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PREFACE

The first international conference on the theory of groups held in Korea, "Groups – Korea 1983", was reported on in the Springer–Verlag Lecture Notes in Mathematics, Volume 1098. "Groups – Korea 1988" is the second such conference. It was held in the Commodore Hotel, Pusan, in August 1988. There were 16 invited one–hour lectures and 17 contributed half-hour seminar–type talks. They are listed in Appendix A. The participants are listed in Appendix B.

"Groups – Korea 1988" was financially supported by the Korean Educational Ministry, the Korea Science and Engineering Foundation, the Korean Mathematical Society, the Pusan Chamber of Commerce & Industry, and the Pusan National University. We record our thanks to these institutions, and to their officers, to whose support the success of the conference owes much. We are also grateful to Dr Jost–Gert Glombitza, of the Deutsche Forschungsgemeinschaft, and to the Deutsche Forschungsgemeinschaft itself, for supporting the five invited speakers from the Federal Republic of Germany. Our thanks also go to the President of the Pusan Chamber of Commerce & Industry, Mr Chung Whan Choi, for special financial contributions, and to Dr Jung Rae Cho, a member of the organizing committee, who typed all manuscripts for the proceedings with great care. Finally, we thank all the staff of the Commodore Hotel, who proved very helpful before and after the conference, and showed wonderful hospitality during the conference itself.

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INFINITE FACTORIZED GROUPS

Bernhard Amberg

A group G is called *factorized* if it can be written as the product of two of its subgroups A and B:

$$G = AB = \{ab \mid a \in A \text{ and } b \in B\}.$$

In the past thirty-five years an increasing number of facts about factorized groups have been discovered particularly for infinite groups. The following survey is a slightly extended version of [14] (see also Robinson [71] and N.S. Chernikov [48]). The notation is standard and may for instance be found in [68] and [70].

I. Existence of Factorizations.

Every group G has at least one factorization $G = AB$, where one of the two subgroups A and B is trivial and the other is the whole group. It is also clear that every infinite group which has all its proper subgroups finite, cannot have any other non-trivial factorization. Thus, in particular, the Prüfer groups of type p^∞ and also the so-called Tarski p-group have only the trivial factorization (A *Prüfer group of type* p^∞ for the prime p is a group which is isomorphic with the multiplicative group of complex p^n-th roots of unity for $n = 0, 1, 2, 3, \cdots$; all proper subgroups of such a group are cyclic of order p^n for some n and they form a chain. A *Tarski p-group* is an infinite group which has all its proper subgroups cyclic of order p for a very large prime p; the existence of such has been established by Ol'shanskij and Rips).

It is well-known that an abelian group which has no non-trivial factorization must be a cyclic p-group or a Prüfer group of type p^∞ for some prime p (see [75], 13.1.6). Using this it is not difficult to see the following:

Every uncountable soluble group and every infinite nilpotent group which is not a Prüfer group of type p^∞ always has at least one non-trivial factorization.

In [67] Rédei considers the structure of groups which are the product of two cyclic subgroups. The groups which are the product of two locally cyclic, torsion-free subgroups are described in Sysak [82].

For some simple groups all possible factorizations are known. For instance in [83] Thiel has found all the factorizations of the groups PSL(2,K), PSL(3,K) and PSU(3,K), where K is a locally finite field. For finite K this is also contained in Itô [58] and Blaum [28]. There are many more papers on the factorizations of the various finite simple groups.

II. The Main Problem.

The main problem about factorized groups is the following question:

Let $G = AB$ be a given factorized group and suppose that the structure of the two subgroups A and B is known. What can then be said about the structure of the factorized group $G = AB$? If, for instance, A and B have a certain group theoretical property $\mathfrak{X}$, when does it follow that G has a certain group theoretical property $\mathfrak{Y}$?

Statements of this type can be trivial: for instance, if A and B are finite (finitely generated, countable), then $G = AB$ is finite (finitely generated, countable). For many other group theoretical properties $\mathfrak{X}$ almost nothing can be said. In particular it is difficult to find examples and counterexamples. In the following we consider some group theoretical properties $\mathfrak{X}$ for which some interesting theorems can be proved.

First we remark that the above problems usually become trivial when one of the two subgroups A and B is subnormal in $G = AB$. This can be seen from the following simple lemma.

Lemma. *Let the group theoretical property $\mathfrak{X}$ be inherited by subgroups, epimorphic images and extensions. If the group $G = AB$ is the product of two $\mathfrak{X}$-subgroups A and B, one of which is subnormal in G, then G is an $\mathfrak{X}$-group.*

Proof. Suppose that the subgroup A is subnormal in G and induct on the length of some subnormal series from A to G. If A is normal in G, then

$$G/A = AB/A \cong B/(A \cap B)$$

is an $\mathfrak{X}$-group. Since also A is an $\mathfrak{X}$-group, G is an $\mathfrak{X}$-group.

