Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Preface

Determinantal rings and varieties have been a central topic of commutative algebra and algebraic geometry. Their study has attracted many prominent researchers and has motivated the creation of theories which may now be considered part of general commutative ring theory. A coherent treatment of determinantal rings is lacking however.

We are algebraists, and therefore the subject will be treated from an algebraic point of view. Our main approach is via the theory of algebras with straightening law. Its axioms constitute a convenient systematic framework, and the standard monomial theory on which it is based yields computationally effective results. This approach suggests (and is simplified by) the simultaneous treatment of the coordinate rings of the Schubert subvarieties of Grassmannians, a program carried out very strictly.

Other methods have not been neglected. Principal radical systems are discussed in detail, and one section each is devoted to invariant and representation theory. However, free resolutions are (almost) only covered for the "classical" case of maximal minors.

Our personal view of the subject is most visibly expressed by the inclusion of Sections 13–15 in which we discuss linear algebra over determinantal rings. In particular the technical details of Section 15 (and perhaps only these) are somewhat demanding.

The bibliography contains several titles which have not been cited in the text. They mainly cover topics not discussed: geometric methods and ideals generated by minors of symmetric matrices and Pfaffians of alternating ones.

We have tried hard to keep the text as self-contained as possible. The basics of commutative algebra supplied by Part I of Matsumura's book [Mt] (and some additions given in Section 16) suffice as a foundation for Sections 3–7, 9, 10, and 12. Whenever necessary to draw upon notions and results not covered by [Mt], for example divisor class groups and canonical modules in Section 8, precise references have been provided. It is no surprise that multilinear algebra plays a role in a book on determinantal rings, and in Sections 2 and 13–15 we expect the reader not to be frightened by exterior and symmetric powers. Even Section 11 which connects our subject and the representation theory of the general linear groups, does not need an extensive preparation; the linear reductivity of these groups is the only essential fact to be imported. The rudiments on Ext and Tor contained in every introduction to homological algebra will be used freely, though rarely, and some familiarity with affine and projective varieties, as developped in Chapter I of Hartshorne's book [Ha.2], is helpful.

We hope this text will serve as a reference. It may be useful for seminars following a course in commutative ring theory. A vast number of notions, results, and techniques can be illustrated significantly by applying them to determinantal rings, and it may even be possible to reverse the usual sequence of "theory" and "application": to learn abstract commutative algebra through the exploration of the special class which is the subject of this book.

Each section contains a subsection "Comments and References" where we have collected the information on our sources. The references given should not be considered assignments of priority too seriously; they rather reflect the authors' history in learning the subject and give credit to the colleagues in whose works we have participated. While it is impossible to mention all of them here, it may be fair to say that we could not have written this text without the fundamental contributions of Buchsbaum, de Concini, Eagon, Eisenbud, Hochster, Northcott, and Procesi.

The first author gave a series of lectures on determinantal rings at the Universidade federal de Pernambuco, Recife, Brazil, in March and April 1985. We are indebted to Aron Simis who suggested to write an extended version for the IMPA subseries of the Lecture Notes in Mathematics. (By now it has become a very extended version).

Finally we thank Petra Düvel, Werner Lohmann and Matthias Varelmann for their help in the production of this book. We are grateful to the staff of the Computing Center of our university, in particular Thomas Haarmann, for generous cooperation and providing excellent printing facilities.

Vechta, January 1988 WINFRIED BRUNS UDO VETTER

Contents

1. Preliminaries

This section serves two purposes. Its Subsections A and B list the ubiquitous basic notations. In C and D we introduce the principal objects of our investigation and relate them to their geometric counterparts.

A. Notations and Conventions

Generally we will use the terminology of [Mt] which seems to be rather standard now. In some inessential details our notations differ from those of [Mt]; for example we try to save parentheses whenever they seem dispensable. A main difference is the use of the attributes "local" and "normal": for us they always include the property of being noetherian. In the following we explain some notations and list the few conventions the reader is asked to keep in mind throughout.

All rings and algebras are commutative and have an element 1. Nevertheless we will sometimes list "commutative" among the hypotheses of a proposition or theorem in order to signalize that the ring under consideration is only supposed to be an *arbitrary* commutative ring. A *reduced* ring has no nilpotent elements. The *spectrum of a ring* A, $\operatorname{Spec} A$ for short, is the set of its prime ideals endowed with the Zariski topology. The *radical* of an ideal I is denoted $\operatorname{Rad} I$. The *dimension* of A is denoted $\dim A$, and the *height* of I is abbreviated $\operatorname{ht} I$.

