfrak{A}/\mathfrak{B}$.*

The notation of quotients used in Noetherian ideal theory will be useful for us. We shall employ it for $\mathfrak{A}$-modules in the following way: If $\mathfrak{N}$ is an $\mathfrak{A}$-submodule of $\mathfrak{M}$ and S is a subset of $\mathfrak{M}$ then

$$(\mathfrak{N}:S) = \{b \mid b \in \mathfrak{A}, \quad sb \in \mathfrak{N}, \quad \text{for all } s \in S\}.$$

$(\mathfrak{N}:S)$ is a right ideal in $\mathfrak{A}$. If $x \in \mathfrak{M}$, the right ideal $(0:x)$ is called the *order* or *order ideal* of x. Evidently

$$(0:\mathfrak{M}) = \bigcap \{(0:x) \mid x \in \mathfrak{M}\}$$

and this is an ideal which is evidently the kernel of the representation of $\mathfrak{A}$ determined by $\mathfrak{M}$. Hence the representation of $\mathfrak{A}$ is faithful (i.e. 1-1) if and only if $(0:\mathfrak{M}) = \{0\}$. We remark also that if $\mathfrak{N}$ is a submodule then $(\mathfrak{N}:\mathfrak{M}) = (0:\mathfrak{M} - \mathfrak{N})$. We note finally the following result which follows from the fundamental theorem of homomorphisms.

Proposition 2. *Let $\mathfrak{M}$ be an $\mathfrak{A}$-module and x an element of $\mathfrak{M}$. Then the set $x\mathfrak{A} = \{xa \mid a \in \mathfrak{A}\}$ is a submodule of $\mathfrak{M}$ and $x\mathfrak{A} \cong \mathfrak{A} - (0:x)$.*

2. Fundamental Definitions

Suppose we have a set Σ of $\mathfrak{A}$-modules. With each $\mathfrak{M} \in \Sigma$ there is associated the kernel $(0:\mathfrak{M})$ of the representation it defines. The set $\bigcap \{(0:\mathfrak{M}) \mid \mathfrak{M} \in \Sigma\}$ is an ideal which we shall call the *kernel* of Σ. The kernel of Σ is equal to $\{0\}$ if and only if for each $a \neq 0$ in $\mathfrak{A}$ there is an $\mathfrak{M} \in \Sigma$ such that the image of a in the representation determined by $\mathfrak{M}$ is not the zero endomorphism. If the kernel is $\{0\}$ then we say that the set Σ is *faithful*. (This terminology will be used also for a single module.)

The fundamental concepts of this chapter depend on the concept of an irreducible module. For our purposes it is convenient to adopt the following

Definition 1. An $\mathfrak{A}$-module $\mathfrak{M}$ is called *irreducible* and the associated representation is called *irreducible* if and only if

(i) $\mathfrak{M}\mathfrak{A} = \{\Sigma x_i a_i \mid x_i \in \mathfrak{M}, a_i \in \mathfrak{A}\} \neq \{0\}$.

(ii) There is no proper $\mathfrak{A}$-submodule of $\mathfrak{M}$ other than $\{0\}$. Evidently (i) implies that $\mathfrak{M} \neq \{0\}$.

We can now give the basic definitions of the structure theory which we shall develop, as follows:

Definition 2. A ring $\mathfrak{A}$ is called *(right) primitive* if and only if it has a faithful irreducible module. If $\mathfrak{A}$ is arbitrary let I be the set of irreducible $\mathfrak{A}$-modules. Then the kernel of I is called the *radical* of $\mathfrak{A}$. If I is faithful then $\mathfrak{A}$ is called *semi-simple*. It is understood that if I is vacuous, then $\mathfrak{A}$ is its own radical, in which case, we say that $\mathfrak{A}$ is a *radical ring.*

An ideal $\mathfrak{P}$ of a ring $\mathfrak{A}$ is called *primitive* if and only if $\mathfrak{A}/\mathfrak{P}$ is a primitive ring. Since a primitive ring is necessarily $\neq \{0\}$, $\mathfrak{P}$ is a proper ideal. *Left primitivity,* the *left radical* and *left semi-simplicity* are defined in the obvious way using left modules. It is not known whether or not primitivity implies left primitivity. It seems unlikely that it does, but no examples of primitive rings which are not left primitive are known. On the other hand, we shall see later that the left radical coincides with the radical and hence that left semi-simplicity is equivalent to semi-simplicity.

Theorem 1. *The radical of a ring $\mathfrak{A}$ is the intersection of its primitive ideals.*

Proof. It suffices to show that $\mathfrak{P}$ is a primitive ideal in $\mathfrak{A}$ if and only if $\mathfrak{P} = (0:\mathfrak{M})$ where $\mathfrak{M}$ is an irreducible $\mathfrak{A}$-module. Suppose $\mathfrak{P}$ is a primitive ideal and $\mathfrak{M}$ is a faithful irreducible $\mathfrak{A}/\mathfrak{P}$-module. Then according to Proposition 1.1, $\mathfrak{M}$ is an irreducible $\mathfrak{A}$-module, and $\mathfrak{P} = (0:\mathfrak{M})$ considering $\mathfrak{M}$ as an $\mathfrak{A}$-module. Conversely, suppose $\mathfrak{P} = (0:\mathfrak{M})$ where $\mathfrak{M}$ is an irreducible $\mathfrak{A}$-module. Then by

Proposition 1.1, $\mathfrak{M}$ is an irreducible $\mathfrak{A}/\mathfrak{P}$-module which is faithful for $\mathfrak{A}/\mathfrak{P}$. Hence $\mathfrak{A}/\mathfrak{P}$ is primitive.