If A is not normal in G, then A is normal in some subnormal subgroup S of G which has a subnormal series in G with length less than that of A. Then $G = SB$ and since $B \cap S$ is an $\mathfrak{X}$-group, also

$$(S \cap B)/(A \cap B) = (S \cap B)/(A \cap B \cap S) \cong (S \cap B)A/A = (S \cap AB)/A = S/A$$

is an $\mathfrak{X}$-group. Since A is an $\mathfrak{X}$-group, also S is an $\mathfrak{X}$-group. By induction it follows that $G = SB$ is an $\mathfrak{X}$-group. This proves the lemma.

The most satisfying result about products of groups is the following famous theorem.

(1) (Itô [57]). *If $G = AB$ where A and B are abelian subgroups of the group G, then G is metabelian.*

This is proved by a surprisingly short commutator calculation (see also [75]. 13.3.2, or [56], p.674). This calculation so far could not be extended to products of nilpotent groups (of class 2).

However, in a slightly more general situation, the (almost) solubility of the group $G = AB$ can still be obtained. A group G is *almost soluble* (*almost central*) if it contains a soluble (central) subgroup of finite index.

(2) (Chernikov [35]). *If $G = AB$ where A and B are almost central subgroups of the group G, then G is almost soluble.*

The proof is by induction on the sum of the finite indices $|A/Z(A)|$ and $|B/Z(B)|$ of the centers $Z(A)$ and $Z(B)$ of A and B; the beginning of the induction is Itô's theorem. It can also be shown that the solubility length of the soluble subgroup of G of finite index is bounded by a function of $|A/Z(A)|$ and $|B/Z(B)|$. However it remains open whether the group $G = AB$ is in fact almost metabelian. – In this connection Wilson has shown in [92] that every residually finite product of two almost abelian subgroups is almost metabelian. (A group G is *residually finite* if the intersection $J(G)$ of all subgroups of finite index in G is trivial).

There is another non-simplicity criterion which holds for arbitrary factorized groups. Recall that a group is an *FC-group* if all its elements have only finitely many conjugates in the group. A group G is *hyperabelian* if it has an ascending invariant series leading from 1 to G with abelian factors.

(3) (Zajtsev [97] and [99] and Chernikov [47]). *Let the infinite group $G = AB$ be the product of an abelian subgroup A and an FC-subgroup B with non-trivial center $Z(B)$. Then the normal closure $C = Z(B)^G$ of $Z(B)$ has an ascending G-invariant series with abelian factors; in particular $AC = A(B \cap AC)$ is hyperabelian and G has a non-trivial abelian normal subgroup.*

The most famous theorem on products of finite groups is the following.

(4) (Wielandt [87] and Kegel [60]). *If $G = AB$ where A and B are finite nilpotent subgroups of the group G, then G is soluble.*

The proof makes use of many techniques of finite group theory (see also [75], 13.2.9, and [56], p.674). Some extensions of the Kegel-Wielandt theorem to certain classes of locally finite groups can for instance be found in Kegel [61], Amberg [7] and Chernikov [34]. However, it seems to be unknown at present whether the theorem of Kegel and Wielandt holds for arbitrary infinite groups.

Using the classification of finite simple groups it was possible to prove a famous conjecture of Szép (see also Kazarin [59]).

(5) (Fisman and Arad [50]). *If $G = AB$ where A and B are finite subgroups with non-trivial centers $Z(A)$ and $Z(B)$, then G is not simple.*

In the situation of the Kegel-Wielandt theorem it is not known how the solubility length of the (finite) soluble group $G = AB$ depends on the nilpotency classes α and β of the nilpotent subgroups A and B. Is it perhaps $\alpha + \beta$? Some indications in this direction are contained in the following result (for an extension to certain locally finite groups see Amberg [7]).

(6) (Pennington [65]). *If $G = AB$ where A and B are finite nilpotent subgroups of the group G with nilpotency classes α and β, then the $(\alpha+\beta)$-th term $G^{(\alpha+\beta)}$ of the derived series of G is a π-group where $\pi = \pi A \cap \pi B$.*

Here, as usual πY denotes the set of primes p for which there exists an element of order p in the group Y.

In view of all the above results one is tempted to ask the following question.

Problem. Let the group $G = AB$ be the product of the subgroups A and B which contain nilpotent normal subgroups A_1 of A and B_1 of B with classes α and β and finite indices $|A : A_1|$ and $|B : B_1|$. Does then G have a soluble normal subgroup of finite index with derived length a function of α and β?

