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# PREFACE

The characteristic polynomial of a linear map is one of the most basic of mathematical objects, and under the guise of the Alexander polynomial has been much studied by knot theorists. The rational homology of the infinite cyclic cover of a knot complement is indeed determined by a family of such polynomials. The finer structure of the integral homology, or the homology of covers of a link complement (corresponding to a set of commuting linear maps) is reflected in the Alexander ideals. These notes are intended to survey what is presently known about the Alexander ideals of classical links, and where possible to give "coordinate free" arguments, avoiding explicit presentations and using only the general machinery of commutative and homological algebra. This has been done to clarify the concepts; in computing examples it is convenient to use Wirtinger presentations and the free differential calculus, Seifert surfaces or surgery descriptions of links. (The avoidance of techniques peculiar to the fundamental group or to 3-dimensional topology means also that these arguments may apply to links in higher dimensions, but little is said on this topic after Chapter II.)

This work grew out of part of my 1978 A.N.U. Ph.D. thesis. However although most of the proofs are mine, a number of the results (mostly in Chapters I, IV, VII and VIII) are due to others. Some of the latter results have been quoted without proof, as the only proofs known to me are very different in character from the rest of these notes.

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CONTENTS

# PRELIMINARIES

In these notes we shall generally follow the usage of Bourbaki [ 13 ] for commutative algebra, Crowell and Fox [ 43 ] for combinatorial group theory, and Rourke and Sanderson [ 159 ] for geometric topology. The book of Magnus, Karrass and Solitar [ 123 ] is a more comprehensive reference for combinatorial group theory, while the books of Hempel [ 69 ], Rolfsen [157] and Spanier [ 177 ] are useful for other aspects of topology.

All manifolds and maps between them shall be assumed PL unless otherwise stated. The expression $A \approx B$ means that the objects A and B are isomorphic in some category appropriate to the context. When there is a canonical isomorphism, or after a particular isomorphism has been chosen, we shall write $A = B$. (For instance the fundamental group of a circle is isomorphic to the additive group of the integers $\mathbb{Z}$, but there are two possible isomorphisms, and choosing one corresponds to choosing an orientation for the circle).

Qualifications and subscripts shall often be omitted, when there is no risk of ambiguity. In particular "$\mu$-component n-link" may be abbreviated to "link", and the symbols $\Lambda_\mu$, X(L), G(L) may appear as $\Lambda$, X and G.

# CHAPTER I LINKS AND LINK GROUPS

This chapter is principally a resumé of standard definitions and theorems, without proofs. Although our main concern is with the classical case, we have framed our definitions so as to apply also in higher dimensions. We begin with definitions of links and of the most important equivalence relations between them. Next we consider link groups and homology boundary links. There follows a section on the equivariant homology of covering spaces of link exteriors, and we conclude with some comments on the construction of such covering spaces.

Let $\mu$ and $n$ be positive integers. If $X$ is a topological space, let $\mu X$ be the space $X \times \{1, \ldots, \mu\}$, the disjoint union of $\mu$ copies of $X$. Let $D^n = \{<x_1, \ldots, x_n> \text{ in } \mathbb{R}^n \mid \sum_{1 \leq i \leq n} x_i^2 \leq 1\}$ be the n-disc and let $S^n = \partial D^{n+1}$ be the n-sphere. The standard orientation of $\mathbb{R}^n$ [ 159 ; page 44] induces an orientation of $D^n$, and hence of $S^{n-1}$ by the convention that the boundary of an oriented manifold be oriented compatibly with taking the inward normal last (cf [ 159 ; page 45]).

<u>Definition</u> A $\mu$-<u>component</u> n-<u>link</u> is an embedding $L: \mu S^n \longrightarrow S^{n+2}$. The $i^{th}$ component of $L$ is the n-knot (1-component n-link) $L_i = L \mid S^n \times \{i\}$. A <u>link type</u> is an equivalence class of links under the relation of being ambient isotopic.

Notice that with this definition, and with the above conventions on the orientation of the spheres, all links are oriented. A 1-link is locally flat (essentially because there are no knotted embeddings of $S^o$ in $S^2$), but embeddings of higher dimensional manifolds in codimension 2 need not be locally flat [ 161 ; page 59].

Definition An I-equivalence between two embeddings $f,g:A \to B$ is an embedding $F:A \times [0,1] \longrightarrow B \times [0,1]$ such that $F|A \times \{0\} = f$, $F|A \times \{1\} = g$ and $F^{-1}(B \times \{0,1\}) = A \times \{0,1\}$.