THEOREM 2. *If $\mathfrak{R}$ is the radical of a ring $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{R}$ is semi-simple.*

PROOF. Let I be the set of irreducible $\mathfrak{A}$-modules. Since $\mathfrak{R} \subseteq (0:\mathfrak{M})$ for all $\mathfrak{M} \in I$, by Proposition 1.1, each $\mathfrak{M}$ in I is an irreducible $\mathfrak{A}/\mathfrak{R}$-module. Clearly this set of $\mathfrak{A}/\mathfrak{R}$-modules is faithful. Hence $\mathfrak{A}/\mathfrak{R}$ is semi-simple.

3. Strictly Cyclic Modules. Modular Right Ideals

It is important to relate the external notions of primitivity and the radical, which we have defined by properties of $\mathfrak{A}$-modules to internal ones in the ring $\mathfrak{A}$ itself. This is particularly useful in applications where it is necessary to deduce properties of a primitive ring from other given properties of the ring. We shall consider now a type of module which can be related in a direct manner with a certain type of right ideal in a ring.

DEFINITION 1. An $\mathfrak{A}$-module $\mathfrak{M}$ is called *strictly cyclic* if and only if there exists a $u \in \mathfrak{M}$ such that $\mathfrak{M} = u\mathfrak{A}$. The element u is called a *generator* (in the strict sense) for $\mathfrak{M}$.

DEFINITION 2 (I. E. SEGAL). A right ideal $\mathfrak{J} \subseteq \mathfrak{A}$ is called *modular*[2] if and only if there exists an $e \in \mathfrak{A}$ such that for all $a \in \mathfrak{A}$, $a - ea \in \mathfrak{J}$. The element e is called a *left identity modulo* $\mathfrak{J}$.

We remark that if $\mathfrak{A}$ has an identity 1 then every right ideal of $\mathfrak{A}$ is modular. Moreover, if $\mathfrak{M}$ is unital, then a submodule $\mathfrak{N}$ of $\mathfrak{M}$ is strictly cyclic with u as generator if and only if $\mathfrak{N}$ is the smallest submodule containing the element u.

PROPOSITION 1. *A module $\mathfrak{M}$ is strictly cyclic if and only if $\mathfrak{M} \cong \mathfrak{A} - \mathfrak{J}$ where $\mathfrak{J}$ is a modular right ideal. A right ideal $\mathfrak{J}$ is modular if and only if $\mathfrak{J} = (0:u)$ where u is a generator of a strictly cyclic $\mathfrak{A}$-module.*

PROOF. Let $\mathfrak{M}$ be a strictly cyclic $\mathfrak{A}$-module with u as a generator. Then every element of $\mathfrak{M}$ has the form ua. The mapping $a \to ua$ is a homomorphism of $\mathfrak{A}$, considered as an $\mathfrak{A}$-module, onto $\mathfrak{M}$. The kernel is the right ideal $(0:u) = \mathfrak{J}$, and $\mathfrak{M} \cong \mathfrak{A} - \mathfrak{J}$. The element u can be expressed in the form $u = ue$ where $e \in \mathfrak{A}$. Then for $a \in \mathfrak{A}$, $ua = uea$ or $u(a - ea) = 0$. Hence $a - ea \in \mathfrak{J}$ for all $a \in \mathfrak{A}$, which means $\mathfrak{J}$ is a modular right ideal. It remains to show that, if $\mathfrak{J}$ is modular, then $\mathfrak{M} = \mathfrak{A} - \mathfrak{J}$ is strictly cyclic and $\mathfrak{J} = (0:u)$ for a suitable generator of $\mathfrak{M}$. Let $\mathfrak{J}$ be a modular right ideal in $\mathfrak{A}$ with e as a left identity modulo $\mathfrak{J}$. Consider the $\mathfrak{A}$-module $\mathfrak{M} = \mathfrak{A} - \mathfrak{J}$. Since $a + \mathfrak{J} = (e + \mathfrak{J})a$, $\mathfrak{M}$ is a strictly

[2] The terminology in the literature is "regular". This is, however, in conflict with the equally standard "quasi-regular" which we shall use later on (§5). For this reason we have introduced the substitute "modular" for "regular".

cyclic $\mathfrak{A}$-module with generator $e + \mathfrak{I}$. Moreover, it is immediate that the order $(0:e + \mathfrak{I}) = \mathfrak{I}$.

COROLLARY. *If $\mathfrak{I}$ is a modular right ideal of $\mathfrak{A}$, then* $\mathfrak{I} \supseteq (\mathfrak{I}:\mathfrak{A})$.

PROOF. $\mathfrak{I} = (0:u) \supseteq (0:\mathfrak{M}) = (0:\mathfrak{A} - \mathfrak{I}) = (\mathfrak{I}:\mathfrak{A})$.