All the modules M considered will be unitary, i.e. $1x = x$ for all $x \in M$. $\operatorname{Ann} M$ is the *annihilator* of M, and the *support* of M is given by

$$\operatorname{Supp} M = \{P \in \operatorname{Spec} A\colon M_P \neq 0\}.$$

We use the notion of *associated prime ideals* only for finitely generated modules over noetherian rings:

$$\operatorname{Ass} M = \{P \in \operatorname{Spec} A\colon \operatorname{depth} M_P = 0\}.$$

The *depth* of a module M over a local ring is the length of a maximal M-sequence in the maximal ideal. The *projective dimension* of a module is denoted $\operatorname{pd} M$. We remind the reader of the equation of Auslander and Buchsbaum for finitely generated modules over local rings A:

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} A \qquad \text{if} \quad \operatorname{pd} M < \infty$$

(cf. [Mt], p. 114, Exercise 4). If a module can be considered a module over different rings (in a natural way), an index will indicate the ring with respect to which an invariant is formed: For example, $\operatorname{Ann}_A M$ is the annihilator of M as an A-module. Instead of Matsumura's $\operatorname{depth}_I(M)$ we use $\operatorname{grade}(I, M)$ and call it, needless to say, the *grade of I with respect to M*; cf. 16.B for a discussion of grade. The *rank* $\operatorname{rk} F$ of a free module F is the number of elements of one of its bases. We discuss a more general concept of rank in 16.A: M has *rank* r if $M \otimes Q$ is a free Q-module of rank r, Q denoting the total ring of fractions of A. The rank of a linear map is the rank of its image. The *length* of a module M is indicated by $\lambda(M)$.

The notations of homological algebra concerning Hom, $\otimes$, and their derived functors seem to be completely standardized; for them we refer to [Rt]. Let A be a ring, M and N A-modules, and $f\colon M \to N$ a homomorphism. We put

$$M^* = \operatorname{Hom}_A(M, A)$$

and

$$f^* = \operatorname{Hom}_A(f, A)\colon N^* \to M^*.$$

M^* and f^* are called the *duals* of M and f.

For the *symmetric* and *exterior powers* of M (cf. [Bo.1] for multilinear algebra) we use the symbols

$$\bigwedge^i M \qquad \text{and} \qquad \mathrm{S}_j(M)$$

resp. Sometimes we shall have to refer to bases of F^*, $\bigwedge^i F$ and $\bigwedge^i F^*$, given a basis $e_1, \dots, e_n$ of the free module F. The *basis* of F^* *dual to* $e_1, \dots, e_n$ is denoted by $e_1^*, \dots, e_n^*$. For $I = (i_1, \dots, i_k)$ the notation e_I is used as an abbreviation of $e_{i_1} \wedge \dots \wedge e_{i_k}$, whereas e_I^* expands into $e_{i_1}^* \wedge \dots \wedge e_{i_k}^*$. (The notation e_I will be naturally extended to arbitrary families of elements of a module.)

We need some combinatorial notations. A subset $I \subset \mathbf{Z}$ also represents the sequence of its elements in ascending order. For subsets $I_1, \dots, I_n \subset \mathbf{Z}$ we let

$$\sigma(I_1, \dots, I_n)$$

denote the signum of the permutation $I_1 \dots I_n$ (given by iuxtaposition) of $I_1 \cup \dots \cup I_n$ relative to its natural order, provided the I_i are pairwise disjoint; otherwise $\sigma(I_1, \dots, I_n) = 0$. A useful formula:

$$\sigma(I_1, \dots, I_n) = \sigma(I_1, \dots, I_{n-1})\sigma(I_1 \cup \dots \cup I_{n-1}, I_n).$$

For elements $i_1, \dots, i_n \in \mathbf{Z}$ we define

$$\sigma(i_1, \dots, i_n) = \sigma(\{i_1\}, \dots, \{i_n\}).$$

The cardinality of a set I is denoted $|I|$. For a set I we let

$$\mathrm{S}(m, I) = \{J\colon J \subset I, |J| = m\}.$$

Last, not least, by

$$1, \dots, \widehat{i}, \dots, n$$

we indicate that i is to be omitted from the sequence $1, \dots, n$.