In this definition we do not assume that the data are PL (so here an embedding is a 1-1 map inducing an homeomorphism with its image). Recent results of Giffen suggest that wild I-equivalences have a rôle even in the context of PL links [ 54a ]. Clearly isotopic embeddings are I-equivalent.

A locally flat isotopy is an ambient isotopy [ 159 ; page 58], but even an isotopy of 1-links need not be locally flat. For instance any knot is isotopic to the unknot, but no such isotopy of a non trivial knot can be ambient. However a theorem of Rolfsen shows that the situation for links is no more complicated.

Definition Two $\mu$-component n-links L and L' are locally isotopic if there is an embedding $j:D^{n+2} \longrightarrow S^{n+2}$ such that $D = L^{-1}(j(D^{n+2}))$ is an n-disc in one component of $\mu S^n$ and such that $L|\mu S^n - D = L'|\mu S^n - D$.

Theorem (Rolfsen [ 153 ]) Two n-links L and L' are isotopic if and only if L' may be obtained from L by a finite sequence of local isotopies and an ambient isotopy.

In other words L and L' are isotopic if and only if L' may be obtained from L by successively suppressing or inserting small knots in one component at a time.

Definition A concordance between two $\mu$-component n-links L and L' is a locally flat I-equivalence $\mathscr{L}$ between L and L'. A link is null concordant (or slice) if it is concordant to the trivial link.

A link L is a slice link if and only if it extends to a locally flat embedding $C:\mu D^{n+1} \longrightarrow D^{n+3}$ such that $C^{-1}(S^{n+2}) = \mu S^n$.

<u>Definition</u> Two $\mu$-component n-links L and L' are <u>link-homotopic</u> if there is a map $H:\mu S^n \times [0,1] \longrightarrow S^{n+2}$ such that $H|\mu S^n \times \{0\} = L$, $H|\mu S^n \times \{1\} = L'$ and $H(S^n \times \{t\} \times \{i\}) \cap H(S^n \times \{t\} \times \{j\}) = \emptyset$ for all t in $[0,1]$ and for all $1 \leq i \neq j \leq \mu$.

In other words a link-homotopy is a homotopy of the maps L and L' such that at no time do the images of distinct components of $\mu S^n$ intersect (although self intersections of components are allowed). Milnor [129] has given a thorough investigation of homotopy of 1-links. Giffen [55] and Goldsmith [57] have recently shown that concordant 1-links are link-homotopic. (Giffen [54a] has also shown that I-equivalent links need not be PL I-equivalent). For other results on isotopy of links and related equivalence relations see [26, 82, 111, 130, 154, 155,175].

The link group

The basic algebraic invariant of a link is the fundamental group of its complement, and most of these notes are concerned with the structure of metabelian quotients of the groups of 1-links.

<u>Definition</u> The <u>exterior</u> of a $\mu$-component n-link L is $X(L) = S^{n+2} - N$, where N is an open regular neighbourhood of the image of L. The <u>group</u> of the link L is G(L), the fundamental group of X(L).

The exterior of L is a deformation retract of $S^{n+2} - L$, the complement of L, and is a compact connected PL (n+2)-manifold with $\mu$ boundary components. By Alexander duality $H_1(X(L);\mathbb{Z}) \approx \mathbb{Z}^\mu$, $H_i(X(L);\mathbb{Z}) = 0$

for $1 < i < n+1$ and $H_{n+1}(X(L);\mathbb{Z}) \approx \mathbb{Z}^{\mu-1}$. We shall assume henceforth that all links are locally flat. Then $\partial X(L) = \mu S^n \times S^1$. A meridianal curve for the $i^{th}$ component of L is an oriented curve in the boundary of X(L) which bounds a disc in $S^{n+2} — X(L_i)$ having algebraic intersection +1 with $L_i$. The image of such a curve in the link group G is well defined up to conjugation, and any element of G in this conjugacy class is called an $i^{th}$ meridian. The images of the meridians in the abelianization $G/G' = H_1(X:\mathbb{Z})$ are well defined and freely generate it, inducing an isomorphism with $\mathbb{Z}^\mu$.