We recall that a right ideal $\mathfrak{I}$ is called *maximal* if and only if $\mathfrak{I}$ is proper and $\mathfrak{I}' \supseteq \mathfrak{I}$, for a proper right ideal $\mathfrak{I}'$, implies $\mathfrak{I}' = \mathfrak{I}$. We observe now that if $\mathfrak{I}$ is a modular right ideal with e a left identity modulo $\mathfrak{I}$ and $\mathfrak{I}'$ is a right ideal containing $\mathfrak{I}$ and e, then $\mathfrak{I}' = \mathfrak{A}$. Thus if $e \in \mathfrak{I}'$ then $ea \in \mathfrak{I}'$ for all $a \in \mathfrak{A}$. Hence every $a = (a - ea) + ea \in \mathfrak{I}'$. We use this remark to prove

PROPOSITION 2. *If $\mathfrak{I}$ is a proper modular right ideal in $\mathfrak{A}$, then $\mathfrak{I}$ can be imbedded in a maximal (necessarily modular) right ideal.*

PROOF. Let e be a left identity modulo $\mathfrak{I}$. Consider the class S of right ideals $\mathfrak{I}'$ such that (i) $\mathfrak{I}' \supseteq \mathfrak{I}$ and (ii) $e \notin \mathfrak{I}'$, partially ordered by inclusion. S is not empty since $\mathfrak{I} \in S$. Let T be an ordered subclass of S. It is easy to verify that $\bigcup\{\mathfrak{I}' \mid \mathfrak{I}' \in T\}$ is an upper bound for T. Hence by Zorn's lemma S has a maximal element $\mathfrak{I}^*$. It follows from the remark made before that $\mathfrak{I}^*$ is a maximal right ideal containing $\mathfrak{I}$. Since every right ideal containing a modular one is modular, $\mathfrak{I}^*$ is modular.

4. CHARACTERIZATIONS OF IRREDUCIBLE MODULES. COMMUTATIVE PRIMITIVE RINGS

We shall now give two characterizations of irreducible modules.

PROPOSITION 1. *$\mathfrak{M}$ is an irreducible $\mathfrak{A}$-module if and only if* (1) $\mathfrak{M} \neq \{0\}$ *and* (2) *$\mathfrak{M}$ is strictly cyclic with every non-zero element as a generator.*

PROOF. Assume that $\mathfrak{M}$ is irreducible. Then $\mathfrak{M} \neq \{0\}$. Consider the subset $\mathfrak{Z}$ of $\mathfrak{M}$ of elements z such that $za = 0$ for all $a \in \mathfrak{A}$. Evidently $\mathfrak{Z}$ is a submodule; hence either $\mathfrak{Z} = \{0\}$ or $\mathfrak{Z} = \mathfrak{M}$ and the latter implies $\mathfrak{M}\mathfrak{A} = \{0\}$ contrary to assumption. Hence $\mathfrak{Z} = \{0\}$. Thus if $u \in \mathfrak{M}$ and $u \neq 0$, then $u\mathfrak{A}$ is a non-zero submodule of $\mathfrak{M}$. Consequently, $u\mathfrak{A} = \mathfrak{M}$. Conversely, assume (1) and (2). Let u be a non-zero element of $\mathfrak{M}$. Then $u\mathfrak{A} = \mathfrak{M}$ and $\mathfrak{M}\mathfrak{A} = \mathfrak{M} \neq \{0\}$. Suppose there is an $\mathfrak{A}$-submodule of $\mathfrak{M}$, call it $\mathfrak{N}$, such that $\{0\} \subset \mathfrak{N} \subset \mathfrak{M}$. If we choose $u \neq 0$ in $\mathfrak{N}$, then $\mathfrak{M} = u\mathfrak{A} \subseteq \mathfrak{N} \subset \mathfrak{M}$, which is a contradiction.

PROPOSITION 2. *$\mathfrak{M}$ is an irreducible $\mathfrak{A}$-module if and only if $\mathfrak{M} \cong \mathfrak{A} - \mathfrak{I}$ where $\mathfrak{I}$ is a modular maximal right ideal.*

PROOF. Let $\mathfrak{M}$ be an irreducible $\mathfrak{A}$-module and let $0 \neq u \in \mathfrak{M}$. Then $\mathfrak{M} = u\mathfrak{A} \cong \mathfrak{A} - (0:u)$ where $(0:u) = \mathfrak{I}$ is a modular right ideal. $\mathfrak{I}$ is maximal because of the well-known correspondence between $\mathfrak{A}$-submodules of $\mathfrak{M} = \mathfrak{A} - \mathfrak{I}$ and right ideals containing $\mathfrak{I}$. Conversely, if $\mathfrak{I}$ is a modular maximal right ideal, then $\mathfrak{M} = \mathfrak{A} - \mathfrak{I}$ contains no proper $\mathfrak{A}$-submodules $\neq \{0\}$. Since $\mathfrak{M}\mathfrak{A}$ is a submodule, $\mathfrak{M}\mathfrak{A} = \{0\}$ or $\mathfrak{M}\mathfrak{A} = \mathfrak{M}$. If $\mathfrak{M}\mathfrak{A} = \{0\}$ then $\mathfrak{A} \subseteq (0:\mathfrak{M}) = (0:\mathfrak{A} - \mathfrak{J}) = (\mathfrak{J}:\mathfrak{A}) \subseteq \mathfrak{J}$, by the corollary to Proposition 3.1. Since $\mathfrak{J}$ is maximal this is false. Hence $\mathfrak{M}\mathfrak{A} = \mathfrak{M}$.