B. Minors and Determinantal Ideals

Let $U = (u_{ij})$ be an $m \times n$ matrix over a ring A. For indices $a_1, \dots, a_t$, $b_1, \dots, b_t$ such that $1 \le a_i \le m$, $1 \le b_i \le n$, $i = 1, \dots, t$, we put

$$[a_1, \dots, a_t | b_1, \dots, b_t] = \det \begin{pmatrix} u_{a_1 b_1} & \cdots & u_{a_1 b_t} \\ \vdots & & \vdots \\ u_{a_t b_1} & \cdots & u_{a_t b_t} \end{pmatrix}.$$

We do not require that $a_1, \dots, a_t$ and $b_1, \dots, b_t$ are given in ascending order. The symbol $[a_1, \dots, a_t | b_1, \dots, b_t]$ has a twofold meaning: $[a_1, \dots, a_t | b_1, \dots, b_t] \in A$ as just defined, and

$$[a_1, \dots, a_t | b_1, \dots, b_t] \in \mathbf{N}^t \times \mathbf{N}^t$$

as an ordered pair of t-tuples of non-negative integers. Clearly $[a_1, \dots, a_t | b_1, \dots, b_t] = 0$ if $t > \min(m, n)$. For systematic reasons it is convenient to let

$$[\emptyset | \emptyset] = 1.$$

If $a_1 \le \cdots \le a_t$ and $b_1 \le \cdots \le b_t$ we say that $[a_1, \dots, a_t | b_1, \dots, b_t]$ is a *t-minor* of U. Of course, as an element of A every $[a_1, \dots, a_t | b_1, \dots, b_t]$ is a t-minor up to sign. We call t the *size* of $[a_1, \dots, a_t | b_1, \dots, b_t]$.

Very often we shall have to deal with the case $t = \min(m, n)$. Our standard assumption will be $m \le n$ then, and we use the simplified notation

$$[a_1, \dots, a_m] = [1, \dots, m | a_1, \dots, a_m].$$

The m-minors are called the *maximal minors*, those of size $m - 1$ the *submaximal minors.* (In section 9 the notion "maximal minor" will be used in a slightly more general sense.)

The ideal generated by the *t-minors* of U is denoted

$$\mathrm{I}_t(U).$$

The reader may check that $\mathrm{I}_t(U)$ is invariant under invertible linear transformations:

$$\mathrm{I}_t(U) = \mathrm{I}_t(VUW)$$

for invertible matrices V, W of formats $m \times m$ and $n \times n$ resp.

Sometimes we will need the *matrix of cofactors* of an $m \times m$ matrix:

$$\operatorname{Cof} U = (\, c_{ij} \,),$$
$$c_{ij} = (-1)^{i+j} [1, \dots, \widehat{j}, \dots, m | 1, \dots, \widehat{i}, \dots, m].$$

C. Determinantal Rings and Varieties

Let B be a commutative ring, and consider an $m \times n$ matrix

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{m1} & \cdots & X_{mn} \end{pmatrix}$$

whose entries are independent indeterminates over B. The principal objects of our study are the residue class rings

$$\mathrm{R}_t(X) = B[X]/\mathrm{I}_t(X),$$

$B[X]$ of course denoting the polynomial ring $B[X_{ij}: i = 1, \dots, m, j = 1, \dots, n]$. The ideal $\mathrm{I}_t(X)$ is generated by the t-minors of X, cf. B. Whenever we shall discuss properties of $\mathrm{R}_t(X)$ which are usually defined for noetherian rings only (for example the dimension or the Cohen-Macaulay property), it will be assumed that B is noetherian.

Over an algebraically closed field $B = K$ of coefficients one can immediately associate a geometric object with the ring $\mathrm{R}_t(X)$. Having chosen bases in an m-dimensional vector space V and an n-dimensional vector space W one identifies $\mathrm{Hom}_K(V, W)$ with the mn-dimensional affine space of $m \times n$ matrices, of which $K[X]$ is the coordinate ring. Under this identification the subvariety defined by $\mathrm{I}_t(X)$ corresponds to

$$\mathrm{L}_{t-1}(V, W) = \{\, f \in \mathrm{Hom}_K(V, W) \colon \ \mathrm{rk}\, f \le t - 1 \,\}.$$

We want to associate the letter r with "rank", and so we replace t by $r+1$. Furthermore we put $\mathrm{L}(V, W) = \mathrm{Hom}_K(V, W)$.