III. Periodic Groups.

A group is *periodic* if all its elements have finite order, it is *locally finite* if its finitely generated subgroups are finite. The product of two periodic (locally finite) groups need not be periodic (locally finite), as the following example shows.

(7) (Suchkov [79] and [80]). *There exists a countable group $G = AB$ where A and B are locally finite subgroups of G, but G is not periodic. In fact, G contains every countable free group and also 2-subgroups which are not locally finite.*

Construction. The construction is as follows. Let $\mathbf{Z}$ be the set of integers. For each k in $\mathbf{Z}$ and $n = 0, 1, 2, \cdots$ we define the following subsets of $\mathbf{Z}$:

$$\begin{aligned} U_n^{(k)} &= \{z \mid z \in \mathbf{Z},\ 3^n(2k-1)+1 \le z \le 3^n(2k+1)\} \\ V_n^{(k)} &= \{z \mid z \in \mathbf{Z},\ 3^n \cdot 2k+1 \le z \le 3^n \cdot 2(k+1)\}. \end{aligned}$$

It is easy to see that each of these sets contains $2 \cdot 3^n$ elements and that for each n the $U_n^{(k)}$ for various k and also the $V_n^{(k)}$ for various k form partitions of $\mathbf{Z}$. It is also clear that for $n > 0$

$$\begin{aligned} U_n^{(k)} &= U_{n-1}^{(3k-1)} \cup U_{n-1}^{(3k)} \cup U_{n-1}^{(3k+1)} \\ V_n^{(k)} &= V_{n-1}^{(3k)} \cup V_{n-1}^{(3k+1)} \cup V_{n-1}^{(3k+2)}. \end{aligned} \qquad (*)$$

The following are subgroups of the symmetric group $S(\mathbf{Z})$ on $\mathbf{Z}$:

$$\begin{aligned} A_n &= \{g \mid g \in S(\mathbf{Z}),\ U_n^{(k)}g = U_n^{(k)},\ k \in \mathbf{Z}\} \\ B_n &= \{g \mid g \in S(\mathbf{Z}),\ V_n^{(k)}g = V_n^{(k)},\ k \in \mathbf{Z}\} \end{aligned}$$

Then A_n and B_n are for each n isomorphic with certain cartesian products of symmetric groups $S_{2\cdot 3^n}$ of degree $2 \cdot 3^n$. By the above relations (*) for each $n > 0$ we have $A_{n-1} \subset A_n$ and $B_{n-1} \subset B_n$.

It follows that

$$A = \bigcup_{n=0}^{\infty} A_n \qquad \text{and} \qquad B = \bigcup_{n=0}^{\infty} B_n$$

are locally finite subgroups of $S(\mathbf{Z})$. Now it is not difficult to prove the following

Lemma. *$A_{n-1}B_{n-1} \subset B_nA_n$ and $B_{n-1}A_{n-1} \subset A_nB_n$ for $n > 0$.*

The lemma implies that $AB = BA = G$ is a group. To see that it contains elements of infinite order we consider

$$U_0^{(k)} = \{2k, 2k+1\} \quad \text{and} \quad V_0^{(k)} = \{2k+1, 2k+2\}.$$

Define the element $a_0 \in A_0$ by

$$(2k)a_0 = 2k+1 \;\; \text{and} \;\; (2k+1)a_0 = 2k \;\; \text{for each } k \in \mathbf{Z},$$

and define the element $b_0 \in B_0$ by

$$(2k+1)b_0 = 2k+2 \;\; \text{and} \;\; (2k+2)b_0 = 2k+1.$$

Then

$$(2k)a_0b_0 = 2k+2 \;\; \text{for each } k \in \mathbf{Z},$$

so that the element $a_0b_0 \in AB = G$ has infinite order.

This construction will now be slightly modified to produce an example $G^* = A^*B^*$ which is countable. Since A and B are locally finite, we can find finite subgroups A_0^* of A and B_0^* of B such that $a_0 \in A_0^*$ and $b_0 \in B_0^*$. Write

$$A_0^*B_0^* = \{b_1a_1, \cdots, b_ka_k \mid a_i \in A,\; b_i \in B \text{ for } i = 1, \cdots, k\}.$$

Since A and B are locally finite, the subgroups

$$A_1^* = \langle A_0^*, a_1, \cdots, a_k \rangle \;\; \text{and} \;\; B_1^* = \langle B_0^*, b_1, \cdots, b_k \rangle$$

are finite. Clearly

$$A_0^* \subset A_1^*, \quad B_0^* \subset B_1^*, \quad A_0^*B_0^* \subset B_1^*A_1^*.$$