The foregoing result gives an internal characterization of primitive ideals and hence of the notion of primitivity of a ring. This is the following

Corollary. *$\mathfrak{P}$ is a primitive ideal in $\mathfrak{A}$ if and only if $\mathfrak{P} = (\mathfrak{J}:\mathfrak{A})$ where $\mathfrak{J}$ is a modular maximal right ideal.*

Proof. By the proof of Theorem 2.1, $\mathfrak{P}$ is primitive if and only if $\mathfrak{P} = (0:\mathfrak{M})$ where $\mathfrak{M}$ is an irreducible $\mathfrak{A}$-module. By Proposition 2, $\mathfrak{M}$ is irreducible if and only if $\mathfrak{M} \cong \mathfrak{A} - \mathfrak{J}$ where $\mathfrak{J}$ is a modular maximal right ideal. Since $(0:\mathfrak{A} - \mathfrak{J}) = (\mathfrak{J}:\mathfrak{A})$, the result is clear.

It is clear from this criterion that a ring $\mathfrak{A}$ is primitive if and only if $\mathfrak{A}$ contains a modular maximal right ideal $\mathfrak{J}$ such that $(\mathfrak{J}:\mathfrak{A}) = \{0\}$. For evidently $\mathfrak{A}$ is primitive if and only if $\{0\}$ is a primitive ideal in $\mathfrak{A}$. We note also that if $\mathfrak{J}$ is a modular right ideal then $(\mathfrak{J}:\mathfrak{A})$ is the largest ideal of $\mathfrak{A}$ contained in $\mathfrak{J}$. Thus, we have seen that $(\mathfrak{J}:\mathfrak{A}) \subseteq \mathfrak{J}$, and if $\mathfrak{B}$ is an ideal $\subseteq \mathfrak{J}$ and $b \in \mathfrak{B}$, then $\mathfrak{A}b \subseteq \mathfrak{B} \subseteq \mathfrak{J}$ implies that $b \in (\mathfrak{J}:\mathfrak{A})$.

We can use these results to give a precise identification of the commutative primitive rings, namely, we have the following

Theorem 1. *A commutative ring is primitive if and only if it is a field.*

Proof. Let $\mathfrak{A}$ be a commutative primitive ring and let $\mathfrak{J}$ be a modular maximal right ideal such that $(\mathfrak{J}:\mathfrak{A}) = \{0\}$. Since $\mathfrak{A}$ is commutative, $\mathfrak{J}$ is an ideal. Hence $\mathfrak{J} = (\mathfrak{J}:\mathfrak{A}) = \{0\}$ and $\{0\}$ is a modular maximal right ideal in the commutative ring $\mathfrak{A}$. The modularity of $\{0\}$ implies that $\mathfrak{A}$ has an identity. This, together with the maximality of $\{0\}$, implies, as is well known, that $\mathfrak{A}$ is a field. To prove the converse, we observe that the following stronger result holds: Any division ring is primitive. This is clear since the regular representation of a division ring is faithful and irreducible.

5. Quasi-Regularity and the Circle Composition

With each element e of $\mathfrak{A}$ one can associate a least modular right ideal having e as a left identity, namely, the set $\{a - ea \mid a \in \mathfrak{A}\}$. One verifies that this is a right ideal. We denote this set as $(1 - e)\mathfrak{A}$ even when $\mathfrak{A}$ does not have an identity. Any modular right ideal having e as left identity contains $(1 - e)\mathfrak{A}$. It may happen that $(1 - z)\mathfrak{A} = \mathfrak{A}$; in this case we say that the element z is *right quasi-regular* (r.q.r.). It is clear that z is r.q.r. if and only if $(1 - z)\mathfrak{A}$ contains z, or equivalently, $-z$. The condition for this is that there exists an element z' in $\mathfrak{A}$ such that $z' - zz' = -z$, or

$$z + z' - zz' = 0. \tag{1}$$

An element z' satisfying this condition will be called a *right quasi-inverse* (r.q.i.)

for z. In a similar fashion we define *left quasi-regularity* (l.q.r.) and *left quasi-inverse* (l.q.i.).

The study of the concepts of quasi-regularity in a ring can be facilitated by the introduction of the "circle" composition in $\mathfrak{A}$. If $a, b \in \mathfrak{A}$ we define

$$a \circ b = a + b - ab.$$

It is easy to verify that $(\mathfrak{A}, \circ)$ is a semi-group, that is, $\circ$ is an associative binary composition in $\mathfrak{A}$. Moreover, 0 acts as the identity in $(\mathfrak{A}, \circ)$. If $\mathfrak{A}$ has an identity then the circle composition corresponds to the ring multiplication under the 1-1 mapping $a \to a\sigma = 1 - a$ in $\mathfrak{A}$. Thus

$$a \circ b = (a\sigma \cdot b\sigma)\sigma^{-1} = 1 - (1 - a)(1 - b) = a + b - ab.$$

Since any ring can be imbedded in a ring with an identity, this observation explains the associativity of $\circ$ and the fact that 0 is the identity.