An application of van Kampen's theorem shows that G is the normal closure of the set of its meridians. (The normal closure of a subset S of a group is the smallest normal subgroup containing S, and shall be denoted $\langle\langle S \rangle\rangle$). Thus the group of a $\mu$-component n-link is a finitely presentable group G which is normally generated by $\mu$ elements, with abelianization $\mathbb{Z}^\mu$ and, if $n \geq 2$, with $H_2(G;\mathbb{Z}) = 0$ (since by Hopf's theorem $H_2(G;\mathbb{Z})$ is the cokernel of the Hurewicz homomorphism $\pi_2(X) \longrightarrow H_2(X;\mathbb{Z})$ [ 83 ] ). Conversely Kervaire has shown that if $n \geq 3$ these four conditions characterize the group of a $\mu$-component n-link [ 96 ]. If $n = 2$ these conditions are neccessary but not sufficient, even for $\mu = 1$ [ 71 ]; if the last condition is replaced by "the group has a presentation of deficiency $\mu$" then Kervaire showed also that it is the group of a link in some homotopy 4-sphere, but this stronger condition is not neccessary.

The case of 1-links with $\mu > 1$ is quite different. For then $H_2(G;\mathbb{Z}) \approx \mathbb{Z}^{\mu-1}$ unless $\pi_2(X) \neq 0$, in which case by the Sphere Theorem [ 147 ] the link is splittable. (An n-link L is splittable if there is an (n+1)-sphere $S^{n+1} \subsetneq S^{n+2} - L$ such that L meets each complementary ball, that is, each component of $S^{n+2} - S^{n+1}$). This is related to the presence of longitudes, non trivial elements of the group commuting with meridians.

Let L be a $\mu$-component 1-link. An $i^{th}$ longitudinal curve for L is a closed curve in the boundary of X(L) which is parallel to $L_i$ (and so in particular intersects an $i^{th}$ meridianal curve in just one point), and which is null homologous in $X(L_i)$. The $i^{th}$ meridian and $i^{th}$ longitude of L, the images of such curves in G(L), are well defined up to simultaneous conjugation. If X(L) has been given a basepoint *, then representatives of the conjugacy classes of the meridians and longitudes in $\pi_1(X(L),*) \approx G(L)$ may be determined on choosing paths joining each component of the boundary to the base point. The linking number $\ell_{ij}$ of the $i^{th}$ component of L with the $j^{th}$ is the image of an $i^{th}$ longitude of L in $H_1(X(L_j);\mathbb{Z}) = \mathbb{Z}$; it is not hard to show that $\ell_{ij} = \ell_{ji}$. (Notice that $\ell_{ii} = 0$).

When chosen as above, the $i^{th}$ longitude and $i^{th}$ meridian commute, since they both come from the fundamental group of the $i^{th}$ boundary component, which is a torus. In the case of higher dimensional links there is no analogue of longitude in the link group, because spheres of dimension greater than or equal to 2 are simply connected, while in knot theory the longitudes are often overlooked, as for 1-knots they always lie in the second commutator subgroup G" (See below). The presence of the longitudes gives the study of classical links and their groups much of its special character.

If the $i^{th}$ longitude is equal to 1 in G(L), then $L_i$ extends to an embedding of a disc disjoint from the other components of L, by the Loop Theorem [ 147 ]. A link is trivial if all the longitudes equal 1.

<u>Theorem 1</u> <u>A</u> 1-<u>link</u> L <u>is trivial if and only if</u> G(L) <u>is free</u>.

<u>Proof</u> (Note that the rank of G(L) must equal the number of components of L). Since a free group contains no noncyclic abelian subgroups [123; page 42], the $i^{th}$ longitude and $i^{th}$ meridian must lie in a common cyclic group. On considering the images in $H_1(X(L_i);\mathbb{Z}) = \mathbb{Z}$, we conclude that the $i^{th}$ longitude must be null homotopic. Hence using the Loop Theorem inductively we see that the link is trivial. The argument in the other direction is immediate. //

This result may be restated as "An n-link is trivial if and only if the homotopy groups $\pi_j(X)$ are those of a trivial link for $j \leq [\frac{n+1}{2}]$" and in this form remains true for n-knots whenever $n \geq 3$. (Of course the proofs are quite different. See Levine [ 114 ] for $n \geq 4$ and for $n = 3$ see Shaneson [ 169 ] in conjunction with Milnor duality [ 132 ]). However it is false for all $\mu \geq 2$ and $n \geq 2$, as was first shown by Poenaru [ 149 ]. (See also Sumners [ 181 ] and the remarks following Theorem II.6 below).

