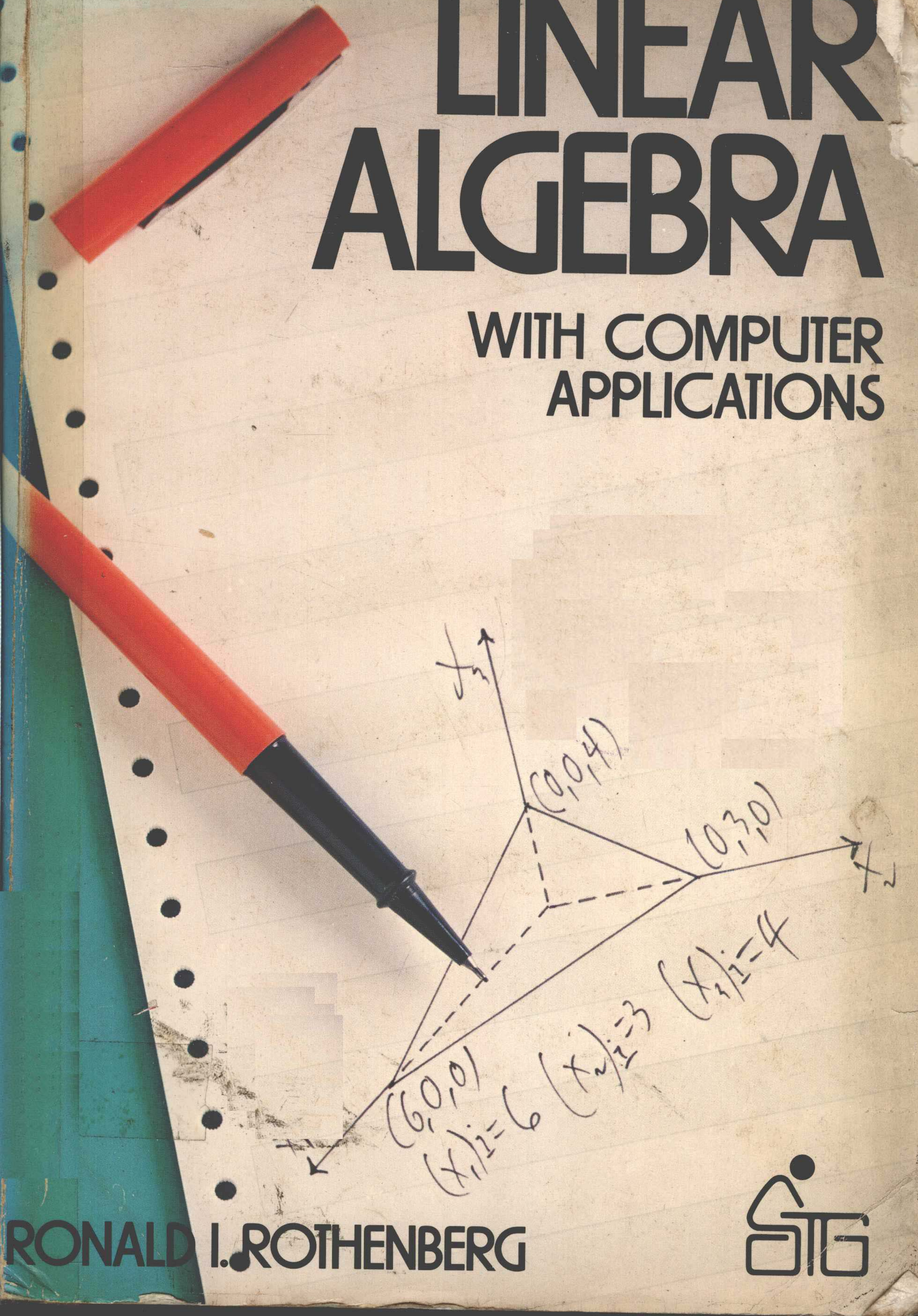
LINEAR ALGEBRA

WITH COMPUTER APPLICATIONS



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Linear Algebra with Computer Applications

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CHAPTER ONE

Elementary Matrix Algebra

Matrix algebra is a branch of *linear algebra*. Matrices (plural of matrix) are useful in studying systems of linear equations and are used for formulation and analysis of many subjects that arise in economics, business, the natural sciences, the social sciences, and computer science. In this chapter we shall study some of the fundamental rules, properties, and manipulations of matrix algebra.