We have remarked that an element $z \in \mathfrak{A}$ is r.q.r. if and only if there exists a $z' \in \mathfrak{A}$ such that $z + z' - zz' = 0$. This says that z has a right inverse relative to $\circ$. Similarly an element z is l.q.r. if and only if there exists a z' in $\mathfrak{A}$ such that $z' \circ z = 0$. Now we shall say that the element z is *quasi-regular* (q.r.) if there exists a z' in $\mathfrak{A}$ such that $z \circ z' = 0 = z' \circ z$. z' is called a *quasi-inverse* (q.i.) of z.

It is easy to see that if z is right and left quasi-regular then z is quasi-regular. The quasi-inverse is unique. It is well known that the elements which possess two-sided inverses in any semi-group with an identity form a subgroup. In particular, the quasi-regular elements of a ring form a group under the circle composition. If $\mathfrak{A}$ has an identity this group is isomorphic to the group of units of $\mathfrak{A}$ under the correspondence $a \to a\sigma = 1 - a$.

We shall say that a right ideal $\mathfrak{J}$ of $\mathfrak{A}$ is *quasi-regular* (q.r.) if every element of $\mathfrak{J}$ is r.q.r. Then we have the following

PROPOSITION 1. *A quasi-regular right ideal $\mathfrak{J}$ in a ring $\mathfrak{A}$ is a subgroup of the group of q.r. elements of $(\mathfrak{A}, \circ)$.*

PROOF. Let $z \in \mathfrak{J}$ and let z' be a r.q.i. of z. Then $z + z' - zz' = 0$, so that $z' = zz' - z \in \mathfrak{J}$. Hence z' has a r.q.i. z''. We have

$$z = z \circ 0 = z \circ (z' \circ z'') = (z \circ z') \circ z'' = z''.$$

Hence $z \circ z' = 0 = z' \circ z$ and z is q.r.

We recall that a one-sided or two-sided ideal in a ring is called *nil* if and only if all of its elements are nilpotent. If z is nilpotent, there exists a positive integer n such that $z^n = 0$. Let $z' = -z - z^2 - \cdots - z^{n-1}$. Then $z \circ z' = 0 = z' \circ z$. Thus z is q.r. Evidently this implies

PROPOSITION 2. *Every nil right ideal is quasi-regular.*

6. Characterizations of the Radical

We can now give the important internal characterizations of the radical. In the following theorems, as always, it is understood that the intersection of a vacuous collection of subsets of a set is the whole set.

THEOREM 1. (1) *The radical $\mathfrak{R}$ of a ring $\mathfrak{A}$ is the intersection of the modular maximal right ideals of the ring.*

(2) *The radical $\mathfrak{R}$ of a ring $\mathfrak{A}$ is a quasi-regular ideal which contains every quasi-regular right ideal.*

PROOF. (1) If $\mathfrak{I}$ is a modular maximal right ideal, then $(\mathfrak{I}:\mathfrak{A})$ is a primitive ideal and $\mathfrak{I} \supseteq (\mathfrak{I}:\mathfrak{A})$. Hence $\bigcap\{\mathfrak{I} \mid \mathfrak{I}$, modular maximal right ideal$\} \supseteq \bigcap\{(\mathfrak{I}:\mathfrak{A}) \mid \mathfrak{I}$, modular maximal right ideal$\} \supseteq \mathfrak{R}$. On the other hand, if $\mathfrak{M}$ is an irreducible $\mathfrak{A}$-module, $(0:\mathfrak{M}) = \bigcap\{(0:u) \mid u \in \mathfrak{M}\}$ and for $u \neq 0$, $(0:u)$ is a modular maximal right ideal. Hence we have $\mathfrak{R} = \bigcap\{(0:\mathfrak{M}) \mid \mathfrak{M}$ irreducible$\} \supseteq \bigcap\{\mathfrak{I} \mid \mathfrak{I}$, modular maximal right ideal$\}$. Thus $\mathfrak{R} = \bigcap\{\mathfrak{I} \mid \mathfrak{I}$, modular maximal right ideal$\}$.

(2) Suppose $z \in \mathfrak{R}$ and z is not r.q.r. Then $(1 - z)\mathfrak{A} \neq \mathfrak{A}$ and by Proposition 3.2, $(1 - z)\mathfrak{A}$ can be imbedded in a modular maximal right ideal $\mathfrak{I}$. By (1), $z \in \mathfrak{I}$. Hence $\mathfrak{I} = \mathfrak{A}$ which is a contradiction. Therefore $\mathfrak{R}$ is q.r. Next let $\mathfrak{Z}$ be any q.r. right ideal and let $z \in \mathfrak{Z}$. Then za is r.q.r. for all $a \in \mathfrak{A}$. Let $\mathfrak{M}$ be an irreducible $\mathfrak{A}$-module. Suppose $z \notin (0:\mathfrak{M})$. Then there is a $u \in \mathfrak{M}$ such that $uz \neq 0$. uz is thus a strict generator of $\mathfrak{M}$ and hence there is an $a \in \mathfrak{A}$ such that $uza = u$. If $\mathfrak{A}$ has an identity 1 then this reads $u(1 - za) = 0$. Since $1 - za$ has an inverse $1 - z'$ this leads to $u = 0$. If $\mathfrak{A}$ does not have an identity then we can replace this argument by one using quasi-inverses. Thus let z' be a r.q.i. for za. Then $0 = u - uza - (u - uza)z' = u - u(za + z' - zaz') = u$. This contradicts $uz \neq 0$. Hence $z \in (0:\mathfrak{M})$ where $\mathfrak{M}$ is any irreducible $\mathfrak{A}$-module; consequently $z \in \mathfrak{R}$. Hence $\mathfrak{Z} \subseteq \mathfrak{R}$.