In this way we construct an ascending series of finite subgroups with

$$A_0^* \subset A_1^* \subset A_2^* \subset A_3^* \subset \cdots \quad \text{and} \quad B_0^* \subset B_1^* \subset B_2^* \subset B_3^* \subset \cdots$$

such that

$$A_0^*B_0^* \subset B_1^*A_1^* \subset A_2^*B_2^* \subset B_3^*A_3^* \subset \cdots.$$

Then

$$A^* = \bigcup_{i=0}^{\infty} A_i^* \qquad \text{and} \qquad B^* = \bigcup_{i=0}^{\infty} B_i^*$$

are locally finite subgroups, and $A^*B^* = B^*A^* = G^*$ is a countable group with the desired properties, since it contains the element a_0b_0 of infinite order.

It is shown in [80] that G^* even contains every countable free group and 2-subgroups which are not locally finite.

On the other hand soluble products of two periodic groups are always periodic. This follows from the following theorem. A group G is a *π-group* if the orders of all its elements contain only primes from the set of primes π.

(8) (Sysak [81]) *If $G = AB$ is a hyperabelian group where A and B are π-subgroups of G for a set of primes π, then G is a π-group.*

This result has an interesting consequence.

Corollary. *If $G = AB$ is a soluble group where A and B are subgroups of finite exponent , then G has finite exponent.*

Proof. Let G^*, A^* and B^* be cartesian products of countably many copies of G, A and B respectively. Then $G^* = A^*B^*$ is soluble and the subgroups A^* and B^* have finite exponent. By Theorem 8 G^* is periodic. But this implies that the group G must have finite exponent.

Some further results on products of abelian subgroups of finite exponent can be found in Brisley and MacDonald [29], Holt and Howlett [54] and Howlett [55].

In [81] Sysak constructs a locally soluble group $G = AB$ where A and B are p'-groups for the complementary set p' of the prime p, but G is not a p'-group. The following questions remain open.

Problems. (a) Is every locally soluble product of two periodic subgroups periodic? Is every product of two soluble and periodic subgroups periodic?

(b) Let $\mathfrak{X}$ be a class of groups which is closed under the forming of subgroups, epimorphic images and extensions. Is every soluble product $G = AB$ of two $\mathfrak{X}$-subgroups A and B likewise an $\mathfrak{X}$-group? (Note that Suchkov's example shows that for non-soluble groups $G = AB$ and the class $\mathfrak{X}$ of periodic groups the answer to the last question is negative).

IV. Minimum and Maximum Conditions.

The following lemma is elementary (see Amberg [2] or [3]).

Lemma. *Let the group $G = AB$ be the product of its subgroups A and B.*

(a) *If A and B satisfy the maximum condition on subgroups, then G satisfies the maximum condition on normal subgroups.*

(b) *If A and B satisfy the minimum condition on subgroups, then G satisfies the minimum condition on normal subgroups.*

Proof. (of (a), the proof of (b) is similar). If N_i is an ascending chain of normal subgroups of G, then

$$A \cap N_i = A \cap N_{i+1} \quad \text{and} \quad B \cap AN_i = B \cap AN_{i+1} \quad \text{for almost all } i.$$

By the modular law

$$AN_i = AN_i \cap AB = A(B \cap AN_i) = A(B \cap AN_{i+1}) = AN_{i+1}$$

for almost all i. Therefore

$$N_i = N_i(A \cap N_i) = N_i(A \cap N_{i+1}) = N_{i+1} \cap AN_i = AN_{i+1} \cap N_{i+1} = N_{i+1}$$

for almost all i. This proves the lemma.

The lemma raises the question whether every product of two subgroups $G = AB$ with maximum (minimum) condition on subgroups always has the maximum (minimum) condition on subgroups. This question is open even when A and B are soluble. However it has a positive answer when the whole group G is soluble. We first consider the minimum condition.

A soluble group G satisfies the minimum condition on subgroups if and only if it is a *Chernikov group* [(*)], i.e. there exists a normal subgroup $J(G)$ of G such that the factor group $G/J(G)$ is finite and $J(G)$ is the direct product of finitely many Prüfer groups of type p^∞ for finitely many primes p. (In any group G denote by $J(G)$ the intersection of all subgroups of finite index in G and call this the *finite residual* of G).

In [76] Sesekin has shown that the product $G = AB$ of two abelian subgroups A and B with minimum condition is a (metabelian) Chernikov group. In Amberg [2] and [3] it is shown that every hyper-(almost-abelian) product $G = AB$ of two Chernikov groups is a Chernikov group and that $J(G) = J(A)J(B)$; for soluble groups G this was proved by Kegel already around 1965 but not published. The most general result on products of Chernikov groups obtained so far concerns.so-called locally graduated groups. A group G is *locally graduated* if every non-trivial finitely generated subgroup of G has a non-trivial finite epimorphic image. The class of these groups is very large and contains the locally finite, the locally soluble, the residually finite and the linear groups.