OBJECTIVES

When you complete this chapter you should be able to

- Identify the entries of a matrix and know how to perform the operations of matrix addition and multiplication of a matrix by a number.
- Perform operations with column and row vectors.
- Multiply matrices.
- Apply matrix methods to practical situations, such as classification of industrial and business data.
- Find the transpose of a matrix and understand its properties.

A. INTRODUCTION TO MATRICES

1. A *matrix* is a rectangular array of numbers.* The following are matrices:

$$\text{(a)} \begin{bmatrix} 3 & 2 & 1 \\ 4 & 6 & 5 \end{bmatrix}, \qquad \text{(b)} \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \qquad \text{(c)} \ [7, \ 6, \ 5], \qquad \text{(d)} \begin{bmatrix} 6 & 4 & 5 \\ 2 & 1 & -3 \\ 8 & -2 & 7 \end{bmatrix}$$

* The "numbers" referred to here are to be considered as real numbers unless otherwise stated. Often, when referring to real numbers, we shall omit the word *real*. Actually, most of the linear algebra properties considered in this book apply equally well to complex numbers. A complex number is a number of the form $c = a + ib$, where a and b are real numbers and where $i = \sqrt{-1}$.

The decision to limit most of the development in this book to the real number case is based on the fact that in the analysis of most practical situations through linear algebra, it will be sufficient to use real numbers.

The numbers in the matrix are called *entries* or *elements*. A matrix is said to have a *size* $m \times n$ (or m by n), where m denotes the number of rows (horizontal lines) and n denotes the number of columns (vertical lines).

For example, the size for the matrix cited as (a) above is 2×3. What are the sizes for the other three preceding matrices?

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(b) 2×1, (c) 1×3, (d) 3×3.

2. We will usually denote matrices by uppercase letters (A, B, M, etc.). Lowercase letters, such as a, b, c, r, q, usually will denote numbers; in discussions on matrices it is common to use the word *scalar* for a number. In a general discussion of a matrix A, we refer to the entry in row i and column j as a_{ij}, which is read "a sub i,j." Thus, the general 2×3 matrix A is written

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

For example, a_{12} is the entry in row 1 and column 2 and a_{23} is the entry in row 2 and column 3. Here, a_{23} is read "a sub 2, 3."

Note that the row label increases in the "top to bottom direction" (row 1, row 2, etc.) while the column label increases in the "left to right" direction (column 1, column 2, etc.). Thus, row 2 is the second horizontal line from the top and column 3 is the third vertical line from the left.

Suppose the matrix cited as (d) in frame 1 is called B. Identify all the entries of B by using the b_{ij} notation.

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$b_{11} = 6$, $b_{12} = 4$, $b_{13} = 5$, $b_{21} = 2$, $b_{22} = 1$, $b_{23} = -3$, $b_{31} = 8$, $b_{32} = -2$, and $b_{33} = 7$. For example, $b_{31} = 8$ is the entry in row 3 and column 1.

3. Matrices arise in a natural way when one tries to record information that involves a two-way classification of data. Suppose an appliance store stocks a particular brand of television set in three sizes—small (12 inches), medium (17 inches), and large (21 inches)—and in the two types—black & white and color. The inventory (list of goods available in stock) at the end of May could be recorded in matrix form as follows, where the matrix is arbitrarily denoted as S:

$$\begin{array}{cc} & \begin{array}{ccc} \text{Sm.} & \text{Med.} & \text{Lg.} \end{array} \\ S = & \begin{bmatrix} 10 & 18 & 14 \\ 17 & 15 & 11 \end{bmatrix} \begin{array}{l} \text{B \& W} \\ \text{Color} \end{array} \end{array}$$

For example, $s_{11} = 10$ indicates the store has available 10 small, black & white sets in stock, while s_{23} indicates that 11 large, color sets are in stock.
Interpret the entries s_{13} and s_{22}.