<u>Definition</u> A $\mu$-component n-link L is a <u>boundary link</u> if there is an embedding $P:W = \cup W_i \longrightarrow S^{n+2}$ of $\mu$ disjoint orientable (n+1)-manifolds each with a single boundary component, such that $L = P|\partial W$.

All knots are boundary links, and conversely many arguments and results about knots proved by means of such "Seifert surfaces" carry over readily to arbitrary boundary links.

Theorem (Smythe [ 174 ]) A $\mu$-component 1-link is a boundary link if and only if there is a map of G(L) onto $F(\mu)$, the free group of rank $\mu$, carrying some set of meridians to a basis of $F(\mu)$.

Gutiérrez extended Smythe's theorem to n-links and characterized the trivial n-links for $n \geq 4$ as the boundary links whose complement has the correct homology groups $\pi_j(X)$ for $j \leq [\frac{n+1}{2}]$ [ 61 ]. The splitting theorem of Cappell shows that this is also the correct criterion for $n = 3$ [ 22 ]. (Little is known about the case $n = 2$, even for knots. See Swarup [ 187 ]).

Definition A $\mu$-component link L is an homology boundary link if there is an epimorphism $G(L) \longrightarrow F(\mu)$.

Note that there is no assumption on the meridians. The kernel of such an epimorphism is necessarily $G_\omega = \bigcap_{n \geq 0} G_n$, the intersection of the intersection of the terms of the lower central series of G. (See below in this section). Smythe showed also that a 1-link L is an homology boundary link if and only if there are $\mu$ disjoint orientable surfaces $U_i$ in X(L) with $\partial U_i \subset \partial X(L)$ and such that $\partial U_i$ is homologous to the $i^{th}$ longitude in $\partial X(L)$. (Such surfaces shall be referred to as "singular Seifert surfaces"). For an homology boundary link, the longitudes lie in $G_\omega$, since a free group contains no noncyclic abelian subgroups. For a boundary link, they lie in $(G_\omega)'$, since they bound surfaces which lift to the maximal free cover of the link complement. (See also the next section).

Any 1-link is ambient isotopic to a link L with image lying strictly above the hyperplane $\mathbb{R}^2 \times 0$ in $\mathbb{R}^3 = S^3 - \{\infty\}$ and for which the composition poL with the projection $p: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is local embedding with finitely many double points. Given such a link, a presentation for the link group (the Wirtinger presentation) may be found in the following way. For each component of the link minus the lower member of each double point pair assign a generator. (This will correspond to a loop coming in on a straight line from $\infty$, going once around this component, and returning to $\infty$). For the double point corresponding to the arc x crossing over the point separating arcs y and z, there is a relation $xyx^{-1} = z$ , where the arcs are oriented as in Figure 1.

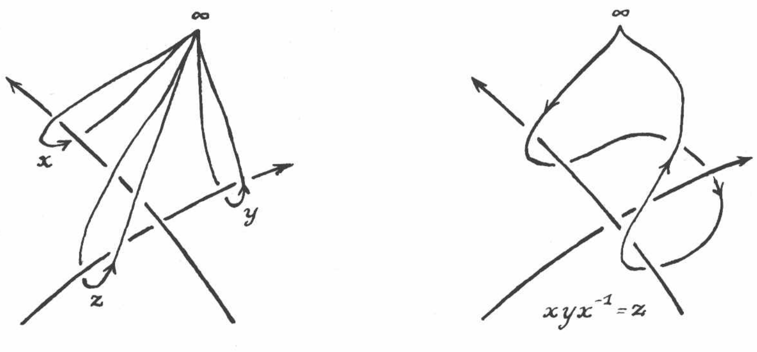


Figure 1

This gives a presentation of deficiency 0 for G(L), of the form $\{x_{ij}, 1 \leq j \leq j(i), 1 \leq i \leq \mu \mid u_{ij}\, x_{ij}\, u_{ij}^{-1} = x_{ij+1}, 1 \leq j \leq j(i), 1 \leq i\}$ (where $u_{ij} = x_{pq}^{\pm 1}$ for some p, q and $x_{ij(i)+1} = x_{i1}$). It is not hard to show that one of these relations is redundant. (See Crowell and Fox [43; pages 72-86] for details). Thus a 1-link group has a presentation of deficiency 1. For a knot group this is clearly best possible.

Theorem 2 The group G of a link L has a presentation of deficiency greater than 1 if and only if L is splittable.