It is not surprising that the geometry of $\mathrm{L}_r(V, W)$ reflects certain properties of the linear maps $f \in \mathrm{L}_r(V, W)$. Let us consider the following two elementary statements which will lead us quickly to some nontrivial information on $\mathrm{L}_r(V, W)$: (a) The map f can be factored through K^r. (b) Let $U \subset V$ be a vector subspace of dimension r and $\widetilde{U}$ a supplement of V, i.e. $V = U \oplus \widetilde{U}$; if $f|U$ is injective, then there exist unique linear maps $g\colon \widetilde{U} \to U$, $h\colon U \to W$ such that $f(u \oplus \widetilde{u}) = h(u) + h(g(\widetilde{u}))$ for all $u \in U$, $\widetilde{u} \in \widetilde{U}$ (in fact, $h = f|U$).

Statement (a) shows that the morphism

$$\mathrm{L}(V, K^r) \times \mathrm{L}(K^r, W) \longrightarrow \mathrm{L}_r(V, W),$$

given by the composition of maps, is surjective. Being an epimorphic image of an irreducible variety, $\mathrm{L}_r(V, W)$ is irreducible itself. An application of (b): It is easy to see that the subset

$$M = \{\, f \in \mathrm{L}_r(V, W) \colon f|U \text{ injective} \,\}$$

is a nonempty open subvariety of $\mathrm{L}_r(V, W)$: One chooses a basis of V containing a basis of U; then M is the union of subsets of $\mathrm{L}_r(V, W)$ each of which is defined by the non-vanishing of a determinantal function. By property (b) we have an isomorphism

$$\mathrm{L}(\widetilde{U}, U) \times \big(\mathrm{L}(U, W) \setminus \mathrm{L}_{r-1}(U, W)\big) \longrightarrow M.$$

Since the variety on the left is an open subvariety of $\mathrm{L}(\tilde{U},U)\times \mathrm{L}(U,W)$, we conclude at once that

$$\dim \mathrm{L}_r(V,W) = \dim M = \dim\left(\mathrm{L}(\tilde{U},U)\times \mathrm{L}(U,W)\right) = (m-r)r + rn$$
$$= mr + nr - r^2.$$

Furthermore M is non-singular. Varying U one observes that all the points $f \in \mathrm{L}_r(V,W)\setminus \mathrm{L}_{r-1}(V,W)$ are non-singular:

(1.1) Proposition. (a) $\mathrm{L}_r(V,W)$ *is an irreducible subvariety of* $\mathrm{L}(V,W)$.
(b) *It has dimension* $mr+nr-r^2$.
(c) *It is non-singular outside* $\mathrm{L}_{r-1}(V,W)$.

The only completely satisfactory information on $\mathrm{R}_{r+1}(X)$ we can draw from (1.1), is its dimension:

$$\dim \mathrm{R}_{r+1}(X) = mr + nr - r^2$$

Part (a) only shows that the radical of $\mathrm{I}_{r+1}(X)$ is prime, and unfortunately there seems to be no easy way to prove that $\mathrm{I}_{r+1}(X)$ is a radical ideal itself (over every reduced ring B of coefficients). Once this is known one can of course directly reverse (c): The generators of the ideal of $\mathrm{L}_r(V,W)$ have all their partial derivatives in $\mathrm{I}_r(X)$, and the Jacobi criterion (or the definition of non-singularity, depending on ones point of view) implies in conjunction with (c) that $\mathrm{L}_{r-1}(V,W)$ is the singular locus of $\mathrm{L}_r(V,W)$.

Proposition (1.1) and its proof have been included not only in order to enrich these introductory considerations by some substantial results. We shall encounter algebraic versions of the ideas underlying its proof several times again.

It would be very difficult (for us, at least) to investigate the rings $\mathrm{R}_t(X)$ without viewing them as the most prominent members of a larger class of rings of type $B[X]/I$ which we call *determinantal rings*. Their defining ideals I can be described as follows: Given integers

$$1 \le u_1 < \cdots < u_p \le m, \qquad 0 \le r_1 < \cdots < r_p < m,$$

and

$$1 \le v_1 < \cdots < v_q \le n, \qquad 0 \le s_1 < \cdots < s_q < n,$$

the ideal I is generated by the

$$(r_i + 1)\text{-minors of the first } u_i \text{ rows}$$

and the

$$(s_j + 1)\text{-minors of the first } v_j \text{ columns},$$

$i = 1,\dots,p$, $j = 1,\dots,q$. Later on we shall introduce a systematic notion for determinantal rings which is hard to motivate at this stage.