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The s_{13} entry indicates 14 large, black & white sets in stock while the s_{22} entry indicates 15 medium, color sets in stock.

4. If a matrix has m rows and n columns, we say the size of the matrix is $m \times n$. Thus, matrices A and B have the same size if they have the same number of rows, and columns, respectively. Two matrices A and B are said to be *equal*, indicated by $A = B$, if they have the same size and if their corresponding entries are equal.
For example,

$$\text{if} \quad A = \begin{bmatrix} 7 & 2 & 5 \\ 6 & 4 & -9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & 2 & 5 \\ 6 & b_{22} & -9 \end{bmatrix},$$

then for A and B to be equal, we must have $b_{11} = 7$ and $b_{22} = 4$.
Consider the three matrices C, D, and E below. Indicate whether or not equality exists between any pair. Give reasons for your answers.

$$C = \begin{bmatrix} 5 & -7 \\ 2 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 2 \\ -7 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 5 & -7 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

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$C \neq D$ ($\neq$ means *not* equal), since corresponding entries are not equal. $C \neq E$, since C and E have different sizes; likewise, $D \neq E$.

5. A matrix with one row is called a *row vector*. Thus, a row vector with k entries is the same as a $1 \times k$ matrix. A matrix with one column is called a *column vector*. Thus, a column vector with k entries is the same as a $k \times 1$ matrix.
We shall often denote vectors by lowercase letters with boldface type, as in $\mathbf{x}$ or $\mathbf{b}$. We read $\mathbf{x}$ as "vector x." Other symbols used in practice are a letter with an arrow above, as in $\vec{c}$, a letter with a caret above, as in $\hat{c}$, or a letter with a bar above, as in $\bar{c}$.
The entries of vectors are often referred to as components. In addition, a single subscript is usually used. Thus, for a k-component row vector $\mathbf{a}$, we have the symbolism

$$\mathbf{a} = [a_1, a_2, \ldots, a_k],$$

where the components are separated by commas.

The general k-component column vector **b** is denoted by

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \qquad \text{where the components are } b_1, b_2, \text{etc.}$$

Here, as for row vectors, we use a single subscript for components.
Which of the matrices of frame 1 are vectors?

$\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ is a two-component column vector, and [7, 6, 5] is a three-component row vector. If the latter is called **c**, then $c_1 = 7$, $c_2 = 6$, and $c_3 = 5$.

6. A zero vector is a vector for which every component is zero. We will denote a zero vector by the boldface zero symbol, **0**. Thus,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{0} = [0, 0, \ldots, 0]$$

are symbols for a zero column vector and row vector, respectively. The number of components, and whether a zero vector is a row or column vector, will be clear from the nature of the discussion. A three-component zero column vector is

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we consider the definition for matrix addition (or sum).

Definition 1.1 (Matrix Addition):

Given two matrices A and B of the same size. The sum of A and B, written as $A + B$, is the matrix obtained by adding corresponding entries of A and B. Addition (i.e., sum) is *not defined* for matrices with different sizes.

Consider the 2×3 matrices A and B:

$$A = \begin{bmatrix} 2 & 6 & 1 \\ 4 & 3 & 8 \end{bmatrix}, \qquad B = \begin{bmatrix} 6 & -3 & 4 \\ 2 & 5 & 1 \end{bmatrix}$$

The row 1, column 1, entry of $A + B$ is $2 + 6 = 8$. Determine the entire matrix $A + B$.

$$A + B = \begin{bmatrix} 2+6 & 6+(-3) & 1+4 \\ 4+2 & 3+5 & 8+1 \end{bmatrix} = \begin{bmatrix} 8 & 3 & 5 \\ 6 & 8 & 9 \end{bmatrix}$$

Here the size of $A + B$ is also 2×3.