(9) (Chernikov [32]). *If the locally graduated group $G = AB$ is the product of two Chernikov subgroups A and B, then G is a Chernikov group, so that*

$$J(G) = J(A)J(B).$$

A similar theorem holds for products of periodic groups with minimum condition on p-subgroups for every prime p. A soluble group G is periodic with minimum condition on p-subgroups for every prime p if and only if the finite residual $J(G)$ is the direct product of Prüfer groups of type p^∞, for each prime p only finitely many (but possibly infinitely many primes), and $G/J(G)$ has finite maximal p-subgroups for each prime p (see [63], 3.18). In particular we note

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(*) named after S.N.Chernikov

(10) (Amberg [8] and [9], Chernikov [38] and [48]). *If the soluble group $G = AB$ is the product of two periodic subgroups A and B with minimum condition on p-subgroups for every prime p, then G is periodic with minimum condition on p-subgroups for every prime p and*

$$J(G) = J(A)J(B).$$

We turn now to the maximum condition. A soluble group G satisfies the maximum condition on subgroups if and only if it is *polycyclic,* i.e. there exists a finite series G_i of G

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = G$$

such that the factors G_i/G_{i-1} are cyclic. The number of infinite cyclic factors G_i/G_{i-1} in any such series is an invariant of G, called its *torsion-free rank* or *Hirsch-number* $r_0(G)$. – A group G is *almost polycyclic* if it contains a polycyclic normal subgroup N of finite index in G. In this case put $r_o(G) = r_o(N)$.

It was shown around 1972 by Amberg and independently by Sesekin that every product of two finitely generated abelian groups is (metabelian and) polycyclic (see [77], [2] and [3]). In the last two papers it is even shown that every soluble product $G = AB$ of two polycyclic subgroups A and B, one of which is nilpotent, is always polycyclic. Finally the condition that A or B is nilpotent, was removed in [64] and [95]. The proof in [64] even holds for an almost soluble group G.

(11) (Lennox and Roseblade [64], see also Zajtsev [95]). *If the almost soluble group $G = AB$ is the product of two (almost) polycyclic subgroups A and B, then G is (almost) polycyclic.*

(11+) (Amberg [2], [4] and [12]). *If $G = AB$ is almost polycyclic, then the torsion-free rank $r_0(G)$ of G satisfies*

$$r_0(G) = r_0(A) + r_0(B) - r_0(A \cap B).$$

Comments on the proof of Theorem 11.

In order to explain the main reduction arguments that are used in proving theorems about (almost) soluble factorized groups we discuss briefly the proof of the following special case of Theorem 11:

(*) *If $G = AB$ is a soluble group where A and B are polycyclic and A or B is nilpotent, then G is polycyclic* (Amberg [2] and [3]).

Assume that the group G in (*) is not polycyclic. Let $G = AB$ be a counterexample with minimal derived length, and let $K \neq 1$ be the last non-trivial term of the derived series of G. Then $G/K = (AK/K)(BK/K)$ is polycyclic, so that the abelian normal subgroup K of G cannot be finitely generated.

In this situation it is convenient to consider the so-called *factorizer* of K in G; this is the subgroup

$$X = X(K) = AK \cap BK$$

It is not difficult to see that it has the '*triple factorization*'

$$X = K(A \cap BK) = K(B \cap AK) = (A \cap BK)(B \cap AK).$$

Since K is not finitely generated, X is also a counterexample. Therefore we may suppose without loss of generality that

$$(\alpha) \qquad G = AK = BK = AB$$

Clearly $C = (A \cap K)(B \cap K)$ is normal in $AK = BK = G$. Since $A \cap K$ and $B \cap K$ are finitely generated, C is abelian and finitely generated. Therefore also $G/C = (AC/C)(BC/C)$ is not polycyclic. Since this group is also a counter example with a triple factorization of type (α) we may in addition suppose that

$$(\beta) \qquad A \cap K = B \cap K = 1$$

Therefore A and B are complements of the normal subgroup K of G. In particular A and B are isomorphic and therefore both nilpotent.

In this situation one usually tries to apply cohomological arguments to show that A and B are conjugate, which immediately implies $A = B = G$, a contradiction. For this, in particular some 'vanishing theorems' of D. Robinson play a decisive role (see [69] and [73]). These guarantee the vanishing of the first cohomology group $H^1(A, K)$ under certain conditions.This is for example the case when A is nilpotent, K is a noetherian A-module and $K = [K, A]$ (see [69 or [73]).