In order to relate the general class of determinantal rings just introduced to the geometric description of $\mathrm{R}_{r+1}(X)$ given above, one chooses bases $d_1,\dots,d_m$ and $e_1,\dots,e_n$ of V and W resp., K being an algebraically closed field, V and W vector spaces of dimensions m and n. Let

$$V_k = \sum_{i=1}^{k} Kd_i \qquad \text{and} \qquad W_k^* = \sum_{i=1}^{k} Ke_i^*$$

($e_1^*,\dots,e_n^*$ is the basis dual to $e_1,\dots,e_n$, cf. A above).

Then the ideal I above defines the *determinantal variety*

$$\{ f \in \operatorname{Hom}_K(V,W): \quad \operatorname{rk} f|V_{u_i} \le r_i, \ \operatorname{rk} f^*|W^*_{v_j} \le s_j, \quad i=1,\dots,p, \ j=1,\dots,q \}.$$

The reader may try to find and to prove the analogue of (1.1) for the variety just defined. It will of course be included in the main results of the Sections 5 and 6.

D. Schubert Varieties and Schubert Cycles

In the sections 4–9 we shall treat a second class of rings simultaneously with the determinantal rings: the homogeneous coordinate rings of the Schubert varieties (generalized to an arbitrary ring of coefficients) which we call *Schubert cycles* for short. There are two reasons for our treatment of Schubert cycles: (i) They are important objects of algebraic geometry. (ii) Their combinatorial structure is simpler than that of determinantal rings, and most often it is easier to prove a result first for Schubert cycles and to descend to determinantal rings afterwards. Algebraically one can consider every determinantal ring as a dehomogenization of a Schubert cycle (cf. 16.D and (5.5)). In geometric terms one passes from a (projective) Schubert variety to an (affine) determinantal variety by removing a hyperplane "at infinity".

The first step in the construction of the Schubert varieties is the description of the Grassmann varieties in which they are embedded as subvarieties. While a projective space gives a geometric structure to the set of one-dimensional subspaces of a vector space, a Grassmann variety does this for the set of m-dimensional subspaces, m fixed. Let K be an algebraically closed field, V an n-dimensional vector space over K, and $e_1,\dots,e_n$ a basis of V. In a first attempt to assign "coordinates" to a vector subspace W, $\dim W = m$, one chooses a basis $w_1,\dots,w_m$ of W and represents $w_1,\dots,w_m$ as linear combinations of $e_1,\dots,e_n$:

$$w_i = \sum_{j=1}^{n} x_{ij} e_j, \qquad i=1,\dots,m.$$

Unfortunately the assignment $W \to (x_{ij})$ is not well-defined, since (x_{ij}) depends on the basis $w_1,\dots,w_m$ of W. Exactly the matrices

$$T \cdot (x_{ij}), \qquad T \in \operatorname{GL}(m,K),$$

represent W. However, the *Plücker coordinates*

$$p = ([a_1,\dots,a_m]: 1 \le a_1 < \dots < a_m \le n)$$

formed by the m-minors of (x_{ij}) remains almost invariant if (x_{ij}) is replaced by $T \cdot (x_{ij})$; it is just replaced by a scalar multiple: The point of projective space with homogeneous coordinates p depends only on W! Thus one has found a well-defined map

$$\mathcal{P}: \{ W \subset V : \dim W = m \} \longrightarrow \mathbf{P}^N(K), \qquad N = \binom{n}{m} - 1.$$

It is called the *Plücker map*.

This construction can of course be given in more abstract terms. With each subspace W, $\dim W = m$, one associates the embedding

$$i_W : W \longrightarrow V.$$

Then the m-th exterior power

$$\bigwedge^m i_W : \bigwedge^m W \longrightarrow \bigwedge^m V$$

maps $\bigwedge^m W$ onto a one-dimensional subspace of $\bigwedge^m V$ which in turn corresponds to a point in $\mathbf{P}(\bigwedge^m V) \cong \mathbf{P}^N(K)$.