7. In general, if A and B are both $m \times n$ matrices, then the sum matrix $A + B$ will also be $m \times n$.

Refer to the appliance store situation of frame 3, which presents the matrix S, for television sets in stock at the end of May. Suppose the store gets a shipment of television sets on June 1, as given by the matrix J:

$$J = \begin{array}{c} \begin{matrix} \text{Sm.} & \text{Med.} & \text{Lg.} \end{matrix} \\ \begin{bmatrix} 6 & 10 & 8 \\ 8 & 12 & 7 \end{bmatrix} \end{array} \begin{matrix} \text{B \& W} \\ \text{Color} \end{matrix}$$

The matrix representing the total amounts of sets in stock after the shipment arrives is given by $S + J$, where

$$S + J = \begin{array}{c} \begin{matrix} \text{Sm.} & \text{Med.} & \text{Lg.} \end{matrix} \\ \begin{bmatrix} 16 & 28 & 22 \\ 25 & 27 & 18 \end{bmatrix} \end{array} \begin{matrix} \text{B \& W} \\ \text{Color} \end{matrix}$$

Thus, for example, after the shipment arrives, there are 27 medium-sized, color T.V.'s in stock.

Certain properties for matrix addition follow directly from properties for addition of numbers. These are summarized in the following theorem.

Theorem 1.1

For matrices A, B, and C, all of the same size,

(a) $A + B = B + A$; i.e., addition of matrices is *commutative*.
(b) $(A + B) + C = A + (B + C)$; i.e., addition of matrices is *associative*.

Because of the equality in (b), we define the triple matrix sum $A + B + C$ as equal to the common value given in (b).

The proof of this theorem depends merely on the commutative and associative properties for addition of real numbers and the definition of matrix addition.

The parentheses in $(A + B) + C$ means that *first* A and B are added, and then the resulting matrix is added to C. Suppose

$$A = \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 2 \\ -3 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 9 & 2 \end{bmatrix}$$

Then,

$$(A + B) + C = \left(\begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix} + \begin{bmatrix} 8 & 2 \\ -3 & 7 \end{bmatrix} \right) + \begin{bmatrix} 1 & 0 \\ 9 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -1 \\ 3 & 12 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} 13 & -1 \\ 12 & 14 \end{bmatrix}$$

Show that computation of $A + (B + C)$ results in the same final matrix. Note that the parentheses here means that *first* B and C are added and then the resulting matrix is added to A.

$$A + (B + C) = \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix} + \left(\begin{bmatrix} 8 & 2 \\ -3 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 9 & 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 2 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} 13 & -1 \\ 12 & 14 \end{bmatrix}$$

8. A matrix for which all the entries are zero is called a zero matrix and is denoted by **O**. The following are zero matrices:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad [0]$$

The sizes are $2 \times 2, 3 \times 2, 2 \times 4$, and 1×1, respectively. If A is any matrix and **O** is a zero matrix of the same size, then

$$A + \mathbf{O} = \mathbf{O} + A = A$$

Because of this last property, the matrix **O** can be thought of as the additive identity for matrix addition. Note that the number 0 is the additive identity for ordinary addition of numbers since $k + 0 = 0 + k = k$ for any number k.

Definition 1.2 (Scalar Multiplication):

The *product of a scalar* (number) k *and a matrix* A, written as kA, is the matrix obtained by multiplying each entry of A by k.

Thus, if A is a general 2×3 matrix, we have

$$kA = k\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{bmatrix}$$

Note that A and kA will have the same size.