In the original proof of (*) a contradiction was obtained in the following way. By an earlier result of Kegel [62] a finite triply factorized group of type (α) with three nilpotent factors A, B and K is likewise nilpotent; see also Theorem 21 below. Thus every finite epimorphic image of the counterexample $G = AB$ is nilpotent. Since A and B are polycyclic, $G = AB$ is a finitely generated soluble group. Now a well-known result of Robinson and Wehrfritz yields that G is nilpotent and hence polycyclic (see [70] p.459). This contradiction proves (*).

In the proof of the general statement of Theorem 11 one needs much deeper results about finitely generated modules over polycyclic groups, in particular the theorem of Roseblade in [74] that a simple module over a polycyclic group is finite. Thus the Theorem of Lennox-Roseblade-Zajtsev cannot be regarded as an elementary result.

The following problems remain open.

Problems. (a) Is every product of two polycyclic groups always polycyclic?
(b) Is every product of two Chernikov groups always a Chernikov group?

V. Nilpotent-by-Polycyclic Groups.

A group G is called *nilpotent-by-polycyclic* if it contains a nilpotent normal subgroup with polycyclic factor group. In such a group G also the Fitting subgroup Fit G is a nilpotent characteristic subgroup of G with polycyclic factor group $G/\text{Fit } G$ (Fit G denotes the product of all nilpotent normal subgroups of G). It follows that an extension of a nilpotent-by-polycyclic group by a polycyclic group is likewise nilpotent-by-polycyclic.

Theorems 11 and 11+ may be used to prove the following theorem.

(12) (Amberg, Franciosi, de Giovanni [21]). *Let the soluble group $G = AB$ of derived length n be the product of a nilpotent-by-polycyclic subgroup A and a polycyclic subgroup B. Then G is nilpotent-by-polycyclic and, if G is not abelian, the torsion-free rank of the Fitting factor group of G satisfies*

$$r_0(G/\text{Fit } G) \leq (2n-3)(r_0(B) + r_0(A/\text{Fit } A)).$$

Comments on the proof.

The proof of the rank inequalities uses Theorem 11+ as well as some facts about the automorphism group of finitely generated abelian groups that depend ultimately on algebraic number theory, in particular on Dirichlet's Unit Theorem. (For the case when A and B are abelian, see Heineken [52]). The following proof of the fact that G is nilpotent-by-polycyclic is easier and depends mainly on Theorem 11:

Assume that G is not nilpotent-by-polycyclic. Let $G = AB$ be a counterexample with minimal derived length and let $K \neq 1$ be the last term of the derived series of G. Then there exists a nilpotent normal subgroup L/K of G/K with polycyclic factor group G/L. Let $F = \text{Fit } A$ be the Fitting subgroup of A and c the nilpotency class of F.

1. *The case $F \cap K = 1$.*

In this case

$$A \cap K \simeq (A \cap K)F/F \subseteq A/F \quad \text{and} \quad (A \cap BK)/(A \cap K) \simeq (A \cap BK)K/K \subseteq BK/K$$

are polycyclic, so that also $A \cap BK$ is polycyclic. By Theorem 11 the factorized group $BK = B(A \cap BK)$ is also polycyclic. In particular K is finitely generated. As a soluble group of automorphisms of a polycyclic group also $G/C_G(K)$ is polycyclic (see [68], Part 1, p.82). Then also $G/(L \cap C_G(K)$ is polycyclic. Since K is contained in the center of $L \cap C_G(k)$ and $(L \cap C_G(K))/K \subseteq L/K$ is nilpotent, also $L \cap C_G(K)$ is nilpotent. Therefore G is nilpotent-by-polycyclic.

2. *The general case.*

Note that $F \cap K$ is normal in AK and that $F/(F \cap K)$ is a nilpotent normal subgroup of $A/(F \cap K)$ with polycyclic factor group A/F. Consider the factorized group $AK = A(B \cap AK)$ and its factor group

$$AK/(F\cap K) = (A/(F\cap K))((B\cap AK)(F\cap K)/(F\cap K)).$$

By the first case $AK/(F\cap K)$ and also its subgroup $FK/(F\cap K)$ are nilpotent-by-polycyclic. Since $F\cap K = Z_c(F)\cap K$ is contained in the c-th term $Z_c(FK)$ of the upper central series of FK, also FK is nilpotent-by-polycyclic. Since $AK/FK \simeq A/(A\cap FK)$ is polycyclic, also AK is nilpotent-by-polycyclic.