It is easy to see that the Plücker map is injective. Let $p = \mathcal{P}(W) = \mathcal{P}(\widetilde{W})$. For reasons of symmetry we may assume that the first coordinate of p is nonzero. Then we can find bases $w_1, \ldots, w_m$ and $\tilde{w}_1, \ldots, \tilde{w}_m$ of W and $\widetilde{W}$ resp. such that

$$w_i = e_i + \sum_{j=m+1}^{n} x_{ij} e_j, \quad \tilde{w}_i = e_i + \sum_{j=m+1}^{n} \tilde{x}_{ij} e_j, \qquad i = 1, \ldots, m.$$

Looking at the m-minors $[1, \ldots, \widehat{i}, \ldots, m, k]$ of the $m \times n$ matrices of coefficients appearing in the preceding equations one sees immediately that $w_i = \tilde{w}_i$ for $i = 1, \ldots, m$, hence $W = \widetilde{W}$.

It takes considerably more effort to describe the image of $\mathcal{P}$. The map $\mathcal{P}$ is induced by a morphism $\widetilde{\mathcal{P}}$ of affine spaces; $\widetilde{\mathcal{P}}$ assigns to each $m \times n$ matrix the tuple of its m-minors. Let X be an $m \times n$ matrix of indeterminates, and let $Y_{[a_1, \ldots, a_m]}$, $1 \le a_1 \cdots < a_m \le n$, denote the coordinate functions of $\mathbf{A}^{N+1}(K)$. Then the homomorphism of coordinate rings associated with $\widetilde{\mathcal{P}}$ is given as

$$\varphi : K[Y_{[a_1, \ldots, a_m]} : 1 \le a_1 < \cdots < a_m \le n] \longrightarrow K[X],$$
$$Y_{[a_1, \ldots, a_m]} \longrightarrow [a_1, \ldots, a_m],$$

$[a_1, \ldots, a_m]$ specifying an m-minor of X now. We denote the image of φ by

$$\mathrm{G}(X);$$

it is the K-subalgebra of $K[X]$ generated by the m-minors of X. By construction it is clear that the affine variety defined by the ideal $\operatorname{Ker} \varphi$ is the Zariski closure of $\operatorname{Im} \widetilde{\mathcal{P}}$, whereas the corresponding projective variety is the closure of $\operatorname{Im} \mathcal{P}$. Much more is true:

(1.2) Theorem. (a) *$\mathcal{P}$ maps the set of m-dimensional subspaces of V bijectively onto the projective variety with homogeneous coordinate ring $\mathrm{G}(X)$.*
(b) *$\widetilde{\mathcal{P}}$ maps the mn-dimensional affine space of $m \times n$ matrices over K surjectively onto the affine variety with coordinate ring $\mathrm{G}(X)$.*

Part (a) obviously follows from (b). In order to prove (b) one first has to describe the variety belonging to $\mathrm{G}(X)$ as a subvariety of $\mathbf{A}^{N+1}(K)$. This problem will be solved in (4.7). Secondly one has to show the surjectivity of $\widetilde{\mathcal{P}}$, a question which will naturally come across us in Section 7, cf. (7.14).

The projective variety appearing in (1.2),(a) is usually denoted by $\mathrm{G}_m(V)$ and called the *Grassmann variety* of m-dimensional subspaces of V. (A different choice of a basis for V only yields a different embedding into $\mathbf{P}^N(K)$; all these embeddings are projectively equivalent.)

The argument which showed the injectivity of $\mathcal{P}$ helps us to determine the dimension of $\mathrm{G}_m(V)$: the open affine subvariety of $\mathrm{G}_m(V)$ complementary to the hyperplane given by the vanishing of $Y_{[1,\ldots,m]}$, is isomorphic to the affine space of dimension $m(\dim V - m)$, hence

$$\dim \mathrm{G}_m(V) = m(\dim V - m).$$

(Note that we are using (1.2) here!) Varying the hyperplane one furthermore sees that $\mathrm{G}_m(V)$ is non-singular. The non-singularity of $\mathrm{G}_m(V)$ can also be deduced from another basic fact. The group $\mathrm{GL}(V)$ of automorphisms of V acts transitively on $\mathrm{G}_m(V)$, since two m-dimensional subspaces of V differ by an automorphism of V only. On the other hand this action is induced by the natural action of $\mathrm{GL}(V)$ on $\mathbf{P}(\bigwedge^m V)$ (via $\bigwedge^m V$); so $\mathrm{GL}(V)$ operates transitively as a group of automorphisms on the Grassmann variety $\mathrm{G}_m(V)$.