Proof If G has a presentation with a generators and b relations, then G is the fundamental group of a 2-dimensional cell complex Z with 1 0-cell, a 1-cells, and b 2-cells, so rank $H_2(G;\mathbb{Z}) \leq$ rank $H_2(Z;\mathbb{Z})$ = rank $H_1(Z;\mathbb{Z})+b-a$. Therefore if $a - b > 1$ then rank $H_2(G;\mathbb{Z}) \leq \mu - 2 <$ rank $H_2(X(L);\mathbb{Z})$ so $\pi_2(X(L)) \neq 0$, and so by the Sphere Theorem X(L) contains an embedded essential 2-sphere which must split L. The argument in the other direction is immediate, since the group of a splittable link is the free product of 2 link groups. //

As in [ 62 ] the group G(L) can be given a "preabelian" presentation [ 123 ; page 149] of the form

$\{x_i, y_{ij}, 2 \leq j \leq j(i), 1 \leq i \leq \mu \mid [v_{ij}, x_i]y_{ij}, [w_i, x_i], 2 \leq j \leq j(i), 1 \leq i \leq \mu\}$

where the $v_{ij}$ and $w_i$ are words in the generators $x_i$ and $y_{ij}$ and where the word $w_i$ represents an $i^{th}$ longitude in G(L). (Notice that the generator $x_i$ here, and all the generators $x_{ij}$ for $1 \leq j \leq j(i)$ in the Wirtinger presentation are representatives of the $i^{th}$ meridians).

Theorem (Milnor [ 130 ]) The nilpotent quotient $G/G_n$ of a link group G has a presentation of the form

$\{x_i, 1 \leq i \leq \mu \mid [w_i^{(n)}, x_i], 1 \leq i \leq \mu, F(\mu)_n\}$ where $w_i^{(n)}$ is a word in the generators representing the image of the $i^{th}$ longitude.

These nilpotent quotients are of particular interest because of the following result.

Theorem (Stallings [ 178 ]) If $f:H \longrightarrow K$ is an homomorphism inducing an isomorphism on first homology (abelianization) and an epimorphism on second homology (with coefficients in the trivial module $\mathbb{Z}$) then $f$ induces isomorphisms on all the nilpotent quotients $f_n:H/H_n \xrightarrow{\simeq} K/K_n$. Consequently, if $\mathscr{L}$ is an I-equivalence of two links $L_o$ and $L_1$, then the natural maps $G(L_o)/G(L_o)_n \xrightarrow{\simeq} G(\mathscr{L})/G(\mathscr{L})_n$ are isomorphisms, and so the nilpotent quotients of the link group are invariant under I-equivalence.

Here $G(\mathscr{L})$ denotes $\pi_1(S^3 \times [0,1] - \mathscr{L})$. If L is an homology boundary link, the epimorphism $G(L) \longrightarrow F(\mu)$ satisfies the hypotheses of Stallings' theorem, and so $G/G_n \xrightarrow{\simeq} F(\mu)/F(\mu)_n$ for all integers $n \geq 1$. Hence $G/G_\omega \twoheadrightarrow F(\mu)/F(\mu)_\omega = F(\mu)$, since free groups are residually nilpotent [ 145 ; page 112 ]. If G is the group of a higher dimensional link then the inclusion of a set of meridians induces a map $F(\mu) \longrightarrow G$ which also satisfies the hypotheses of Stallings' theorem, so again $G/G_n \approx F(\mu)/F(\mu)_n$. In this case however we cannot assume that the map $F(\mu) \longrightarrow G/G_\omega$ is onto, although it is 1-1.

## Equivariant (co)homology

Let L be a $\mu$-component n-link and let $p:X' \to X$ be the maximal abelian cover of the exterior of L. On choosing fixed lifts of the cells of X to X' we obtain a finite free basis for $C_*$, the cellular chain complex of X', as a $\mathbb{Z}[G/G']$-module. The isomorphism determined by the meridians enables us to identify $\mathbb{Z}[G/G']$ with $\Lambda_\mu = \mathbb{Z}[\mathbb{Z}^\mu] = \mathbb{Z}[t_1,t_1^{-1}, \ldots t_\mu,t_\mu^{-1}]$, the ring of integral Laurent polynomials in $\mu$ variables. This ring is a regular noetherian domain of dimension $\mu + 1$, and in particular is factorial. As a group ring $\Lambda_\mu$ has a natural involution, denoted by an overbar, sending each $t_i$ to $\bar{t}_i = t_i^{-1}$, and augmentation $\varepsilon:\Lambda \to \mathbb{Z}$, which sends each $t_i$ to 1.