The result (2) and its left analogue imply that the radical and the left radical (which is defined to be the intersection of the $(0:\mathfrak{M}')$ where $\mathfrak{M}'$ is an irreducible left $\mathfrak{A}$-module) coincide. We prefer to state this explicity as follows:

Theorem 2. *Let $\mathfrak{R}$ be the radical of a ring $\mathfrak{A}$. Then*
(1) $\mathfrak{R} = \bigcap\{(0:\mathfrak{M}') \mid \mathfrak{M}'$, *irreducible left-module*$\}$.
(2) $\mathfrak{R} = \bigcap\{\mathfrak{I}' \mid \mathfrak{I}'$, *modular maximal left ideal of* $\mathfrak{A}\}$.
(3) $\mathfrak{R}$ *contains every q.r. left ideal.*

COROLLARY. *The radical contains all nil one-sided ideals.*

We note one further left-right symmetric element characterisation of the radical.

PROPOSITION 1. *Let $\mathfrak{A}$ be a ring with radical $\mathfrak{R}$. Then $\mathfrak{R} = \{z \mid bza$ is q.r. for all $a, b \in \mathfrak{A}\}$.*

PROOF. Since $\mathfrak{R}$ is an ideal, if $z \in \mathfrak{R}$ then so does bza for all $a, b \in \mathfrak{A}$. Since $\mathfrak{R}$ is a q.r. right ideal bza is q.r. Conversely, let z be an element of $\mathfrak{A}$ such that bza is q.r. for all $a, b \in \mathfrak{A}$. Let $\mathfrak{M}$ be an irreducible $\mathfrak{A}$-module. The argument used in Theorem 1 (2) shows that $bz \in (0:\mathfrak{M})$ for all $b \in \mathfrak{A}$. If $0 \neq u \in \mathfrak{M}$, then $\mathfrak{M} = u\mathfrak{A}$ and $\mathfrak{M}z = u\mathfrak{A}z = \{0\}$ so that $z \in (0:\mathfrak{M})$. Hence $z \in \mathfrak{R}$.

7. Radical of Related Rings

We consider now the following type of problem: Let $\mathfrak{A}$ be a given ring and let $\mathfrak{B}$ be a ring which is related to $\mathfrak{A}$ in some natural manner. Obtain the radical of $\mathfrak{B}$ as a function of the radical of $\mathfrak{A}$. We shall consider this problem here for the following types of rings $\mathfrak{B}$: (1) $\mathfrak{B}$ an ideal in $\mathfrak{A}$, (2) $\mathfrak{B}$ the ring obtained by adjoining an identity to $\mathfrak{A}$ in the usual way, (3) $\mathfrak{B} = \mathfrak{A}_n$ the n-rowed matrix ring with elements in the given ring $\mathfrak{A}$, (4) $\mathfrak{B} = \mathfrak{A}[\lambda]$ the ring of polynomials in an indeterminate λ with coefficients in $\mathfrak{A}$. Later (§3.7) we shall consider the same problem for rings of the form $e\mathfrak{A}e$, where e is an idempotent element ($e^2 = e$) in $\mathfrak{A}$. We note first the following

PROPOSITION 1. (1) *The radical* $\mathfrak{R}$ *of a ring* $\mathfrak{A}$ *is a radical ring.* (2) *Let* S *be a homomorphism of a ring* $\mathfrak{A}$ *with radical* $\mathfrak{R}$ *onto a ring* $\mathfrak{B}$ *with radical* $\mathfrak{R}'$. *Then the image* $\mathfrak{R}^S \subseteq \mathfrak{R}'$.[3] *Hence if* $\mathfrak{B}$ *is semi-simple, then* $\mathfrak{R}$ *is in the kernel of* S.

PROOF. (1) If $z \in \mathfrak{R}$, z is q.r. and its q.i. is in $\mathfrak{R}$. Hence $\mathfrak{R}$ is a radical ring by Theorem 6.1 (2). (2) Since $\mathfrak{R}$ is a q.r. ideal in $\mathfrak{A}$, $\mathfrak{R}^S$ is a q.r. ideal in $\mathfrak{A}^S = \mathfrak{B}$. Hence $\mathfrak{R}^S \subseteq$ radical of $\mathfrak{B}$.

PROPOSITION 2. *Let* $\mathfrak{A}$ *be a semi-simple ring and* $\mathfrak{B}$ *a right ideal in* $\mathfrak{A}$. *Considering* $\mathfrak{B}$ *as a ring (subring of* $\mathfrak{A}$*), let* $\mathfrak{R}(\mathfrak{B})$ *be its radical. Then* $\mathfrak{R}(\mathfrak{B}) = \{z \mid z \in \mathfrak{B}, z\mathfrak{B} = \{0\}\}$. *In other words,* $\mathfrak{R}(\mathfrak{B})$ *is the left annihilator of* $\mathfrak{B}$ *in* $\mathfrak{B}$.