The *negative of a matrix* A, denoted by $-A$, is defined to be the scalar -1 times A. That is,

$$-A = (-1)A$$

Suppose $A = \begin{bmatrix} 6 & 5 & 4 \\ 1 & -2 & 3 \end{bmatrix}$. Determine (a) $3A$, and (b) $-A$.

(a) $$3A = 3\begin{bmatrix} 6 & 5 & 4 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 3\cdot 6 & 3\cdot 5 & 3\cdot 4 \\ 3\cdot 1 & 3\cdot(-2) & 3\cdot 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 & 12 \\ 3 & -6 & 9 \end{bmatrix}.$$

(b) $$-A = (-1)\begin{bmatrix} 6 & 5 & 4 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -6 & -5 & -4 \\ -1 & 2 & -3 \end{bmatrix}.$$

9. Next, we consider the definition for matrix subtraction.

Definition 1.3 (Matrix Subtraction):

The *difference* between matrices A and B of the same size, denoted by $A - B$, and referred to as A *minus* B, is defined by $A - B = A + (-B)$. That is, $A - B$ is equivalent to adding A and $(-B)$.

Suppose that $A = \begin{bmatrix} 6 & 5 & 4 \\ 1 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 8 & 5 \\ 2 & 1 & 1 \end{bmatrix}$. Determine (a) $-B$, (b) $A - B$, and (c) $A - A$.

(a) $$-B = (-1)\begin{bmatrix} 3 & 8 & 5 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -8 & -5 \\ -2 & -1 & -1 \end{bmatrix}.$$

(b) $$A - B = A + (-B) = \begin{bmatrix} 6 & 5 & 4 \\ 1 & -2 & 3 \end{bmatrix} + \begin{bmatrix} -3 & -8 & -5 \\ -2 & -1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -3 & -1 \\ -1 & -3 & 2 \end{bmatrix}.$$

(c) $$A - A = \begin{bmatrix} 6 + (-6) & 5 + (-5) & 4 + (-4) \\ 1 + (-1) & (-2) + 2 & 3 + (-3) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}.$$

10. Note that for any matrix A, we have that $A - A$ is the zero matrix of the same size. Observe that a similar result holds with respect to numbers, namely, that $k - k = 0$, for any number k. Also, observe that the difference $A - B$ can be calculated by subtracting each entry of B from the corresponding entry of A. Thus, for the previous example,

$$A - B = \begin{bmatrix} 6 & 5 & 4 \\ 1 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 8 & 5 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & -1 \\ -1 & -3 & 2 \end{bmatrix}$$

Several properties involving multiplication of a matrix by a scalar are given in the next theorem.

Theorem 1.2

For matrices A and B of the same size, and for any two scalars h and k, the following rules hold:

(a) $(h + k)A = hA + kA$,
(b) $h(A + B) = hA + hB$,
(c) $hk(A) = h(kA)$.

Verify property (b) with respect to the matrices A and B of frame 9 if $h = 3$. The left side of (b) indicates that you should first form the matrix sum $A + B$ and then multiply by 3, while the right side of (b) indicates that you should first form the matrices $3A$ and $3B$ and then add $3A$ and $3B$ together.

Left side: $$3(A + B) = 3\left(\begin{bmatrix} 6 & 5 & 4 \\ 1 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 8 & 5 \\ 2 & 1 & 1 \end{bmatrix}\right) = 3\begin{bmatrix} 9 & 13 & 9 \\ 3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 27 & 39 & 27 \\ 9 & -3 & 12 \end{bmatrix}$$

Right side: $$3A + 3B = 3\begin{bmatrix} 6 & 5 & 4 \\ 1 & -2 & 3 \end{bmatrix} + 3\begin{bmatrix} 3 & 8 & 5 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 15 & 12 \\ 3 & -6 & 9 \end{bmatrix} + \begin{bmatrix} 9 & 24 & 15 \\ 6 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 27 & 39 & 27 \\ 9 & -3 & 12 \end{bmatrix}$$

We end up with the same matrix, thus verifying property (b) of Theorem 1.2 for the particular matrices and the scalar 3.