(1.3) Theorem. *$\mathrm{G}_m(V)$ is a non-singular variety of dimension $m(\dim V - m)$.*

To define the Schubert subvarieties one considers the flag of subspaces associated with the given basis $e_1, \ldots, e_n$ of V taken in reverse order:

$$V_j = \sum_{i=n-j+1}^{n} Ke_i, \qquad 0 = V_0 \subset \ldots \subset V_n = V.$$

Let $1 \leq a_1 < \cdots < a_m \leq n$ be a sequence of integers. Then the *Schubert subvariety* $\Omega(a_1, \ldots, a_m)$ of $\mathrm{G}_m(V)$ is defined by

$$\Omega(a_1, \ldots, a_m) = \{\, W \in \mathrm{G}_m(V)\colon\ \dim W \cap V_{a_i} \geq i \quad \text{for} \quad i = 1, \ldots, m \,\}.$$

The varieties thus defined of course depend on the flag of subspaces chosen. But the automorphism group of V acts transitively on the set of flags, and its action induced on $\mathrm{G}_m(V)$ makes corresponding Schubert subvarieties differ by an automorphism of $\mathrm{G}_m(V)$ only. Hence $\Omega(a_1, \ldots, a_m)$ is essentially determined by $(a_1, \ldots, a_m)$. It is indeed justified to call $\Omega(a_1, \ldots, a_m)$ a variety:

(1.4) Theorem. *$\Omega(a_1, \ldots, a_m)$ is the closed subvariety of $\mathrm{G}_m(V)$ defined by the vanishing of all the coordinate functions*

$$Y_{[b_1,\ldots,b_m]}, \qquad b_i < n - a_{m-i+1} + 1 \quad \textit{for some} \quad i,\ 1 \leq i \leq m.$$

PROOF: The proof is simpler if we dualize our notations first. Let $c_i = n - a_i$ and $W_j = \sum_{k=1}^{j} Ke_k$. Then $V = V_{n-j} \oplus W_j$ and there is a projection $\pi_j\colon V \to W_j$, $\operatorname{Ker} \pi_j = V_{n-j}$. By definition

$$\Omega(a_1, \ldots, a_m) = \{\, W \in \mathrm{G}_m(V)\colon\ \dim \pi_{c_i}(W) \leq m - i \quad \text{for} \quad i = 1, \ldots, m \,\}.$$

After the choice of a basis $w_1, \ldots, w_m$, the subspace W is represented by the matrix (x_{uv}), $w_u = \sum_{v=1}^{n} x_{uv} e_v$. One obviously has

$$\dim \pi_{c_i}(W) \leq m - i \qquad \Longleftrightarrow \qquad \mathrm{I}_{m-i+1}(x_{uv}\colon 1 \leq v \leq c_i) = 0,$$

and in case this condition holds, every m-minor which has at least $m-i+1$ of its columns among the first c_i columns of (x_{uv}), vanishes. Thus all the coordinate functions named in the theorem vanish on $\Omega(a_1, \ldots, a_m)$. Conversely, if $\mathrm{I}_{m-i+1}(x_{uv}\colon 1 \leq v \leq c_i) \neq 0$, then there is an m-minor of (x_{uv}) different from zero and having at least $m - i + 1$ of its columns among the first c_i ones of (x_{uv}). —

For arbitrary rings B of coefficients the *Schubert cycle associated with* $\Omega(a_1, \ldots, a_m)$ is the residue class ring of $\mathrm{G}(X)$ with respect to the ideal generated by all the minors $[b_1, \ldots, b_m]$ such that $b_i < n - a_{m-i+1} + 1$ for some i.

E. Comments and References

The references given below have been included to manifest the geometric significance of determinantal and Schubert varieties. We have restricted ourselves to books (with one exception) since any selection of research articles would inevitably turn out superficial and random. (After all, the AMS classification scheme contains the keys "Determinantal varieties" and "Schubert varieties".)

The classical source for "the geometry of determinantal loci" is Room's book [Rm]. It gives plenty of information on the early history of our subject. The decisive treatment of Schubert varieties has been given by Hodge and Pedoe in their monograph [HP]. Among the recent books on algebraic geometry those of Arabello, Cornalba, Griffiths, and Harris [ACGH], Fulton [Fu], and Griffiths and Harris [GH] contain sections on determinantal and/or Schubert varieties. Kleiman and Laksov's article [KmL] may serve as a pleasant introduction.