PROOF. $\mathfrak{R}(\mathfrak{B})\mathfrak{B}$ is a right ideal in $\mathfrak{A}$. Since $\mathfrak{R}(\mathfrak{B})\mathfrak{B} \subseteq \mathfrak{R}(\mathfrak{B})$, the elements of $\mathfrak{R}(\mathfrak{B})\mathfrak{B}$ are q.r. Hence $\mathfrak{R}(\mathfrak{B})\mathfrak{B}$ is in the radical of $\mathfrak{A}$. Hence $\mathfrak{R}(\mathfrak{B})\mathfrak{B} = \{0\}$, which means that $\mathfrak{R}(\mathfrak{B})$ is contained in the left annihilator of $\mathfrak{B}$. On the other hand, the left (right) annihilator in $\mathfrak{B}$ of any ring $\mathfrak{B}$ is a nil ideal and is therefore contained in the radical. Hence $\mathfrak{R}(\mathfrak{B})$ is the left annihilator of $\mathfrak{B}$ in $\mathfrak{B}$.

THEOREM 1. *Let* $\mathfrak{A}$ *be a ring with radical* $\mathfrak{R}$ *and let* $\mathfrak{B}$ *be an ideal in* $\mathfrak{A}$. *Considering* $\mathfrak{B}$ *as a ring, its radical* $\mathfrak{R}(\mathfrak{B}) = \mathfrak{B} \cap \mathfrak{R}$. *In particular, if* $\mathfrak{A}$ *is semi-simple, then so is* $\mathfrak{B}$.

PROOF. Assume first that $\mathfrak{A}$ is semi-simple. Then the radical $\mathfrak{R}(\mathfrak{B})$ is the left annihilator of $\mathfrak{B}$ in $\mathfrak{B}$. Hence $\mathfrak{R}(\mathfrak{B})$ is the intersection with $\mathfrak{B}$ of the left annihilator of $\mathfrak{B}$ in $\mathfrak{A}$. Hence $\mathfrak{R}(\mathfrak{B})$ is an ideal in $\mathfrak{A}$. Evidently $\mathfrak{R}(\mathfrak{B})$ is q.r. so that the

[3] We shall often write images under a ring homomorphism S as x^S etc.

semi-simplicity of $\mathfrak{A}$ forces $\mathfrak{R}(\mathfrak{B}) = \{0\}$. Now let $\mathfrak{A}$ be arbitrary. Then $(\mathfrak{B} + \mathfrak{R})/\mathfrak{R}$ is an ideal in the semi-simple ring $\mathfrak{A}/\mathfrak{R}$. Hence the ring $(\mathfrak{B} + \mathfrak{R})/\mathfrak{R}$ is semi-simple. Since $(\mathfrak{B} + \mathfrak{R})/\mathfrak{R} \cong \mathfrak{B}/(\mathfrak{B} \cap \mathfrak{R})$, the latter is semi-simple. By Proposition 1 (2), $\mathfrak{R}(\mathfrak{B}) \subseteq \mathfrak{R} \cap \mathfrak{B}$. On the other hand, $\mathfrak{R} \cap \mathfrak{B}$ is a q.r. right ideal in $\mathfrak{B}$. Hence $\mathfrak{R} \cap \mathfrak{B} \subseteq \mathfrak{R}(\mathfrak{B})$. Thus $\mathfrak{R}(\mathfrak{B}) = \mathfrak{R} \cap \mathfrak{B}$.

It should be noted that the homomorphic image of a semi-simple ring need not be semi-simple. The simplest example of this type is obtained from the ring of integers. We note first that the ring J of integers is semi-simple, since, if p is a prime, the principal ideal (p) is primitive in J and $\bigcap_p(p) = \{0\}$. On the other hand, if $e > 1$, the homomorphic image $J/(p^e)$ contains the nil ideal $(p)/(p^e)$ and hence is not semi-simple.

Let $\mathfrak{A}$ be an arbitrary ring and J the ring of integers. Then it is well known that we can construct a ring $\mathfrak{A}^* = \mathfrak{A} + J$ such that $\mathfrak{A} \cap J = \{0\}$ and the identity of J is the identity of $\mathfrak{A}^*$. It is clear that every ideal of $\mathfrak{A}$ is an ideal in $\mathfrak{A}^*$. We now prove

THEOREM 2. *Let $\mathfrak{A}$ be a ring, J the ring of rational integers. Let $\mathfrak{A}^* = \mathfrak{A} + J$, $\mathfrak{A} \cap J = \{0\}$ and let the identity of J be the identity of $\mathfrak{A}^*$. Then $\mathfrak{A}$ and $\mathfrak{A}^*$ have the same radical.*

PROOF. If we employ the natural homomorphism of $\mathfrak{A}^*$ onto $\mathfrak{A}^*/\mathfrak{A} \cong J$ and use the semi-simplicity of J we see that the radical of $\mathfrak{A}^*$, $\mathfrak{R}(\mathfrak{A}^*) \subseteq \mathfrak{A}$. On the other hand, $\mathfrak{R}(\mathfrak{A}) = \mathfrak{R}(\mathfrak{A}^*) \cap \mathfrak{A}$. Hence $\mathfrak{R}(\mathfrak{A}) = \mathfrak{R}(\mathfrak{A}^*)$.