11. We don't want to forget that one of the reasons we study matrices is because of their practical applications. We have already focused on two practical examples dealing with using matrices for a two-way classification of data (frames 3 and 7). Now we will make use of matrices S and J listed in frames 3 and 7 in another practical application. These matrices represent the inventory at the end of May and the shipment arriving in June, respectively. Suppose we let the matrix D represent the television sets sold during the month of June. For example, suppose

$$D = \begin{array}{c} \\ \end{array} \begin{array}{c} \text{Sm.} \quad \text{Med.} \quad \text{Lg.} \\ \begin{bmatrix} 9 & 19 & 17 \\ 16 & 21 & 16 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{B \& W} \\ \text{Color} \end{array}$$

This indicates that 9 ($= d_{11}$) small, black & white and 21 ($= d_{22}$) medium-sized, color T.V.'s are sold during June, for example. It should be clear that the matrix given by $S + J - D$ represents the inventory of T.V. sets at the *end* of June. (Recall that inventory refers to goods in stock.) Determine the matrix $S + J - D$ and interpret several entries in the matrix. (Note that one way of computing the entries of $S + J - D$ is by first adding corresponding entries of S and J and then subtracting the corresponding entry of D.)

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$$S + J - D = \begin{bmatrix} 10 & 18 & 14 \\ 17 & 15 & 11 \end{bmatrix} + \begin{bmatrix} 6 & 10 & 8 \\ 8 & 12 & 7 \end{bmatrix} - \begin{bmatrix} 9 & 19 & 17 \\ 16 & 21 & 16 \end{bmatrix}$$

$$= \begin{array}{c} \text{Sm.} \quad \text{Med.} \quad \text{Lg.} \\ \begin{bmatrix} 7 & 9 & 5 \\ 9 & 6 & 2 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{B \& W} \\ \text{Color} \end{array}$$

As a partial interpretation, we see there will be nine medium-sized, black & white T.V. sets and two large, color sets left in stock at the end of June.

B. MATRIX MULTIPLICATION

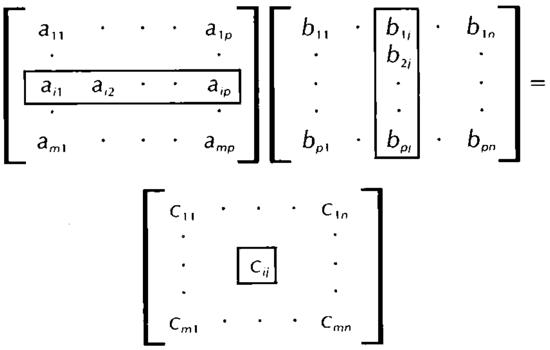
12. We shall now define what is meant by matrix multiplication. Let A and B be matrices in which the number of columns of A is equal to the number of rows of B. Then AB, which is read as the *product of A and B* (or *A times B*), is a matrix that has the same number of rows as A and the same number of columns as B.

Definition 1.4 (Matrix Multiplication):

Suppose A is $m \times p$ and B is $p \times n$. Then the product matrix AB is an $m \times n$ matrix. Let us temporarily rename AB as C. The computation of c_{ij}, the entry in the ith row (i.e., row i) and jth column (i.e., column j) of matrix C, is indicated as follows:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$$

There are p terms in this sum. A schematic diagram showing this multiplication appears on Figure 1.1.



$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$$

FIGURE 1.1 Matrix multiplication: $C = AB$.

As indicated by the boxed-in strips in Figure 1.1, to find the entry c_{ij} we multiply the corresponding entries of row i of A and column j of B and add the resulting products.

Notes: (a) If the number of columns of A is not equal to the number of rows of B—for example, if A is $m \times p$ and B is $r \times n$, where $p \neq r$—then the product AB is *not defined.*

(b) The product BA is usually not equal to the product AB. In fact, if A is $m \times p$ and B is $p \times n$, and m is unequal to n, then AB is an $m \times n$ matrix but BA is not even defined.