Put $H = AK\cap L$. Then $K\subseteq H\subseteq L$ and since L/K is nilpotent, H is a nilpotent-by-polycyclic subnormal subgroup of G. Let

$$H = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_t = G$$

be the standard series of H in G. Since A normalizes H, it is well-known that A also normalizes each H_i. Then $A\subseteq N_G(H_i)$ and so $N_G(H_i) = A(B\cap N_G(H_i))$. Consider the factorized group

$$N_G(H_i)/H_i = (AH_i/H_i)((B\cap N_G(H_i))H_i/H_i).$$

The group AH_i/H_i is an epimorphic image of

$$A/(A\cap H) = A/(A\cap L) \simeq AL/L \subseteq G/L,$$

and hence it is polycyclic. By Theorem 11 the group $N_G(H_i)/H_i$ is also polycyclic. Hence, if H_i is nilpotent-by-polycyclic for some i, also $N_G(H_i)$ and its subgroup H_{i+1} are nilpotent-by-polycyclic. Since $H_0 = H$ is nilpotent-by-polycyclic, also $G = H_t$ is nilpotent-by-polycyclic.

Remarks. (a) For non-soluble groups Theorem 12 becomes false in general. This can be seen from the alternating group of degree 5 which can be written as a product $G = AB$ where A is a cyclic group of order 5 and B is an alternating group of degree 4, but G is not soluble.

(b) Let p be a prime and $A = \text{Aut } B$ be the automorphism group of a Prüfer group B of type p^∞. Then A is an uncountable abelian group. Consider the holomorph $G = AB$. Then $B = \text{Fit } G$ and $G/\text{Fit } G$ is not a Chernikov group, since it is uncountable. Therefore in Theorem 12 'polycyclic' cannot be replaced by 'Chernikov'. However, it is not difficult to show that a soluble product of a hypercentral-by-Chernikov group and a Chernikov group must be an extension of a locally nilpotent group by a Chernikov group.

Problem. Determine the structure of (soluble) groups $G = AB$ where A is a subgroup satisfying some generalized nilpotency condition and the subgroup B satisfies some finiteness condition.

VI. Minimax Groups.

A group G is a *minimax group* if it has a finite series G_i such that

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = G$$

whose factors G_{i+1}/G_i satisfy the maximum or minimum condition on subgroups. It is easy to see that a soluble group G is a minimax group if it has such a series in which each factor G_{i+1}/G_i is cyclic or a Prüfer group of type p^∞ for some prime p. The number of infinite factors in any such series is an invariant of the group G which is called its *minimax rank* $m(G)$.

Problem. Is every product $G = AB$ of two (soluble) minimax subgroups A and B a minimax group?

If the group $G = AB$ is soluble, this question has a positive answer.

(13) (Wilson [88] and [90]). *If $G = AB$ is a soluble group where A and B are minimax subgroups of G, then G is a minimax group and the minimax rank of G satisfies*

$$m(G) = m(A) + m(B) - m(A \cap B).$$

The special case of this theorem, where A or B is nilpotent, was previously proved by Amberg and Robinson in [26] and by Zajtsev in [96]; the case when A and B are abelian was already treated in [93].

The proofs of these results reduce quickly to the situation of a triply factorized group

$$G = AB = AK = BK$$

where A and B are minimax subgroups and K is an abelian normal subgroup of G with $A \cap K = B \cap K = 1$.

Regard K as an A-module. The map δ from A to K assigning to each element a of A the unique element of $(a^{-1}B) \cap K$ is surjective and is easily checked to be a *derivation,* that is

$$(a_1a_2)\delta = (a_1\delta)a_2 + a_2\delta \quad \text{for all } a_1,\ a_2 \text{ in } A.$$

Theorem 13 follows immediately from a fact proved in [90] that if δ is a surjective derivation from a soluble minimax group A to an A-module K, then K, regarded as a group, is minimax (This even holds when the derivation δ is only nearly surjective, that is the set $A\delta$ has a finite complement in K). To see this, after some reductions one has to show that there are no infinite simple modules K for a soluble minimax group A with the property that there is a (nearly) surjective derivation from A to K. Note that all soluble minimax groups which are not polycyclic have infinite simple modules, but apart from some ideas in a paper of Brookes [30] there is little known about these. (See however Theorem 19 below). In the proof also some cohomological results are used.

For the proof of the rank formula for the minimax rank in Theorem 13 number theoretical arguments such as Dirichlet's Unit Theorem are decisive.

Corresponding to Theorem 12 there is the following result. Recall that the *Gruenberg radical* $K(G)$ of the group G is the subgroup generated by all its abelian ascendant subgroups.