If $\mathfrak{A}$ is any ring, the ring of all $n \times n$ matrices with elements taken from $\mathfrak{A}$ will be denoted as $\mathfrak{A}_n$. The processes of taking the radical and forming the matrix ring commute. Thus we have

THEOREM 3. *If $\mathfrak{R}$ is the radical of a ring $\mathfrak{A}$, then the radical $\mathfrak{R}(\mathfrak{A}_n)$ of $\mathfrak{A}_n$ is $\mathfrak{R}_n$.*

PROOF. Consider a matrix of the form

$$Z = \begin{pmatrix} z_{11} & z_{12} & \cdot & \cdot & \cdot & z_{1n} \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

where z_{11} is r.q.r. Then $(1 - z_{11})\mathfrak{A} = \mathfrak{A}$ so that we can find z'_{11} such that $z_{11} \circ z'_{11} = 0$ and also z'_{1i}, $i = 2 \cdots n$, such that $z'_{1i} - z_{11}z'_{1i} = -z_{1i}$. Then if

$$Z' = \begin{pmatrix} z'_{11} & z'_{12} & \cdot & \cdot & \cdot & z'_{1n} \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix},$$

$Z \circ Z' = 0$ so that Z is r.q.r. Now let $\mathfrak{J}_j$ be the set of elements Z of $\mathfrak{A}_n$ with all elements of the jth row being elements of $\mathfrak{R}$ and all elements of the other rows zero. Each $\mathfrak{J}_j$ is a right ideal and an argument similar to the one just used for $j = 1$ shows that $\mathfrak{J}_j$ is q.r. Hence $\mathfrak{J}_j \subseteq \mathfrak{R}(\mathfrak{A}_n)$ for each $j = 1, \cdots, n$. Therefore $\mathfrak{R}_n = \mathfrak{J}_1 + \mathfrak{J}_2 + \cdots + \mathfrak{J}_n \subseteq \mathfrak{R}(\mathfrak{A}_n)$. On the other hand, let

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ \vdots & & & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix}$$

belong to $\mathfrak{R}(\mathfrak{A}_n)$. If a is any element of $\mathfrak{A}$, let A_{pq} be the matrix with a in the (p, q) position and 0's elsewhere. Let a and b be arbitrary in $\mathfrak{A}$. Form

$$\sum_{k=1}^{n} A_{kp} W B_{qk} = \text{diag}\,(aw_{pq}b, \cdots, aw_{pq}b).$$

By hypothesis W is in $\mathfrak{R}(\mathfrak{A}_n)$; hence $\sum A_{kp}WB_{qk}$ is in $\mathfrak{R}(\mathfrak{A}_n)$. Let (w'_{ij}) be its quasi-inverse. This implies $aw_{pq}b \circ w'_{11} = 0 = w'_{11} \circ aw_{pq}b$ so that $aw_{pq}b$ is q.r. for all $a, b \in \mathfrak{A}$. By Proposition 6.1, $w_{pq} \in \mathfrak{R}$. Hence $\mathfrak{R}(\mathfrak{A}_n) \subseteq \mathfrak{R}_n$. Finally $\mathfrak{R}(\mathfrak{A}_n) = \mathfrak{R}_n$.

We consider next the ring $\mathfrak{A}[\lambda]$ of polynomials $a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n$ in the indeterminate λ with coefficients a_i in the ring $\mathfrak{A}$. It does not seem to be an easy matter to determine the radical of $\mathfrak{A}[\lambda]$. At the present time we have only a partial answer (Theorem 4), due to Amitsur, to this problem. We require first the following

Lemma. *Let $\mathfrak{B}$ be a non-zero ideal in $\mathfrak{A}[\lambda]$ and let $p(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^n$, $a_n \neq 0$, be a polynomial of least degree belonging to $\mathfrak{B}$. Suppose $r(\lambda)$ is a polynomial such that $a_n^{\mu}r(\lambda) = 0$, $\mu \geq 1$. Then $a_n^{\mu-1}p(\lambda)r(\lambda) = 0$.*

Proof. The condition $a_n^{\mu}r(\lambda) = 0$ is equivalent to the set of conditions $a_n^{\mu}r_i = 0$, r_i the ith coefficient of $r(\lambda)$. Hence it suffices to prove the lemma for $r(\lambda) = r$ of 0 degree. In this case $a_n^{\mu-1}p(\lambda)r$ has $a_n^{\mu}r$ as coefficient of λ^n. Since this polynomial belongs to $\mathfrak{B}$ and since n is the least degree for non-zero polynomials in $\mathfrak{B}$, $a_n^{\mu-1}p(\lambda)r = 0$.

We can now prove

Theorem 4. *If $\mathfrak{A}$ has no non-zero nil ideals, then $\mathfrak{A}[\lambda]$ is semi-simple.*

Proof. Assume $\mathfrak{R}(\mathfrak{A}[\lambda]) \neq \{0\}$ and let M be the set of non-zero polynomials of least degree belonging to $\mathfrak{R}(\mathfrak{A}[\lambda])$. The leading coefficients of these polynomials and 0 form an ideal $\mathfrak{N} \neq \{0\}$ in $\mathfrak{A}$. We wish to show that $\mathfrak{N}$ is a nil ideal. Let $p(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^n \in M$. Then we know that $p(\lambda)\lambda a_n$ is quasi-regular. Hence there exists a polynomial $q(\lambda)$ such that