The following example will illustrate matrix multipliction.

Example: Suppose the 3×2 matrix A and 2×4 matrix B are given as follows:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 7 & 8 & 9 & 1 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

Determine the product matrix $C = AB$.

Solution: First, observe that C will be 3×4, since A is $\underline{3} \times 2$ and B is $2 \times \underline{4}$. Also, observe that the matrix product BA is not defined, since the number of columns of B (4) is not equal to the number of rows of A (3).

For the calculation of c_{23}, we form products between the entries in row 2 of A and column 3 of B and then add the products. Thus,

$$c_{23} = 3 \cdot 9 + 4 \cdot 4 = 27 + 16 = 43$$

For another example, c_{11} is computed from products between the entries of row 1 of A and column 1 of B. Thus,

$$c_{11} = 1 \cdot 7 + 2 \cdot 2 = 7 + 4 = 11$$

Also

$$c_{12} = 1 \cdot 8 + 2 \cdot 3 = 8 + 6 = 14.$$

Compute the rest of the entries of the matrix $AB = C$ and display the matrix AB.

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Some of the calculations of typical entries are as follows:

$$c_{24} = 3 \cdot 1 + 4 \cdot 5 = 3 + 20 = 23$$

$$c_{31} = 5 \cdot 7 + 6 \cdot 2 = 35 + 12 = 47$$

$$c_{33} = 5 \cdot 9 + 6 \cdot 4 = 45 + 24 = 69$$

Thus, matrix $C = AB$ is the following 3×4 matrix:

$$AB = \begin{bmatrix} 11 & 14 & 17 & 11 \\ 29 & 36 & 43 & 23 \\ 47 & 58 & 69 & 35 \end{bmatrix}$$

13. Here are further notes on the characteristics of matrix multiplication.

Notes: (i) Suppose A is $m \times p$. Then both AB and BA will be defined if B is $p \times m$. In this case, AB will be $m \times m$ and BA will be $p \times p$. (ii) If A and B are both $m \times m$, then AB and BA both will be defined and will have the same size, namely, $m \times m$. Usually, for this case, AB will not equal BA.

For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$, you should verify that

$AB = \begin{bmatrix} 10 & 4 \\ 24 & 10 \end{bmatrix}$ and $BA = \begin{bmatrix} 10 & 16 \\ 6 & 10 \end{bmatrix}$; that is, $AB \neq BA$.

The following theorem summarizes several rules of matrix multiplication. It is assumed that the products and sums are defined.

Theorem 1.3

Matrix multiplication satisfies the following properties:

(a) $(AB)C = A(BC)$ (Associative Law)
(b) $A(B + C) = AB + AC$ (Left Distributive Law)
(c) $(B + C)A = BA + CA$ (Right Distributive Law)
(d) $k(AB) = (kA)B = A(kB)$, where k is a scalar

Note: Because of property (a), we define the triple matrix product ABC as equal to the common value given in (a): thus, $ABC = (AB)C = A(BC)$.

Suppose matrices A, B, and C are given by

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$$

Verify properties (a) and (b) of Theorem 1.3 for these matrices. To help you get started, note that the formulation $(AB)C$ implies that *first* the matrix product AB is computed and then AB is multiplied times C.

Thus,

$$AB = \begin{bmatrix} 1\cdot1 + 2\cdot2 & 1\cdot3 + 2\cdot4 \\ 0\cdot1 + 3\cdot2 & 0\cdot3 + 3\cdot4 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 6 & 12 \end{bmatrix}$$

Then,

$$(AB)C = \begin{bmatrix} 5 & 11 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5\cdot2 + 11\cdot4 & 5\cdot(-3) + 11\cdot1 \\ 6\cdot2 + 12\cdot4 & 6\cdot(-3) + 12\cdot1 \end{bmatrix}$$