(14) (Amberg, Franciosi, de Giovanni [21]). *If the soluble group $G = AB$ of derived length n is the product of a hypercentral-by-minimax subgroup A and a minimax subgroup B, then G is an extension of its Gruenberg radical by a minimax group, and, if G is not abelian, the minimax rank of the Gruenberg factor group satisfies*

$$m(G/K(G)) \leq (2n-3)(m(B) + m(A/H)),$$

where H is any hypercentral normal subgroup of A with minimax factor group A/H.

VII. Finite Rank.

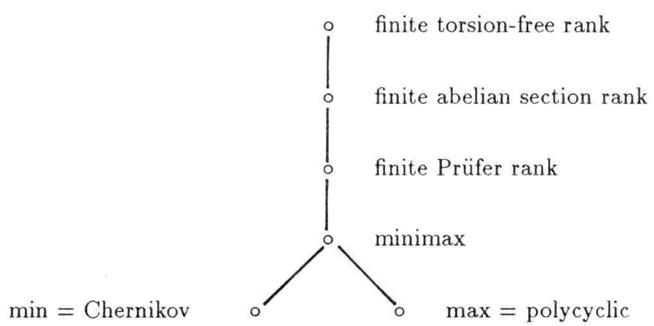
There are several types of groups of finite rank which are considered.

(a) A group G has *finite torsion-free rank* if it has a series of finite length n in which each non-periodic factor is infinite cyclic. The number of infinite cyclic factors in any such series is an invariant of G called its *torsion-free rank* $r_0(G)$.

(b) A group G has *finite abelian section rank* if every elementary abelian section (p-factor) of G is finite for every prime p.

(c) A group G has *finite Prüfer rank* $r = r(G)$ if every finitely generated subgroup of G can be generated by r elements and r is the least such number.

For soluble groups the relations between the various classes are indicated in the following diagram. Simple examples show that even for abelian groups all the inclusions are proper.



Problem. If the group $G = AB$ is the product of two (soluble) subgroups A and B of finite rank in some sense, does then G have finite rank in this same sense?

This problem was first considered in Amberg [4]. The example of Suchkov in Theorem 7 shows that an arbitrary product of two groups with finite torsion-free rank can have infinite torsion-free rank. On the other hand, Sysak has announced that every soluble product $G = AB$ of two subgroups A and B with finite torsion-free rank has likewise finite torsion-free rank. If A and B are abelian, this was known already from [93]. Robinson in [72] considered the case when A or B is nilpotent. Some more general special cases when A and B satisfies some weak nilpotency requirements were treated in Amberg [12], Chernikov [42] and Zajtsev [98]. Also it is shown in [25] that every factorized group $G = AB$ of finite torsion-free rank satisfies

$$r_0(G) \leq r_0(A) + r_0(B) - r_0(A \cap B).$$

It seems to be unknown at present whether in the general case we have an equality here. The difficulty is to prove that the rank equality holds for the torsion-free rank in the case that the group G has the form $G = AB = AT = BT$ where T is a periodic normal subgroup of G.

Every soluble group G with finite abelian section rank has a finite series

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_t = G$$

whose factors G_{i+1}/G_i are infinite cyclic or abelian torsion groups whose primary components satisfy the minimum condition. The number of Prüfer p-groups in any such series is finite and an invariant of G which is called its *p^∞-rank*. The following theorem solves the above problem for soluble groups with finite abelian section rank.

(15) (Wilson [91]. *If the soluble group $G = AB$ is the product of two subgroups A and B of finite abelian section rank, then G has finite abelian section rank.*

The special case of this when A and B are abelian was treated by Zajtsev in [93], Robinson considered the case when A or B is nilpotent in [72]. Some further more general special cases of Theorem 15 can be found in Amberg [4] and [13] and Chernikov[44]. Using the arguments of Wilson [88] it is possible to prove the following additional information on the structure of the factorized group in Theorem 15 (see Amberg and Müller [25]).

(15+) *If $G = AB$ is a soluble group with finite abelian section rank, then for every prime p and for $p = 0$ the ranks $r_p(G)$ satisfy*

$$r_p(G) = r_p(A) + r_p(B) - r_p(A \cap B).$$

From Theorem 15 it follows easily that also every soluble product $G = AB$ of two subgroups A and B with finite Prüfer rank likewise has finite Prüfer rank (see [13] or [91]). Morover, the Prüfer rank of G is bounded by a function of the Prüfer ranks of A and B (see [13]). For finite G this was already shown by Zajtsev in [95] (see also Robinson [72] and Amberg and Robinson [26]).

Again the proofs of these results involve cohomological methods as well as some facts about finitely generated modules over locally almost-polycyclic groups: see Theorem 19 below. In the proof of Theorem 15 theorems about surjective derivations are decisive.